Theme: Multiple Linear Regression
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Multiple linear regression attempts to model the relationship between two or more explanatory variables and a response variable by fitting a linear equation to observed data. Every value of the independent variable x is associated with a value of the dependent variable y. The population regression line for p explanatory variables $x_1, x_2, \ldots, x_p$ is defined to be $\mu_y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$. This line describes how the mean response μ_y changes with the explanatory variables. The observed values for y vary about their means μ_y and are assumed to have the same standard deviation σ. The fitted values $b_0, b_1, \ldots, b_p$ estimate the parameters $\beta_0, \beta_1, \ldots, \beta_p$ of the population regression line.

Since the observed values for y vary about their means μ_y, the multiple regression model includes a term for this variation. In words, the model is expressed as DATA = FIT + RESIDUAL, where the "FIT" term represents the expression $\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots \beta_p x_p$. The "RESIDUAL" term represents the deviations of the observed values y from their means μ_y, which are normally distributed with mean 0 and variance σ. The notation for the model deviations is ε.

Formally, the model for multiple linear regression, given n observations, is

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots \beta_p x_{ip} + \varepsilon_i \text{ for } i = 1, 2, \dots n.$$

In the least-squares model, the best-fitting line for the observed data is calculated by minimizing the sum of the squares of the vertical deviations from each data point to the line (if a point lies on the fitted line exactly, then its vertical deviation is 0). Because the deviations are first squared, then summed, there are no cancellations between positive and negative values. The least-squares estimates b_0, b_1, ... b_p are usually computed by statistical software.

The values fit by the equation $b_0 + b_1 x_{i1} + ... + b_p x_{ip}$ are denoted $\hat{y}_i$, and the residuals e_i are equal to $y_i - \hat{y}_i$, the difference between the observed and fitted values. The sum of the residuals is equal to zero.

The variance σ^2 may be estimated by $s^2 = \dfrac{\sum e_i^2}{n - p - 1}$, also known as the mean-squared error (or MSE).

The estimate of the standard error s is the square root of the MSE.

Example

The dataset "Healthy Breakfast" contains, among other variables, the *Consumer Reports* ratings of 77 cereals and the number of grams of sugar contained in each serving. (*Data source: Free publication available in many grocery stores. Dataset available through the Statlib Data and Story Library (DASL).*)

A simple linear regression model considering "Sugars" as the explanatory variable and "Rating" as the response variable produced the regression line Rating = 59.3 - 2.40 Sugars, with the square of the correlation $r^2 = 0.577$ (see Inference in Linear Regression for more details on this regression).

The "Healthy Breakfast" dataset includes several other variables, including grams of fat per serving and grams of dietary fiber per serving. Is the model significantly improved when these variables are included?

Suppose we are first interested in adding the "Fat" variable. The correlation between "Fat" and "Rating" is equal to -0.409, while the correlation between "Sugars" and "Fat" is equal to 0.271. Since "Fat" and "Sugar" are not highly correlated, the addition of the "Fat" variable may significantly improve the model.

The MINITAB "Regress" command produced the following results:

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Regression Analysis

The regression equation is
Rating = 61.1 - 3.07 Fat - 2.21 Sugars
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Standardized Residuals
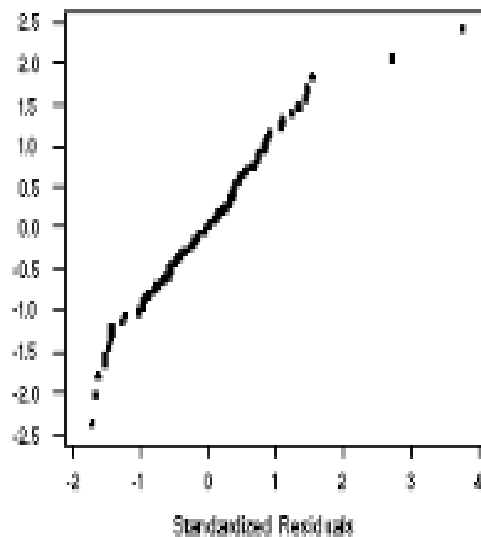
After fitting the regression line, it is important to investigate the residuals to determine whether or not they appear to fit the assumption of a normal distribution. A normal quantile plot of the standardized residuals $y - \hat{y}$ is shown to the left. Despite two large values which may be outliers in the data, the residuals do not seem to deviate from a random sample from a normal distribution in any systematic manner.

The test statistic t is equal to b_j/s_{bj}, the parameter estimate divided by its standard deviation. This value follows a $t(n-p-1)$ distribution when p variables are included in the model.

In the example above, the parameter estimate for the "Fat" variable is -3.066 with standard deviation 1.036 The test statistic is t = -3.066/1.036 = -2.96, provided in the "T" column of the MINITAB output. For a two-sided test, the probability of interest is $2P(T \geq |-2.96|)$ for the $t(77-2-1) = t(74)$ distribution, which is about 0.004. The "P" column of the MINITAB output provides the P-value associated with the two-sided test. Since the P-values for both "Fat" and "Sugar" are highly significant, both variables may be included in the model.

Thank you for attention