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# Numerical study of filtration process of ground and pressure waters in multilayer porous media

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**Abstract.** A article discusses the process of forecasting changes in the level of ground and pressure water. A brief analysis and computational experiments of scientific papers on mathematical and numerical modeling of the object under study are given. For a comprehensive study of the problem under consideration, a mathematical model was developed that takes into account the external source, evaporation, filtration coefficients, active porosity, filtration rate and two-way boundary conditions. An effective numerical algorithm has been developed for predicting changes in the ground water level using a combination of finite-difference schemes and run-through methods. It has been studied that changes in the level of ground and pressure water, filtration permeability, water loss coefficient and filtration rate associated with the water level can have a serious impact on the environmental process.

## 1. Introduction

The main tasks of hydrogeology, including the tasks related to land development, reclamation and irrigation construction, assessment of groundwater reserves and resources, and many other aspects, are to provide a prediction of hydrodynamic and hydromechanical regime of groundwater, which is a closely interconnected element of a single geofiltration system.

In conditions of acute shortage of water resources, water supply issues for the population, especially in ecologically disadvantaged zones of Central Asia, Khiva, Bukhara, including in the territories of Karakalpakstan, are especially relevant. One of the main sources of household and drinking water supply to the population in such conditions is groundwater formed by the construction of groundwater intakes.

Groundwater intakes are designed by performing numerous computational experiments using mathematical models of geofiltration and salt transfer in the underground hydrosphere, in order to determine the main indicators and parameters of the object of study.

The main objective in the issues of stable development of the agricultural sector is to increase crop yields and the quality of the output product, subject to significant savings in labor and energy resources, environmental requirements, etc. Which, in turn, is associated with solving the problems of substantiating the intensity of water reclamation of agrolandscapes, optimizing the calculation of agricultural drainage and managing the water regime of agricultural land. It should be noted that the



volume of drainage and waste water in many irrigation systems in Central Asia, the Caucasus and other areas reaches water intake.

In particular, a decrease in the Aral Sea level caused significant changes in the exploitation of aquifers, groundwater in the coastal zone, and had a negative impact on the environment. In fact, fisheries have been eliminated, groundwater exploitation conditions have worsened, the fauna has become poor, and the surface freed from the sea is subjected to aeolian processes, which entail a decrease in the productivity of coastal pasture lands adjacent to the Aral Sea, etc.

In forecasts of hydrogeological and ameliorative situation for the study of hydrodynamic regime of ground and pressure waters and the salt regime of soils, the following methods are mainly used: the balance, correlation and regression, analytical methods, the methods of identification, of numerical and analog modeling.

To conduct a comprehensive study, forecast and to make managerial decisions on the above mentioned issues, a number of problems have been solved, where the core is a mathematical model, a numerical algorithm and a software-instrumental complex for conducting computer experiment.

The basics of the science of groundwater movement (hydrogeology) are associated with the names of A. Darcy, J. Dupuis, N.E. Zhurkovsky, F. Forchheimer and others. A major role in the development of mathematical methods with the intensive development of the theory and practice of groundwater movement was also played by the works of F.B. Abutaliev, E.B. Abutaliev, P.Ya. Polubarinova-Kochina, V.I. Aravina, S.N. Numerova, G.N. Kamensky, A.I. Silina-Bekchurina, P.P. Klimentova, G.B. Pykhacheva, V.A. Mironenko, I.K. Gavich et al.

The theoretical foundations of the hydrodynamic method for assessing the operational reserves of groundwater are considered in the works of F.M. Bochevera, N.N. Bindeman, N.I. Plotnikova, V.M. Shestakova, L.S. Yazvina, E.N. Bondareva, V.N. Nikolaevsky, V.I. Lavrika, V.I. Penkovsky and others VN Shchelkachev, M. A. Guseyn-zade, V. M. Shestakov, N. N. Verigin, I. A. Charny, F made a great contribution to the development of the theory of the elastic regime in the strata and the study of the problems of fluid flow in the layers. M. Bochever et al. Among foreign researchers, M.S. Hantusha, S.E. Jacob for the first time, apparently, pointed out the need to take into account the elastic regime in a poorly permeable layer.

To conduct a comprehensive study, forecast and to make managerial decisions on the above mentioned issues, a number of problems have been solved, where the core is a mathematical model, a numerical algorithm and a software-instrumental complex for conducting computer experiment.

In [1], the hydrogeological conditions of construction site of an underground complex and a mathematical model of soil base geofiltration were described. The results of computational studies of hydrogeological regime change in construction site when fencing the pit with a wall in the ground were considered.

A stationary model of groundwater filtration developed in [2], was used to quantify and analyze the underground hydrodynamics in the Akaki catchment, paying particular attention to the borehole field that supplies with fresh water the city of Addis Ababa. The simulation was performed in a two-layer unlimited aquifer with a spatially variable recharge and hydraulic conductivity under well-defined boundary conditions. The model was used to predict the pattern of groundwater flow, the interaction of groundwater and surface-water, and the effect of pumping on the borehole field under various scenarios.

In [3], a one-dimensional mathematical model of dissolved substances transport in finite aquifers was considered. The basic equation of dissolved substances transport by an unsteady-state flow of groundwater was solved analytically by the Laplace transform method. Initially, the aquifer was subjected to a spatially dependent concentration of the source with zero order formation. One end of the aquifer receives the concentration of the source and is represented by a mixed-type boundary condition in the time domain of solution. The concentration gradient at the other end of the porous medium is assumed to be zero.

In [4], the mechanism of artificial recharge affecting the groundwater reservoir was considered. Various scenarios of location of the infiltration basin and replenishment intensity model have been

developed based on a generalized groundwater reservoir in a two-dimensional sand reservoir in order to study how to increase the efficiency of artificial replenishment of groundwater reservoir.

The authors in [5] have simulated the process of groundwater filtration taking into account the nonuniform distribution and rarefaction of the aquifer with insufficient data on the object and poor knowledge of their properties. The problem was solved by taking into account the vertical stratification of aquifers of equal thickness.

In [6,18], a mathematical model was developed for predicting groundwater levels in two-layer formations. The authors of the paper consider a two-layer medium consisting of two layers: soil (with low permeability) and water as a mathematical model of the geofiltration process.

The papers [7-8] are devoted to numerical modeling of water and salt transfer process in soil. To conduct a comprehensive study, a mathematical model was proposed taking into account the colmatage of soil pores with fine particles over time; changes in soil permeability coefficient, water loss and filtration coefficient; changes in initial porosity and the porosity of settled mass; an effective numerical algorithm based on the Samarsky-Fryazinov vector scheme with a second order of approximation of differential operators to finite difference one was considered. To derive a mathematical model of salt transfer it was assumed that the pressure gradient in the channel is constant and equal to atmospheric pressure. The calculation results for the proposed algorithms were presented in the form of graphical objects; a detailed analysis of these results was given.

In [9], the computational filtering schemes from canals and water distributors in soil, underlain by a highly permeable pressure aquifer or waterproof base were developed; they allowed a comparative assessment of the role of cross-sectional profile of water source channel and water level in it, water support from underlying horizon or confining layer.

In [10], a model was proposed that allowed obtaining reliable information on groundwater level change and on the intensity of water reclamation of agro-landscapes, to optimize the calculation of agricultural drainage and to adjust the water regime of agricultural lands.

Mathematical modeling proposed in [11], was based on the data of complex studies carried out within the Ararat and Aparan intermountain basins; the issues related to the prevention of environmental consequences caused by large groundwater intakes were considered. As a result of data analysis by the mathematical modeling method, the problems of predicting the regime of groundwater level changes were solved while maintaining a constant load on the existing water intakes.

In [12], a model was given that correctly simulated the behavior of a free aquifer. During the periods of model operation, the values of groundwater level were calculated in a model with piezometers, in accordance with the groundwater level recorded by the piezometers.

A modeling system based on soil moisture content and groundwater dynamics was considered in [13]; numerical solutions were obtained for modeling hydrological processes of replenishment of sediments, surface-water, groundwater, and soil moisture content.

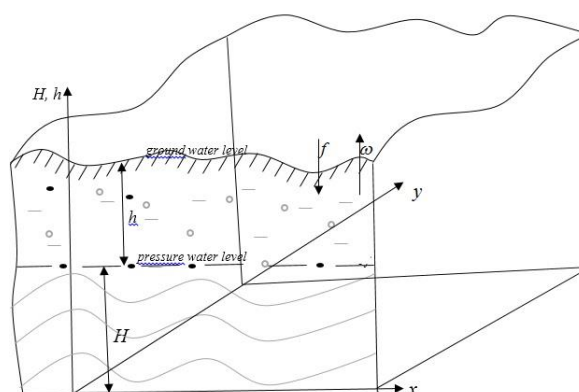
In [14], a method was proposed for numerical solution of non-stationary problems on two-component fluid flow in a porous medium, simulating transport of salt dissolved in groundwater.

In [15], an analytical method for predicting groundwater sequence regimes was considered.

In [16], an urgent problem related to the process of change in underground water level and mineral salt transfer in soils is solved in the paper. The problem is described by a system of partial differential equations and the corresponding initial, internal and boundary conditions of various kinds. To derive a mathematical model of the process under consideration, a detailed review of scientific papers devoted to various aspects and software of the object of study is given. To conduct a comprehensive study of the process of filtration and change in salt regime of groundwater, mathematical models and an effective numerical algorithm are proposed taking into account external sources and evaporation. Since the process is described by a nonlinear system of partial differential equations, it is difficult to obtain an analytical solution.

## 2. Problem Statement

In mathematical modeling of monitoring and predicting the groundwater level and hydrodynamic processes occurring in them, considering the interaction of external factors: evaporation and infiltration, the studied object is presented schematically in the form shown in Fig. 1. According to the results of hydrogeological conditions analysis, the territory by groundwater renewal (GWR) in geofiltration relation should be considered as a two-layer one in medium, consisting of two aquifers.



**Figure 1.** Schematic representation of the object of study

The conditions accepted for predicting the changes in groundwater level (ground and pressure aquifers) during the filtration process give reason to present a mathematical model of the object in the form of a system of nonlinear partial differential equations:

$$\left. \begin{aligned} \mu_1 n_0 \frac{\partial h}{\partial t} &= \frac{\partial}{\partial x} \left( k_1 m \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_1 m \frac{\partial h}{\partial y} \right) + f - \omega, \\ \mu_2 \frac{\partial H}{\partial t} &= \frac{\partial}{\partial x} \left( k_2 m \frac{\partial H}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_2 m \frac{\partial H}{\partial y} \right) - \eta Q. \end{aligned} \right\} \quad (1)$$

where  $h(x, y, t)$ ,  $H(x, y, t)$  - are the groundwater and pressure water levels;  $\mu_1$ ,  $\mu_2$  - water loss coefficients;  $m$  - the capacity of the separating layer;  $k_1$ ,  $k_2$  - filtration coefficients of the upper and lower layers;  $Q$  - flow rate;  $f$  external source;  $\omega$  - evaporation;  $n_0$  - active porosity of soil in the respective zones.  $\eta$  - a coefficient to reduce the model to dimensional form.

System (1) is solved under the following initial and boundary conditions:

$$h|_{t=0} = h_0, \quad H|_{t=0} = H_0, \quad (2)$$

$$\mu_1 m \frac{\partial h}{\partial x} \Big|_{x=0} = -(h - h_0), \quad \mu_1 m \frac{\partial h}{\partial x} \Big|_{x=L_x} = (h - h_0), \quad (3)$$

$$\mu_1 m \frac{\partial h}{\partial y} \Big|_{y=0} = -(h - h_0), \quad \mu_1 m \frac{\partial h}{\partial y} \Big|_{y=L_y} = (h - h_0), \quad (4)$$

$$\mu_2 m \frac{\partial H}{\partial x} \Big|_{x=0} = -(H - H_0), \quad \mu_2 m \frac{\partial H}{\partial x} \Big|_{x=L_x} = (H - H_0), \quad (5)$$

$$\mu_2 m \frac{\partial H}{\partial y} \Big|_{y=0} = -(H - H_0), \quad \mu_2 m \frac{\partial H}{\partial y} \Big|_{y=L_y} = (H - H_0), \quad (6)$$

$$H \Big|_{y=m+0} = h \Big|_{y=m-0}, \quad (7)$$

$$k_2 m \frac{\partial H}{\partial y} \Big|_{y=m+0} = k_1 m \frac{\partial h}{\partial y} \Big|_{y=m-0}. \quad (8)$$

where  $h_0, H_0$  – are the initial values of groundwater and pressure water levels.

To solve the problem (1), introduce the dimensionless variables:

$$h^* = \frac{h}{h_0}, \quad H^* = \frac{H}{H_0}, \quad x^* = \frac{x}{L_x}, \quad y^* = \frac{y}{L_y}, \quad k_1^* = \frac{k_1}{(k_1)_0}, \quad \tau = \frac{(k_1)_0 m_0}{\mu_1 n_0 L_x^2} t, \quad m^* = \frac{m}{m_0}, \quad k_2^* = \frac{k_2}{(k_2)_0}.$$

Then problem (1) - (8) takes the following form:

$$\left. \begin{aligned} \frac{\partial h^*}{\partial \tau} &= \frac{\partial}{\partial x^*} (k_1^* m^* \frac{\partial h^*}{\partial x^*}) + \frac{L_x^2}{L_y^2} \frac{\partial}{\partial y^*} (k_1^* m^* \frac{\partial h^*}{\partial y^*}) + \frac{L_x^2}{(k_1)_0 m_0 h_0} (f - \omega), \\ \frac{\partial H^*}{\partial \tau} &= \frac{\partial}{\partial x^*} (k_2^* m^* \frac{\partial H^*}{\partial x^*}) + \frac{L_x^2}{L_y^2} \frac{\partial}{\partial y^*} (k_2^* m^* \frac{\partial H^*}{\partial y^*}) - \frac{\mu_1 n_0 L_x^2}{\mu_2 (k_1)_0 m_0 H_0} \eta Q. \end{aligned} \right\} \quad (9)$$

under boundary conditions:

$$\frac{\mu_1 m_0 h_0}{L_x} m^* \frac{\partial h^*}{\partial x^*} \Big|_{x^*=0} = -(h_0 h^* - h_0), \quad \frac{\mu_1 m_0 h_0}{L_x} m^* \frac{\partial h^*}{\partial x^*} \Big|_{x^*=1} = (h_0 h^* - h_0), \quad (10)$$

$$\frac{\mu_1 m_0 h_0}{L_y} m^* \frac{\partial h^*}{\partial y^*} \Big|_{y^*=0} = -(h_0 h^* - h_0), \quad \frac{\mu_1 m_0 h_0}{L_y} m^* \frac{\partial h^*}{\partial y^*} \Big|_{y^*=1} = (h_0 h^* - h_0), \quad (11)$$

$$\frac{\mu_2 m_0 H_0}{L_x} m^* \frac{\partial H^*}{\partial x^*} \Big|_{x^*=0} = -(H_0 H^* - H_0), \quad \frac{\mu_2 m_0 H_0}{L_x} m^* \frac{\partial H^*}{\partial x^*} \Big|_{x^*=1} = (H_0 H^* - H_0), \quad (12)$$

$$\frac{\mu_2 m_0 H_0}{L_y} m^* \frac{\partial H^*}{\partial y^*} \Big|_{y^*=0} = -(H_0 H^* - H_0), \quad \frac{\mu_2 m_0 H_0}{L_y} m^* \frac{\partial H^*}{\partial y^*} \Big|_{y^*=1} = (H_0 H^* - H_0), \quad (13)$$

$$H_0 H^* \Big|_{y^* = \frac{m_0 m^* + 0}{L_y}} = h_0 h^* \Big|_{y^* = \frac{m_0 m^* - 0}{L_y}}, \quad (14)$$

$$\frac{(k_2)_0 m_0 H_0}{L_y} k_2^* m^* \frac{\partial H^*}{\partial y^*} \Big|_{y^* = \frac{m_0 m^* + 0}{L_y}} = \frac{(k_1)_0 m_0 h_0}{L_y} k_1^* m^* \frac{\partial h^*}{\partial y^*} \Big|_{y^* = \frac{m_0 m^* - 0}{L_y}}. \quad (15)$$

Later, for simplicity, we omit the “\*” sign in equations and problem (9) - (15) in dimensionless variables is written as follows:

$$\left. \begin{aligned} \frac{\partial h}{\partial \tau} &= \frac{\partial}{\partial x} (k_1 m \frac{\partial h}{\partial x}) + \frac{L_x^2}{L_y^2} \frac{\partial}{\partial y} (k_1 m \frac{\partial h}{\partial y}) + \frac{L_x^2}{(k_1)_0 m_0 h_0} (f - \omega), \\ \frac{\partial H}{\partial \tau} &= \frac{\partial}{\partial x} (k_2 m \frac{\partial H}{\partial x}) + \frac{L_x^2}{L_y^2} \frac{\partial}{\partial y} (k_2 m \frac{\partial H}{\partial y}) - \frac{\mu_1 n_0 L_x^2}{\mu_2 (k_1)_0 m_0 H_0} \eta Q. \end{aligned} \right\} \quad (16)$$

Problem (16) has the following form:

$$\left. \begin{aligned} \frac{\partial h}{\partial \tau} &= \frac{\partial}{\partial x} (k_1 m \frac{\partial h}{\partial x}) + \xi \frac{\partial}{\partial y} (k_1 m \frac{\partial h}{\partial y}) + \xi_1 (f - \omega), \\ \frac{\partial H}{\partial \tau} &= \frac{\partial}{\partial x} (k_2 m \frac{\partial H}{\partial x}) + \xi \frac{\partial}{\partial y} (k_2 m \frac{\partial H}{\partial y}) - \xi_2 \eta Q. \end{aligned} \right\} \quad (16^*)$$

where  $\xi = \frac{L_x^2}{L_y^2}$ ,  $\xi_1 = \frac{L_x^2}{(k_1)_0 m_0 h_0}$ ,  $\xi_2 = \frac{\mu_1 n_0 L_x^2}{\mu_2 (k_1)_0 m_0 H_0}$ .

Under boundary conditions:

$$\frac{\mu_1 m_0 h_0}{L_x} m \frac{\partial h}{\partial x} \Big|_{x=0} = -(h_0 h - h_0), \quad \frac{\mu_1 m_0 h_0}{L_x} m \frac{\partial h}{\partial x} \Big|_{x=1} = (h_0 h - h_0), \quad (17)$$

$$\frac{\mu_1 m_0 h_0}{L_y} m \frac{\partial h}{\partial y} \Big|_{y=0} = -(h_0 h - h_0), \quad \frac{\mu_1 m_0 h_0}{L_y} m \frac{\partial h}{\partial y} \Big|_{y=1} = (h_0 h - h_0), \quad (18)$$

$$\frac{\mu_2 m_0 H_0}{L_x} m \frac{\partial H}{\partial x} \Big|_{x=0} = -(H_0 H - H_0), \quad \frac{\mu_2 m_0 H_0}{L_x} m \frac{\partial H}{\partial x} \Big|_{x=1} = (H_0 H - H_0), \quad (19)$$

$$\frac{\mu_2 m_0 H_0}{L_y} m \frac{\partial H}{\partial y} \Big|_{y=0} = -(H_0 H - H_0), \quad \frac{\mu_2 m_0 H_0}{L_y} m \frac{\partial H}{\partial y} \Big|_{y=1} = (H_0 H - H_0), \quad (20)$$

$$H_0 H \Big|_{y=\frac{m_0 m + 0}{L_y}} = h_0 h \Big|_{y=\frac{m_0 m - 0}{L_y}}, \quad (21)$$

$$\frac{(k_2)_0 m_0 H_0}{L_y} k_2 m \frac{\partial H}{\partial y} \Big|_{y=\frac{m_0 m + 0}{L_y}} = \frac{(k_1)_0 m_0 h_0}{L_y} k_1 m \frac{\partial h}{\partial y} \Big|_{y=\frac{m_0 m - 0}{L_y}}. \quad (22)$$

### 3. Solution Method

To solve problem (16) - (22), the finite difference method [6,7,16] is used. For this, introduce a grid for the region  $D = \{0 \leq x < L_x, 0 \leq y < L_y, 0 \leq t \leq T\}$ , where  $T$  is the maximum time during which the process is studied. To do this, replace the continuous domain of the problem solution by the grid domain:

$$\omega_{\Delta x, \Delta y, \Delta \tau} = \{(x_i, y_j, t_n), x_i = i \Delta x; i = 0, 1, 2, \dots, J; y_j = j \Delta y; j = 0, 1, 2, \dots, J; t_n = n \Delta \tau; n = 0, 1, 2, \dots, N\}$$

Next, approximate equation (24 \*) for the  $n + \frac{1}{2}$  layer and use the implicit scheme on the grid  $\omega_{\Delta x, \Delta y, \Delta \tau}$  in the form [6,7,16-19]:

$$\left. \begin{aligned}
& \frac{0.5\Delta\tau(k_1)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2} h_{i-1,j}^{n+\frac{1}{2}} - \frac{0.5\Delta\tau((k_1)_{i-0.5,j}m_{i-0.5,j} + (k_1)_{i+0.5,j}m_{i+0.5,j}) + \Delta x^2}{\Delta x^2} h_{i,j}^{n+\frac{1}{2}} + \\
& \qquad \qquad \qquad + \frac{0.5\Delta\tau(k_1)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2} h_{i+1,j}^{n+\frac{1}{2}} = \\
& = -(h_{i,j}^n + 0.5\Delta\tau\xi \frac{(k_1)_{i,j-0.5}m_{i,j-0.5}h_{i,j-1}^n - ((k_1)_{i,j-0.5}m_{i,j-0.5} + (k_1)_{i,j+0.5}m_{i,j+0.5})h_{i,j}^n}{\Delta y^2} + \\
& \qquad \qquad \qquad + 0.5\Delta\tau\xi \frac{(k_1)_{i,j+0.5}m_{i,j+0.5}h_{i,j+1}^n}{\Delta y^2} + 0.5\Delta\tau\xi_1(f - \omega)), \\
& \frac{0.5\Delta\tau(k_2)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2} H_{i-1,j}^{n+\frac{1}{2}} - \frac{0.5\Delta\tau((k_2)_{i-0.5,j}m_{i-0.5,j} + (k_2)_{i+0.5,j}m_{i+0.5,j}) + \Delta x^2}{\Delta x^2} H_{i,j}^{n+\frac{1}{2}} + \\
& \qquad \qquad \qquad + \frac{0.5\Delta\tau(k_2)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2} H_{i+1,j}^{n+\frac{1}{2}} = \\
& = -(H_{i,j}^n + 0.5\Delta\tau\xi \frac{(k_2)_{i,j-0.5}m_{i,j-0.5}H_{i,j-1}^n - ((k_2)_{i,j-0.5}m_{i,j-0.5} + (k_2)_{i,j+0.5}m_{i,j+0.5})H_{i,j}^n}{\Delta y^2} + \\
& \qquad \qquad \qquad + 0.5\Delta\tau\xi \frac{(k_2)_{i,j+0.5}m_{i,j+0.5}H_{i,j+1}^n}{\Delta y^2} - 0.5\Delta\tau\xi_2\eta Q).
\end{aligned} \right\} \quad (23)$$

After some transforms and grouping similar terms, the finite-difference system (23) is rewritten in the form:

$$a_{i,j}h_{i-1,j}^{n+\frac{1}{2}} - b_{i,j}h_{i,j}^{n+\frac{1}{2}} + c_{i,j}h_{i+1,j}^{n+\frac{1}{2}} = -d_{i,j}^n, \quad (24)$$

$$\bar{a}_{i,j}H_{i-1,j}^{n+\frac{1}{2}} - \bar{b}_{i,j}H_{i,j}^{n+\frac{1}{2}} + \bar{c}_{i,j}H_{i+1,j}^{n+\frac{1}{2}} = -\bar{d}_{i,j}^n. \quad (25)$$

here

$$\begin{aligned}
a_{i,j} &= \frac{0.5\Delta\tau(k_1)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2}, & b_{i,j} &= \frac{0.5\Delta\tau((k_1)_{i-0.5,j}m_{i-0.5,j} + (k_1)_{i+0.5,j}m_{i+0.5,j}) + \Delta x^2}{\Delta x^2}, \\
c_{i,j} &= \frac{0.5\Delta\tau(k_1)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2}, \\
d_{i,j}^n &= h_{i,j}^n + 0.5\Delta\tau\xi \frac{(k_1)_{i,j-0.5}m_{i,j-0.5}h_{i,j-1}^n - ((k_1)_{i,j-0.5}m_{i,j-0.5} + (k_1)_{i,j+0.5}m_{i,j+0.5})h_{i,j}^n}{\Delta y^2} + \\
& \qquad \qquad \qquad + 0.5\Delta\tau\xi \frac{(k_1)_{i,j+0.5}m_{i,j+0.5}h_{i,j+1}^n}{\Delta y^2} + 0.5\Delta\tau\xi_1(f - \omega), \\
\bar{a}_{i,j} &= \frac{0.5\Delta\tau(k_2)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2}, & \bar{b}_{i,j} &= \frac{0.5\Delta\tau((k_2)_{i-0.5,j}m_{i-0.5,j} + (k_2)_{i+0.5,j}m_{i+0.5,j}) + \Delta x^2}{\Delta x^2}, \\
\bar{c}_{i,j} &= \frac{0.5\Delta\tau(k_2)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2}, \\
\bar{d}_{i,j}^n &= H_{i,j}^n + 0.5\Delta\tau\xi \frac{(k_2)_{i,j-0.5}m_{i,j-0.5}H_{i,j-1}^n - ((k_2)_{i,j-0.5}m_{i,j-0.5} + (k_2)_{i,j+0.5}m_{i,j+0.5})H_{i,j}^n}{\Delta y^2} + \\
& \qquad \qquad \qquad + 0.5\Delta\tau\xi \frac{(k_2)_{i,j+0.5}m_{i,j+0.5}H_{i,j+1}^n}{\Delta y^2} - 0.5\Delta\tau\xi_2\eta Q.
\end{aligned}$$



The resulting systems of equations (24) and (25) with respect to the sought for variables are solved by the sweep method, where sweep coefficients are calculated as:

$$h_{i,j}^{n+\frac{1}{2}} = \alpha_{i+1,j} h_{i+1,j}^{n+\frac{1}{2}} + \beta_{i+1,j}^n, \quad (24^*)$$

$$H_{i,j}^{n+\frac{1}{2}} = \bar{\alpha}_{i+1,j} H_{i+1,j}^{n+\frac{1}{2}} + \bar{\beta}_{i+1,j}^n. \quad (25^*)$$

$\alpha_{i,j}$ ,  $\beta_{i,j}^n$  and  $\bar{\alpha}_{i,j}$ ,  $\bar{\beta}_{i,j}^n$  - are the sweep coefficients:

$$\alpha_{i+1,j} = \frac{c_{i,j}}{b_{i,j} - a_{i,j}\alpha_{i,j}}, \quad \beta_{i+1,j}^n = \frac{d_{i,j}^n + a_{i,j}\beta_{i,j}^n}{b_{i,j} - a_{i,j}\alpha_{i,j}}, \quad \bar{\alpha}_{i+1,j} = \frac{\bar{c}_{i,j}}{b_{i,j} - \bar{a}_{i,j}\bar{\alpha}_{i,j}}, \quad \bar{\beta}_{i+1,j}^n = \frac{\bar{d}_{i,j}^n + \bar{a}_{i,j}\bar{\beta}_{i,j}^n}{b_{i,j} - \bar{a}_{i,j}\bar{\alpha}_{i,j}}.$$

Next the boundary conditions (17) - (22) are approximated:

$$\frac{\mu_1 m_0 h_0}{L_x} m_{1,j} \frac{h_{0,j}^{n+\frac{1}{2}} - 4h_{1,j}^{n+\frac{1}{2}} + 3h_{2,j}^{n+\frac{1}{2}}}{2\Delta x} = -(h_0 h_{1,j}^{n+\frac{1}{2}} - h_0), \quad (26)$$

$$\frac{\mu_1 m_0 h_0}{L_x} m_{l,j} \frac{-3h_{l-1,j}^{n+\frac{1}{2}} + 4h_{l,j}^{n+\frac{1}{2}} - h_{l+1,j}^{n+\frac{1}{2}}}{2\Delta x} = (h_0 h_{l,j}^{n+\frac{1}{2}} - h_0), \quad (27)$$

$$\frac{\mu_1 m_0 h_0}{L_y} m_{i,1} \frac{h_{i,0}^{n+1} - 4h_{i,1}^{n+1} + 3h_{i,2}^{n+1}}{2\Delta y} = -(h_0 h_{i,1}^{n+1} - h_0), \quad (28)$$

$$\frac{\mu_1 m_0 h_0}{L_y} m_{i,J} \frac{-3h_{i,I-1}^{n+1} + 4h_{i,J}^{n+1} - h_{i,J+1}^{n+1}}{2\Delta y} = (h_0 h_{i,J}^{n+1} - h_0), \quad (29)$$

$$\frac{\mu_2 m_0 H_0}{L_x} m_{1,j} \frac{H_{0,j}^{n+\frac{1}{2}} - 4H_{1,j}^{n+\frac{1}{2}} + 3H_{2,j}^{n+\frac{1}{2}}}{2\Delta x} = -(H_0 H_{1,j}^{n+\frac{1}{2}} - H_0), \quad (30)$$

$$\frac{\mu_2 m_0 H_0}{L_x} m_{l,j} \frac{-3H_{l-1,j}^{n+\frac{1}{2}} + 4H_{l,j}^{n+\frac{1}{2}} - H_{l+1,j}^{n+\frac{1}{2}}}{2\Delta x} = (H_0 H_{l,j}^{n+\frac{1}{2}} - H_0), \quad (31)$$

$$\frac{\mu_2 m_0 H_0}{L_y} m_{i,1} \frac{H_{i,0}^{n+1} - 4H_{i,1}^{n+1} + 3H_{i,2}^{n+1}}{2\Delta y} = -(H_0 H_{i,1}^{n+1} - H_0), \quad (32)$$

$$\frac{\mu_2 m_0 H_0}{L_y} m_{i,J} \frac{-3H_{i,I-1}^{n+1} + 4H_{i,J}^{n+1} - H_{i,J+1}^{n+1}}{2\Delta y} = (H_0 H_{i,J}^{n+1} - H_0), \quad (33)$$

$$H_0 H_{i,J}^n = h_0 h_{i,J}^n, \quad (34)$$

$$\frac{(k_2)_0 m_0 H_0}{L_y} (k_2)_{i,J} m_{i,J} \frac{-3H_{i,I-1}^n + 4H_{i,J}^n - H_{i,J+1}^n}{2\Delta y} = \frac{(k_1)_0 m_0 h_0}{L_y} (k_1)_{i,J} m_{i,J} \frac{-3h_{i,I-1}^n + 4h_{i,J}^n - h_{i,J+1}^n}{2\Delta y}. \quad (35)$$

If  $i=1$ , then equation (24) is transformed to equation (36), and as a result of simplification of equation (26), we get (37).

$$h_{2,j}^{n+\frac{1}{2}} = -\frac{a_{1,j}}{c_{1,j}} h_{0,j}^{n+\frac{1}{2}} + \frac{b_{1,j}}{c_{1,j}} h_{1,j}^{n+\frac{1}{2}} - \frac{d_{1,j}^n}{c_{1,j}}, \quad (36)$$

$$h_{2,j}^{n+\frac{1}{2}} = -\frac{1}{3} h_{0,j}^{n+\frac{1}{2}} + \left(\frac{4}{3} - \frac{2\Delta x L_x h_0}{3\mu_1 m_0 h_0 m_{1,j}}\right) h_{1,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x h_0}{3\mu_1 m_0 h_0 m_{1,j}}. \quad (37)$$

Comparing (36) with (37), we get  $h_{0,j}^{n+\frac{1}{2}}$ :

$$h_{0,j}^{n+\frac{1}{2}} = \frac{3\mu_1 m_0 h_0 m_{1,j} b_{1,j} - 4\mu_1 m_0 h_0 m_{1,j} c_{1,j} + 2\Delta x L_x h_0 \mu_1 m_0 h_0 m_{1,j} c_{1,j}}{\mu_1 m_0 h_0 m_{1,j} (3a_{1,j} - c_{1,j})} h_{1,j}^{n+\frac{1}{2}} - \frac{2\Delta x L_x h_0 c_{1,j}}{\mu_1 m_0 h_0 m_{1,j} (3a_{1,j} - c_{1,j})}. \quad (38)$$

If  $i = 0$ , equation (32\*) is transformed to equation (39):

$$h_{0,j}^{n+\frac{1}{2}} = \alpha_{1,j} h_{1,j}^{n+\frac{1}{2}} + \beta_{1,j}^n. \quad (39)$$

Comparing (38) with (39), we get  $\alpha_{1,j}$  and  $\beta_{1,j}^n$ :

$$\alpha_{1,j} = \frac{3\mu_1 m_0 h_0 m_{1,j} b_{1,j} - 4\mu_1 m_0 h_0 m_{1,j} c_{1,j} + 2\Delta x L_x h_0 \mu_1 m_0 h_0 m_{1,j} c_{1,j}}{\mu_1 m_0 h_0 m_{1,j} (3a_{1,j} - c_{1,j})}, \quad \beta_{1,j}^n = -\frac{2\Delta x L_x h_0 c_{1,j}}{\mu_1 m_0 h_0 m_{1,j} (3a_{1,j} - c_{1,j})}.$$

At  $i = I$  equation (24) takes the form (40), as a result of simplification of equation (27) we obtain (41):

$$h_{I+1,j}^{n+\frac{1}{2}} = -\frac{a_{I,j}}{c_{I,j}} h_{I-1,j}^{n+\frac{1}{2}} + \frac{b_{I,j}}{c_{I,j}} h_{I,j}^{n+\frac{1}{2}} - \frac{d_{I,j}^n}{c_{I,j}}, \quad (40)$$

$$h_{I+1,j}^{n+\frac{1}{2}} = -3h_{I-1,j}^{n+\frac{1}{2}} + \frac{4\mu_1 m_0 h_0 m_{I,j} - 2\Delta x L_x h_0}{\mu_1 m_0 h_0 m_{I,j}} h_{I,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x h_0}{\mu_1 m_0 h_0 m_{I,j}}. \quad (41)$$

Comparing (40) with (41), we get  $h_{I-1,j}^{n+\frac{1}{2}}$ :

$$h_{I-1,j}^{n+\frac{1}{2}} = \frac{b_{I,j} \mu_1 m_0 h_0 m_{I,j} - 4\mu_1 m_0 h_0 m_{I,j} c_{I,j} + 2\Delta x L_x h_0 c_{I,j}}{\mu_1 m_0 h_0 m_{I,j} (a_{I,j} - 3c_{I,j})} h_{I,j}^{n+\frac{1}{2}} - \frac{2\Delta x L_x h_0 c_{I,j} + d_{I,j}^n \mu_1 m_0 h_0 m_{I,j}}{\mu_1 m_0 h_0 m_{I,j} (a_{I,j} - 3c_{I,j})}. \quad (42)$$

If  $i = I - 1$ , then equation (24\*) is transformed to equation (42\*):

$$h_{I-1,j}^{n+\frac{1}{2}} = \alpha_{I,j} h_{I,j}^{n+\frac{1}{2}} + \beta_{I,j}^n. \quad (42^*)$$

Comparing (42) with (42\*), we get  $h_{I,j}^{n+\frac{1}{2}}$ :

$$h_{I,j}^{n+\frac{1}{2}} = \frac{\beta_{I,j}^n \mu_1 m_0 h_0 m_{I,j} (a_{I,j} - 3c_{I,j}) + 2\Delta x L_x h_0 c_{I,j} + d_{I,j}^n \mu_1 m_0 h_0 m_{I,j}}{b_{I,j} \mu_1 m_0 h_0 m_{I,j} - 4\mu_1 m_0 h_0 m_{I,j} c_{I,j} + 2\Delta x L_x h_0 c_{I,j} - \alpha_{I,j} \mu_1 m_0 h_0 m_{I,j} (a_{I,j} - 3c_{I,j})}.$$

If  $i = 1$ , then equation (25) is transformed to equation (43), and as a result of simplification of equation (30), we obtain (43\*).

$$H_{2,j}^{n+\frac{1}{2}} = -\frac{\bar{a}_{1,j}}{\bar{c}_{1,j}} H_{0,j}^{n+\frac{1}{2}} + \frac{\bar{b}_{1,j}}{\bar{c}_{1,j}} H_{1,j}^{n+\frac{1}{2}} - \frac{\bar{d}_{1,j}^n}{\bar{c}_{1,j}}, \quad (43)$$

$$H_{2,j}^{n+\frac{1}{2}} = -\frac{1}{3} H_{0,j}^{n+\frac{1}{2}} + \frac{4\mu_2 m_0 m_{1,j} - 2\Delta x L_x}{3\mu_2 m_0 m_{1,j}} H_{1,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x}{3\mu_2 m_0 m_{1,j}}. \quad (43^*)$$

Comparing (43) with (43\*), we get  $H_{0,j}^{n+\frac{1}{2}}$ :

$$H_{0,j}^{n+\frac{1}{2}} = \frac{3(3\mu_2 m_0 m_{1,j} \bar{b}_{1,j} - 4\mu_2 m_0 m_{1,j} \bar{c}_{1,j} + 2\Delta x L_x \bar{c}_{1,j})}{3\bar{a}_{1,j} - \bar{c}_{1,j}} H_{1,j}^{n+\frac{1}{2}} - \frac{3(3\mu_2 m_0 m_{1,j} \bar{d}_{1,j}^n + 2\Delta x L_x \bar{c}_{1,j})}{3\bar{a}_{1,j} - \bar{c}_{1,j}}. \quad (44)$$

If  $i = 0$ , then equation (25\*) is transformed to equation (44\*):

$$H_{0,j}^{n+\frac{1}{2}} = \bar{\alpha}_{1,j} H_{1,j}^{n+\frac{1}{2}} + \bar{\beta}_{1,j}^n. \quad (44^*)$$

Comparing (44) with (44\*), we get  $\bar{\alpha}_{1,j}$  and  $\bar{\beta}_{1,j}^n$ :

$$\bar{\alpha}_{1,j} = \frac{3(3\mu_2 m_0 m_{1,j} \bar{b}_{1,j} - 4\mu_2 m_0 m_{1,j} \bar{c}_{1,j} + 2\Delta x L_x \bar{c}_{1,j})}{3\bar{a}_{1,j} - \bar{c}_{1,j}},$$

$$\bar{\beta}_{1,j}^n = -\frac{3(3\mu_2 m_0 m_{1,j} \bar{d}_{1,j}^n + 2\Delta x L_x \bar{c}_{1,j})}{3\bar{a}_{1,j} - \bar{c}_{1,j}}.$$

At  $i = I$  equation (25) takes the form (45), as a result of simplification of equation (31) we obtain (45):

$$H_{I+1,j}^{n+\frac{1}{2}} = -\frac{\bar{a}_{I,j}}{\bar{c}_{I,j}} H_{I-1,j}^{n+\frac{1}{2}} + \frac{\bar{b}_{I,j}}{\bar{c}_{I,j}} H_{I,j}^{n+\frac{1}{2}} - \frac{\bar{d}_{I,j}^n}{\bar{c}_{I,j}}, \quad (45)$$

$$H_{I+1,j}^{n+\frac{1}{2}} = -3H_{I-1,j}^{n+\frac{1}{2}} + \frac{4\mu_2 m_0 m_{I,j} - 2\Delta x L_x}{\mu_2 m_0 m_{I,j}} H_{I,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x}{\mu_2 m_0 m_{I,j}}. \quad (45^*)$$

Comparing (45) with (45\*), we get  $H_{I-1,j}^{n+\frac{1}{2}}$ :

$$H_{I-1,j}^{n+\frac{1}{2}} = \frac{\mu_2 m_0 m_{I,j} \bar{b}_{I,j} - 4\mu_2 m_0 m_{I,j} \bar{c}_{I,j} + 2\Delta x L_x \bar{c}_{I,j}}{\mu_2 m_0 m_{I,j} (\bar{a}_{I,j} - 3\bar{c}_{I,j})} H_{I,j}^{n+\frac{1}{2}} - \frac{2\Delta x L_x \bar{c}_{I,j} + \mu_2 m_0 m_{I,j} \bar{d}_{I,j}^n}{\mu_2 m_0 m_{I,j} (\bar{a}_{I,j} - 3\bar{c}_{I,j})}. \quad (46)$$

If  $i = I - 1$ , then equation (25\*) is transformed to equation (46\*):

$$H_{I-1,j}^{n+\frac{1}{2}} = \bar{\alpha}_{I,j} H_{I,j}^{n+\frac{1}{2}} + \bar{\beta}_{I,j}^n. \quad (46^*)$$

Comparing (46) with (46\*), we get  $H_{I,j}^{n+\frac{1}{2}}$ :

$$H_{I,j}^{n+\frac{1}{2}} = \frac{2\Delta x L_x \bar{c}_{I,j} + \mu_2 m_0 m_{I,j} \bar{d}_{I,j}^n + \mu_2 m_0 m_{I,j} (\bar{a}_{I,j} - 3\bar{c}_{I,j}) \bar{\beta}_{I,j}^n}{\mu_2 m_0 m_{I,j} \bar{b}_{I,j} - 4\mu_2 m_0 m_{I,j} \bar{c}_{I,j} + 2\Delta x L_x \bar{c}_{I,j} - \mu_2 m_0 m_{I,j} (\bar{a}_{I,j} - 3\bar{c}_{I,j}) \bar{\alpha}_{I,j}}.$$

Next, we approximate equation (16\*) for the  $n+1$  layer and use the implicit scheme on the grid  $\omega_{\Delta x, \Delta y, \Delta \tau}$  in the form [6,7,16]:

$$\left. \begin{aligned}
 & \frac{0.5\Delta\tau\xi(k_1)_{i,j-0.5}m_{i,j-0.5}}{\Delta y^2}h_{i,j-1}^{n+1} - \frac{0.5\Delta\tau\xi((k_1)_{i,j-0.5}m_{i,j-0.5} + (k_1)_{i,j+0.5}m_{i,j+0.5}) + \Delta y^2}{\Delta y^2}h_{i,j}^{n+1} + \\
 & \qquad \qquad \qquad + \frac{0.5\Delta\tau\xi(k_1)_{i,j+0.5}m_{i,j+0.5}}{\Delta y^2}h_{i,j+1}^{n+1} = \\
 & = -\left(\frac{0.5\Delta\tau(k_1)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2}h_{i-1,j}^{n+\frac{1}{2}} - \frac{0.5\Delta\tau((k_1)_{i-0.5,j}m_{i-0.5,j} + (k_1)_{i+0.5,j}m_{i+0.5,j}) - \Delta x^2}{\Delta x^2}h_{i,j}^{n+\frac{1}{2}} + \right. \\
 & \qquad \qquad \qquad \left. + \frac{0.5\Delta\tau(k_1)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2}h_{i+1,j}^{n+\frac{1}{2}} + 0.5\Delta\tau\xi_1(f - \omega)\right), \\
 & \frac{0.5\Delta\tau\xi(k_2)_{i,j-0.5}m_{i,j-0.5}}{\Delta y^2}H_{i,j-1}^{n+1} - \frac{0.5\Delta\tau\xi((k_2)_{i,j-0.5}m_{i,j-0.5} + (k_2)_{i,j+0.5}m_{i,j+0.5}) + \Delta y^2}{\Delta y^2}H_{i,j}^{n+1} + \\
 & \qquad \qquad \qquad + \frac{0.5\Delta\tau\xi(k_2)_{i,j+0.5}m_{i,j+0.5}}{\Delta y^2}H_{i,j+1}^{n+1} = \\
 & = -\left(\frac{0.5\Delta\tau(k_2)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2}H_{i-1,j}^{n+\frac{1}{2}} - \frac{0.5\Delta\tau((k_2)_{i-0.5,j}m_{i-0.5,j} + (k_2)_{i+0.5,j}m_{i+0.5,j}) - \Delta x^2}{\Delta x^2}H_{i,j}^{n+\frac{1}{2}} + \right. \\
 & \qquad \qquad \qquad \left. + \frac{0.5\Delta\tau(k_2)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2}H_{i+1,j}^{n+\frac{1}{2}} - 0.5\Delta\tau\xi_2\eta Q\right).
 \end{aligned} \right\} \quad (47)$$

After some transforms and grouping similar terms, the finite-difference system (47) is rewritten in the form:

$$\bar{a}_{i,j}h_{i,j-1}^{n+1} - \bar{b}_{i,j}h_{i,j}^{n+1} + \bar{c}_{i,j}h_{i,j+1}^{n+1} = -\bar{d}_{i,j}^n, \quad (48)$$

$$\tilde{a}_{i,j}H_{i,j-1}^{n+1} - \tilde{b}_{i,j}H_{i,j}^{n+1} + \tilde{c}_{i,j}H_{i,j+1}^{n+1} = -\tilde{d}_{i,j}^n. \quad (48^*)$$

here

$$\begin{aligned}
 \bar{a}_{i,j} &= \frac{0.5\Delta\tau\xi(k_1)_{i,j-0.5}m_{i,j-0.5}}{\Delta y^2}, \quad \bar{b}_{i,j} = \frac{0.5\Delta\tau\xi((k_1)_{i,j-0.5}m_{i,j-0.5} + (k_1)_{i,j+0.5}m_{i,j+0.5}) + \Delta y^2}{\Delta y^2}, \\
 \bar{c}_{i,j} &= \frac{0.5\Delta\tau\xi(k_1)_{i,j+0.5}m_{i,j+0.5}}{\Delta y^2}, \\
 \bar{d}_{i,j}^n &= \frac{0.5\Delta\tau(k_1)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2}h_{i-1,j}^{n+\frac{1}{2}} - \frac{0.5\Delta\tau((k_1)_{i-0.5,j}m_{i-0.5,j} + (k_1)_{i+0.5,j}m_{i+0.5,j}) - \Delta x^2}{\Delta x^2}h_{i,j}^{n+\frac{1}{2}} + \\
 & \qquad \qquad \qquad + \frac{0.5\Delta\tau(k_1)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2}h_{i+1,j}^{n+\frac{1}{2}} + 0.5\Delta\tau\xi_1(f - \omega), \\
 \tilde{a}_{i,j} &= \frac{0.5\Delta\tau\xi(k_2)_{i,j-0.5}m_{i,j-0.5}}{\Delta y^2}, \quad \tilde{b}_{i,j} = \frac{0.5\Delta\tau\xi((k_2)_{i,j-0.5}m_{i,j-0.5} + (k_2)_{i,j+0.5}m_{i,j+0.5}) + \Delta y^2}{\Delta y^2},
 \end{aligned}$$

$$\begin{aligned}\tilde{c}_{i,j} &= \frac{0.5\Delta\tau\xi(k_2)_{i,j+0.5}m_{i,j+0.5}}{\Delta y^2}, \\ \tilde{d}_{i,j}^n &= \frac{0.5\Delta\tau(k_2)_{i-0.5,j}m_{i-0.5,j}}{\Delta x^2}H_{i-1,j}^{n+\frac{1}{2}} - \frac{0.5\Delta\tau((k_2)_{i-0.5,j}m_{i-0.5,j} + (k_2)_{i+0.5,j}m_{i+0.5,j}) - \Delta x^2}{\Delta x^2}H_{i,j}^{n+\frac{1}{2}} + \\ &\quad + \frac{0.5\Delta\tau(k_2)_{i+0.5,j}m_{i+0.5,j}}{\Delta x^2}H_{i+1,j}^{n+\frac{1}{2}} - 0.5\Delta\tau\xi_2\eta Q.\end{aligned}$$

The resulting systems of equations (48) and (48\*) with respect to the sought for variables are solved by the sweep method, where sweep coefficients are calculated as:

$$h_{i,j}^{n+1} = \bar{\alpha}_{i,j+1}h_{i,j+1}^n + \bar{\beta}_{i,j+1}^n, \quad (49)$$

$$H_{i,j}^{n+1} = \tilde{\alpha}_{i,j+1}H_{i,j+1}^{n+1} + \tilde{\beta}_{i,j+1}^n. \quad (49^*)$$

$\bar{\alpha}_{i,j}$ ,  $\bar{\beta}_{i,j}^n$  and  $\tilde{\alpha}_{i,j}$ ,  $\tilde{\beta}_{i,j}^n$  - are the sweep coefficients:

$$\bar{\alpha}_{i,j+1} = \frac{\bar{c}_{i,j}}{\bar{b}_{i,j} - \bar{a}_{i,j}\bar{\alpha}_{i,j}}, \quad \bar{\beta}_{i,j+1}^n = \frac{\bar{d}_{i,j}^n + \bar{a}_{i,j}\bar{\beta}_{i,j}^n}{\bar{b}_{i,j} - \bar{a}_{i,j}\bar{\alpha}_{i,j}}, \quad \tilde{\alpha}_{i,j+1} = \frac{\tilde{c}_{i,j}}{\tilde{b}_{i,j} - \tilde{a}_{i,j}\tilde{\alpha}_{i,j}}, \quad \tilde{\beta}_{i,j+1}^n = \frac{\tilde{d}_{i,j}^n + \tilde{a}_{i,j}\tilde{\beta}_{i,j}^n}{\tilde{b}_{i,j} - \tilde{a}_{i,j}\tilde{\alpha}_{i,j}}.$$

If  $j = 1$ , then equation (48) is transformed to equation (50) and as a result of simplification of equation (28), we get (50\*).

$$h_{i,2}^{n+1} = -\frac{\bar{a}_{i,1}}{\bar{c}_{i,1}}h_{i,0}^{n+1} + \frac{\bar{b}_{i,1}}{\bar{c}_{i,1}}h_{i,1}^{n+1} - \frac{\bar{d}_{i,1}^n}{\bar{c}_{i,1}}, \quad (50)$$

$$h_{i,2}^{n+1} = -\frac{1}{3}h_{i,0}^{n+1} + \frac{4\mu_1m_0h_0m_{i,1} - 2\Delta yL_yh_0}{3\mu_1m_0h_0m_{i,1}}h_{i,1}^{n+1} + \frac{2\Delta yL_yh_0}{3\mu_1m_0h_0m_{i,1}}, \quad (50^*)$$

Comparing (50) with (50\*), we get  $h_{i,0}^{n+1}$ :

$$h_{i,0}^{n+1} = \frac{3\mu_1m_0h_0m_{i,1}\bar{b}_{i,1} - 4\mu_1m_0h_0m_{i,1}\bar{c}_{i,1} + 2\Delta yL_yh_0\bar{c}_{i,1}}{\mu_1m_0h_0m_{i,1}(3\bar{a}_{i,1} - \bar{c}_{i,1})}h_{i,1}^{n+1} - \frac{3\mu_1m_0h_0m_{i,1}\bar{d}_{i,1}^n + 2\Delta yL_yh_0\bar{c}_{i,1}}{\mu_1m_0h_0m_{i,1}(3\bar{a}_{i,1} - \bar{c}_{i,1})}. \quad (51)$$

If  $j = 0$ , then equation (49) is transformed to equation (51\*):

$$h_{i,0}^{n+1} = \bar{\alpha}_{i,1}h_{i,1}^{n+1} + \bar{\beta}_{i,1}^n. \quad (51^*)$$

Comparing (51) with (51\*), we get  $\bar{\alpha}_{i,1}$  and  $\bar{\beta}_{i,1}^n$ :

$$\begin{aligned}\bar{\alpha}_{i,1} &= \frac{3\mu_1m_0h_0m_{i,1}\bar{b}_{i,1} - 4\mu_1m_0h_0m_{i,1}\bar{c}_{i,1} + 2\Delta yL_yh_0\bar{c}_{i,1}}{\mu_1m_0h_0m_{i,1}(3\bar{a}_{i,1} - \bar{c}_{i,1})}, \\ \bar{\beta}_{i,1}^n &= -\frac{3\mu_1m_0h_0m_{i,1}\bar{d}_{i,1}^n + 2\Delta yL_yh_0\bar{c}_{i,1}}{\mu_1m_0h_0m_{i,1}(3\bar{a}_{i,1} - \bar{c}_{i,1})}.\end{aligned}$$

At  $j = J$  equation (48) takes the form (52), as a result of simplification of equation (29) we obtain (52\*):

$$h_{i,J+1}^{n+1} = -\frac{\bar{\bar{a}}_{i,J}}{\bar{\bar{c}}_{i,J}} h_{i,J-1}^{n+1} + \frac{\bar{\bar{b}}_{i,J}}{\bar{\bar{c}}_{i,J}} h_{i,J}^{n+1} - \frac{\bar{\bar{d}}_{i,J}^n}{\bar{\bar{c}}_{i,J}}, \quad (52)$$

$$h_{i,J+1}^{n+1} = -3h_{i,J-1}^{n+1} + \frac{4\mu_1 m_0 h_0 m_{i,J} - 2\Delta y L_y h_0}{\mu_1 m_0 h_0 m_{i,J}} h_{i,J}^{n+1} + \frac{2\Delta y L_y h_0}{\mu_1 m_0 h_0 m_{i,J}}. \quad (52^*)$$

Comparing (52) with (52\*), we get  $h_{i,J-1}^{n+1}$ :

$$h_{i,J-1}^{n+1} = \frac{\mu_1 m_0 h_0 m_{i,J} \bar{\bar{b}}_{i,J} - 4\mu_1 m_0 h_0 m_{i,J} \bar{\bar{c}}_{i,J} + 2\Delta y L_y h_0 \bar{\bar{c}}_{i,J}}{\mu_1 m_0 h_0 m_{i,J} (\bar{\bar{a}}_{i,J} - 3\bar{\bar{c}}_{i,J})} h_{i,J}^{n+1} - \frac{2\Delta y L_y h_0 \bar{\bar{c}}_{i,J} + \mu_1 m_0 h_0 m_{i,J} \bar{\bar{d}}_{i,J}^n}{\mu_1 m_0 h_0 m_{i,J} (\bar{\bar{a}}_{i,J} - 3\bar{\bar{c}}_{i,J})}. \quad (53)$$

If  $j = J - 1$ , then equation (49) is transformed to equation (53\*):

$$h_{i,J-1}^{n+1} = \bar{\bar{\alpha}}_{i,J} h_{i,J}^{n+1} + \bar{\bar{\beta}}_{i,J}. \quad (53^*)$$

Comparing (53) with (53\*), we get  $h_{i,J}^{n+1}$ :

$$h_{i,J}^{n+1} = \frac{\bar{\bar{\beta}}_{i,J} \mu_1 m_0 h_0 m_{i,J} (\bar{\bar{a}}_{i,J} - 3\bar{\bar{c}}_{i,J}) + 2\Delta y L_y h_0 \bar{\bar{c}}_{i,J} + \bar{\bar{d}}_{i,J}^n \mu_1 m_0 h_0 m_{i,J}}{\bar{\bar{b}}_{i,J} \mu_1 m_0 h_0 m_{i,J} - 4\mu_1 m_0 h_0 m_{i,J} \bar{\bar{c}}_{i,J} + 2\Delta y L_y h_0 \bar{\bar{c}}_{i,J} - \bar{\bar{\alpha}}_{i,J} \mu_1 m_0 h_0 m_{i,J} (\bar{\bar{a}}_{i,J} - 3\bar{\bar{c}}_{i,J})}.$$

If  $j = 1$ , then equation (48\*) is transformed to equation (54), and as a result of simplification of equation (32), we get (54\*):

$$H_{i,2}^{n+1} = -\frac{\tilde{\tilde{a}}_{i,1}}{\tilde{\tilde{c}}_{i,1}} H_{i,0}^{n+1} + \frac{\tilde{\tilde{b}}_{i,1}}{\tilde{\tilde{c}}_{i,1}} H_{i,1}^{n+1} - \frac{\tilde{\tilde{d}}_{i,1}^n}{\tilde{\tilde{c}}_{i,1}}, \quad (54)$$

$$H_{i,2}^{n+1} = -\frac{1}{3} H_{i,0}^{n+1} + \frac{4\mu_2 m_0 m_{i,1} - 2\Delta y L_y}{3\mu_2 m_0 m_{i,1}} H_{i,1}^{n+1} + \frac{2\Delta y L_y}{3\mu_2 m_0 m_{i,1}}. \quad (54^*)$$

Comparing (54) with (54\*), we get  $H_{i,0}^{n+1}$ :

$$H_{i,0}^{n+1} = \frac{3\mu_2 m_0 m_{i,1} \tilde{\tilde{b}}_{i,1} - 4\mu_2 m_0 m_{i,1} \tilde{\tilde{c}}_{i,1} - 2\Delta y L_y \tilde{\tilde{c}}_{i,1}}{\mu_2 m_0 m_{i,1} (3\tilde{\tilde{a}}_{i,1} - \tilde{\tilde{c}}_{i,1})} H_{i,1}^{n+1} - \frac{2\Delta y L_y \tilde{\tilde{c}}_{i,1} + 3\mu_2 m_0 m_{i,1} \tilde{\tilde{d}}_{i,1}^n}{\mu_2 m_0 m_{i,1} (3\tilde{\tilde{a}}_{i,1} - \tilde{\tilde{c}}_{i,1})}. \quad (55)$$

If  $j = 0$ , then equation (49\*) is transformed to equation (55\*):

$$H_{i,0}^{n+1} = \tilde{\tilde{\alpha}}_{i,1} H_{i,1}^{n+1} + \tilde{\tilde{\beta}}_{i,1}^n. \quad (55^*)$$

Comparing (55) with (55\*), we get  $\tilde{\tilde{\alpha}}_{i,1}$  and  $\tilde{\tilde{\beta}}_{i,1}^n$ :

$$\tilde{\tilde{\alpha}}_{i,1} = \frac{3\mu_2 m_0 m_{i,1} \tilde{\tilde{b}}_{i,1} - 4\mu_2 m_0 m_{i,1} \tilde{\tilde{c}}_{i,1} - 2\Delta y L_y \tilde{\tilde{c}}_{i,1}}{\mu_2 m_0 m_{i,1} (3\tilde{\tilde{a}}_{i,1} - \tilde{\tilde{c}}_{i,1})}, \quad \tilde{\tilde{\beta}}_{i,1}^n = -\frac{2\Delta y L_y \tilde{\tilde{c}}_{i,1} + 3\mu_2 m_0 m_{i,1} \tilde{\tilde{d}}_{i,1}^n}{\mu_2 m_0 m_{i,1} (3\tilde{\tilde{a}}_{i,1} - \tilde{\tilde{c}}_{i,1})}.$$

At  $j = J$  equation (48) takes the form (56), as a result of simplification of equation (33) we obtain (56\*):

$$H_{i,J+1}^{n+1} = -\frac{\tilde{\tilde{a}}_{i,J}}{\tilde{\tilde{c}}_{i,J}} H_{i,J-1}^{n+1} + \frac{\tilde{\tilde{b}}_{i,J}}{\tilde{\tilde{c}}_{i,J}} H_{i,J}^{n+1} - \frac{\tilde{\tilde{d}}_{i,J}^n}{\tilde{\tilde{c}}_{i,J}}, \quad (56)$$

$$H_{i,J+1}^{n+1} = -3H_{i,J-1}^{n+1} + \frac{4\mu_1 m_0 m_{i,J} - 2\Delta y L_y}{\mu_1 m_0 m_{i,J}} H_{i,J}^{n+1} + \frac{2\Delta y L_y}{\mu_1 m_0 m_{i,J}}. \quad (56^*)$$

Comparing (56) with (56\*), we get  $H_{i,J-1}^{n+1}$ :

$$H_{i,J-1}^{n+1} = \frac{\mu_1 m_0 m_{i,J} \tilde{b}_{i,J} - 4\mu_1 m_0 m_{i,J} \tilde{c}_{i,J} + 2\Delta y L_y \tilde{c}_{i,J}}{\mu_1 m_0 m_{i,J} (\tilde{a}_{i,J} - 3\tilde{c}_{i,J})} H_{i,J}^{n+1} - \frac{2\Delta y L_y \tilde{c}_{i,J} + \mu_1 m_0 m_{i,J} \tilde{d}_{i,J}^n}{\mu_1 m_0 m_{i,J} (\tilde{a}_{i,J} - 3\tilde{c}_{i,J})}. \quad (57)$$

If  $j = J - 1$ , then equation (49\*) is transformed to equation (57\*):

$$H_{i,J-1}^{n+1} = \tilde{\alpha}_{i,J} H_{i,J}^{n+1} + \tilde{\beta}_{i,J}^n. \quad (57^*)$$

Comparing (57) with (57\*), we get  $H_{i,J}^{n+1}$ :

$$H_{i,J}^{n+1} = \frac{2\Delta y L_y \tilde{c}_{i,J} + \mu_1 m_0 m_{i,J} \tilde{d}_{i,J}^n + \mu_1 m_0 m_{i,J} (\tilde{a}_{i,J} - 3\tilde{c}_{i,J}) \tilde{\beta}_{i,J}^n}{\mu_1 m_0 m_{i,J} \tilde{b}_{i,J} - 4\mu_1 m_0 m_{i,J} \tilde{c}_{i,J} + 2\Delta y L_y \tilde{c}_{i,J} - \mu_1 m_0 m_{i,J} (\tilde{a}_{i,J} - 3\tilde{c}_{i,J}) \tilde{\alpha}_{i,J}}.$$

The convergence of the iterative process is checked using the conditions:

$$\left| (h_{i,j}^n)^{(s+1)} - (h_{i,j}^n)^s \right| < \varepsilon, \quad \left| (H_{i,j}^n)^{(s+1)} - (H_{i,j}^n)^s \right| < \varepsilon, \quad \varepsilon > 0.$$

Here  $\varepsilon$  is the required accuracy of solution,  $S$  is the number of iterations; the initial iterative value is chosen equal to the solution on the previous time layer.

#### 4. Result

Machine algorithm for solving the problem is as follows:

- 1<sup>st</sup> step.** Input of initial (baseline) data (input of constants).
- 2<sup>nd</sup> step.** Calculation of boundary values of the sought for variables from boundary conditions of the problem.
- 3<sup>rd</sup> step.** Calculation of the elements of a tridiagonal transition matrix obtained by approximating differential operators to finite-difference ones.
- 4<sup>th</sup> step.** Calculation of sweep coefficients.
- 5<sup>th</sup> step.** Calculation of values of the sought for variables of the task.
- 6<sup>th</sup> step.** Adequacy verification of the task.
- 7<sup>th</sup> step.** Interpretation of computer experiments results.

#### 5. Conclusion

To predict changes in the level of ground and pressure water, a mathematical model was developed that takes into account the physical and mechanical properties of the geofiltration process, that is, parameters such as an external source, evaporation, filtration coefficients, active porosity, and filtration rate. Based on the theory of groundwater movement, a mathematical model has been developed that takes into account bilateral boundary conditions in a two-layer formation. Using a combination of finite-difference schemes and sweep methods, a method was developed for an approximate solution of the problem of changing the groundwater level in initial, boundary conditions and an effective numerical algorithm. An effective algorithm for conducting numerical experiments based on the developed finite-difference scheme is proposed. A conservative numerical algorithm was developed for computer experiments. Numerical algorithms based on the developed mathematical model significantly reduce the amount of work in the experiment and minimize expensive experimental work.

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