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Energetic processes around electromagnetically charged black hole in the Rastall gravity

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A B S T R A C T

Here we investigate the energetics of the charged-rotating-NUT-Kiselev black hole (CRNK-BH) in the Rastall modified theory of gravity. First, we study the dependence of the ergoregion around the CRNK-BH on the spacetime parameters, which is the key point to explain the energy extraction process from the BH. We show that the effect of the Kerr spin parameter a on this region is still dominating compared to the rest of the spacetime parameters. Then, we show that an increase in the NUT parameter l, electromagnetic charge q, and the parameter of equation-of-state of the quintessence ω, increases the efficiency of the Penrose-process while the effect of the rest of the spacetime parameters is just opposite. Moreover, we study the energy extraction process from the CRNK-BH in the Rastall modified theory of gravity due to the Blandford–Znajek mechanism. We demonstrate that the effect of all the spacetime parameters, except the spin and NUT parameters, on the efficiency of the Blandford–Znajek process is similar to that of the Penrose-process. We also give a comparison of the two energy extracting processes, discussed here, for the CRNK-BH in the Rastall modified theory of gravity.

1. Introduction

The observation of gravitational waves reported by the LIGO-VIRGO collaboration [1] from a binary of BHs, the images of BH shadows in the centre of the galaxy M87 and the Milky Way galaxy released by the network of synchronized radio observatories, known as the Event-Horizon-Telescope (EHT) collaboration [2,3], have encouraged researchers to investigate BH spacetimes with much deeper interest to explore the underlying Physics. BHs could be considered the most exciting objects as they can be studied in different astrophysically interesting scenarios such as the extraction of energy from them. In high-energy astrophysics, the extraction of energy from ultra-compact objects like quasars, X-ray binaries and active galactic nuclei has been remain a fascinating topic. Different mechanisms have been proposed to extract energy from BHs and in particular from rotating BHs. In this regard, the first attempt was made by Penrose to present a mechanism for the energy extraction from a rotating BH, called the Penrose-process [4]. In the Penrose-process, a particle falling from infinity creates two particles in the ergoregion of a BH. One of the two particles created in the ergoregion is bounded with negative energy, whereas the other one with positive energy escapes to infinity. Semiz

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claimed another mechanism based on the notion of Hawking radiation, for the extraction of energy from BHs that need not be rotating [5]. Bardeen et al. [6] and Wald [7], have separately shown that the Penrose-process of extracting energy from rotating BHs is not astrophysically viable for high-energy jets, as for the energy to be high enough for the breakup of the infalling particle must be itself of relativistic nature. Wagh et al. [8], have then shown that the drawback pointed out by Bardeen et al. and Wald, in the Penrose-process could be removed if the BH is immersed in an external electromagnetic field and it is called the magnetic Penrose-process. Another version of the Penrose-process called the collisional Penrose-process, in which instead of a single infalling particle two particles collide inside the ergoregion, was presented by Piran et al. with the efficiency almost same as that of the original Penrose-process [9]. Various BHs have been examined in the literature for their energy extraction by applying the Penrose-processes [10–12]. Negative energy states of the energy extraction processes for the Kerr BH have been discussed by Dhurandhar and Dadhich [13], where a formalism for the energy extraction is presented by them in a separate work [14]. The energetics of the Kerr–Newman black in the presence of an electromagnetic field, using the Penrose-process for charged particles have also been studied in the literature [15]. The possibility of the astrophysical jet collimation for rotating BHs and the existence of high-energy particles due to the Penrose-process, in the absence of magnetic fields, has been shown by Gariel et al. [16]. For the Kerr-Taub-NUT BH the dependence of the NUT parameter on the extracted energy has been discussed in [10], where it has been pointed out that the energy extraction rate due to the Penrose-process increases with the NUT parameter. A study of the efficiency of the Penrose-process for the magnetized Reissner–Nordstrom BH for both the charged and uncharged particles has been carried out in a recent work [17], where the effects of the charge of the BH and magnetic field on the Penrose-process are discussed. The electric Penrose-process and high-energy acceleration of ionized particles by a static and weakly charged BH have also been considered in the literature [18]. A review of the energy extraction processes from rotating BHs has been presented recently, where the magnetic Penrose-process is also revisited [19]. It is worth noting that there are also other mechanisms to explain the energy extraction from the BH environment due to the collision of particles around BHs. In [20] collisions of spinning test particles have been investigated near the event horizon of Deser–Sarioglu–Tekin (DST) BHs. In [21] the centre of mass (CM) energy of head-on colliding massive particles around BHs in the $f(R)$ gravity has been studied. Particle collision near $1+1$ Dimensional Horava–Lifshitz BH has taken place in [22].

Another promising mechanism for the energy extraction from compact objects like rotating BHs is the Blandford–Znajek (BZ) process. This mechanism provides a powerful description of the astrophysical jet formation around the supermassive BHs exist at the centre of galaxies. The recent discovery by NASA of the very long jet of particles in X-rays that coming from a supermassive BH, PJ352-15, at the centre of a distant galaxy, almost 13 billion light-years away from the Earth, with the help of Chandra X-ray Observatory [23], has opened new windows for researchers to study the BZ-process for different BH spacetimes. This discovery by NASA may provide an understanding of the formation of the biggest BH at the early stages of our Universe. The BZ-process has already been discussed for various BH spacetimes to look at the effects of the spacetime parameters on the efficiency of this process [24–27]. The efficiency of the BZ-process has also been compared with that of the Penrose-process for different rotating BHs [28–31]. The structure and strength of the magnetic field in the close vicinity of the BHs surrounded by an accretion disc, which may occur in radio galaxies and quasars, have been analysed in the context of the BZ-process [32]. The BZ-process has also been studied as a central engine consisting of a BH with an external magnetic field, for a gamma-ray burst which is of great interest in observational astrophysics providing information about the occurrence of each burst [33]. In a recent work, the BZ-process in the general stationary axially-symmetric BH spacetime has been discussed [27]. In another recent study, the extraction of energy by the BZ-process from BHs in the dynamical Chern–Simons gravity and scalar Gauss–Bonnet gravity has been investigated by Dong et al. where a comparison with the results in the Einstein gravity has also been given [34]. In [35] the BZ mechanism was explored to determine if the energy extraction rate and the magnetic field configuration around a (non-charged) BH surrounded by its magnetosphere can be modified by non-linear effects. Interested readers may study the useful literature available on the topic of the BZ-process (see e.g. [36–44])

Although the Einstein's theory of gravity has passed several observational tests and is a widely accepted theory of gravity, there are still unresolved mysteries like the current accelerated expansion of our Universe, quantization of gravitational fields on non-flat backgrounds etc. which could not be fully addressed in the framework of Einstein's theory. Different modifications of the Einstein theory of gravity have been proposed to overcome the issues that persist at both the cosmological and plank scale [45]. There are two approaches to having an alternative theory of gravity. One way is to modify the geometric part of the action of the Einstein theory by replacing the scalar curvature with a general function of it or introducing some higher-order curvature terms in the action. The other way is to modify the matter part of the action of the Einstein theory by introducing, e.g dark energy content in it [46]. A modification of Einstein's gravity is the Rastall modified theory of gravity (RMTG) which is widely studied in the literature in different contexts [47–50]. In the RMTG the non-minimally coupling of the matter field with the spacetime geometry is considered [51]. In the RMTG, the divergence of the curvature scalar is proportional to the divergence of the stress–energy–momentum tensor. Visser [52], has rejected the RMTG by saying that it is completely equivalent to Einstein's gravity. However, Darbari et al. [53], have countered the Visser claim, arguing that the Einstein theory and the RMTG are two different theories of gravity and have advocated the RMTG as an alternative theory of gravity. In the RMTG a charged rotating BH solution with the NUT parameter l, in the presence of a particular form of dark energy known as quintessence, which is considered as a candidate for the observed accelerated expansion of our Universe, has been derived [54]. This BH solution of the RMTG can be reduced to the charged-Kiselev BH solution [55], in the absence of the NUT and Rastall parameters. In a recent study the motion of particles and related phenomena for the CRNK-BH in the RMTG have been considered [56], where in particular the effects of the Rastall parameter have been analysed on geodesic motion. In another recent work photon motion and the shadow formed by the CRNK-BH in the RMTG have been investigated [57]. In the present study, we investigate the Penrose and BZ-processes for the CRNK-BH in the Rastall gravity to look at the effects of the spacetime parameters and in particular to examine the effect of the Rastall parameter on the efficiency of the two energy extracting process. In [58] the tunnelling rate of the massive charged W-bosons in a background of the electromagnetic field is calculated in order to investigate the Hawking temperature of BHs surrounded by perfect fluid in the RMTG. The shadow and energy emission rate of a spherically symmetric non-commutative BH in the RMTG has been studied in [59]. Further analyses of properties of several BH solutions have taken place in our previous works [60–68].

This paper is organized as follows: In the next Sec. we give a brief review of the CRNK-BH in the RMTG. In Section 3 we discuss the Penrose-process for the CRNK-BH in the RMTG, where we show that the spin of the BH has a prominent impact on the ergoregion of the BH in comparison to other spacetime parameters. In the same Sec. we show that the efficiency of the Penrose-process increases with the NUT parameter l, electromagnetic charge, and the parameter of equation-of-state of the quintessence. While the other spacetime parameters, including the Rastall parameter, have the opposite effect on the efficiency of the Penrose-process of energy extraction. In Section 4 we study the BZ-process for the CRNK-BH in the RMTG, where we demonstrate that the effect of all the spacetime parameters, except the spin and NUT parameters of the BH, on the efficiency of the BZ-process is similar to that of the Penrose-process. In the last Sec. we present a summary of our work.

2. The charged-rotating-NUT-Kiselev black hole in the Rastall modified theory of gravity

From one side General Relativity (GR) is the classical field theory that leads to the cosmological and astrophysical singularities at the origin which can be removed through the quantization of gravity on a curved background. On the other hand, GR is constrained by observational problems e.g. the late-time accelerated cosmic expansion cannot be well explained in the framework of the classical GR theory. Therefore, in order to resolve such problems, several alternative/modified theories of gravity have been proposed [69], amongst which one is the RMTG. There exist rotating and static BH solutions in the RMTG [70,71]. Sakti et al. [54], have given a rotating BH solution in the RMTG which we briefly discuss here.

The background geometry of the CRNK-BH in the RMTG is described by the spacetime metric (see [54])

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\Sigma}[dt - \{a\sin^2\theta + 2l(1-\cos\theta)\}d\phi]^2 \\ &+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[adt - \{r^2 + (l+a)^2\}d\phi]^2 , \end{aligned} \tag{1}$$

where

$$\Delta = r^2 - 2Mr + g^2 + e^2 + a^2 - l^2 - \alpha r^{\upsilon}, \tag{2}$$

$$\upsilon = \frac{1-3\omega}{1-3\kappa\lambda(\omega+1)}, \tag{3}$$

$$\Sigma = r^2 + (l + a\cos\theta)^2 . \tag{4}$$

The CRNK-BH in the RMTG has seven parameters including the standard ones as the total mass M and the specific angular momentum a of the BH. The rest of the spacetime parameters are: the NUT parameter l, the quintessential intensity parameter α, the Rastall gravity parameter $\kappa\lambda$, the equation-of-state parameter of the quintessence ω, the electric charge e and the magnetic charge g of the BH, respectively [54]. A new parameter $q^2 = g^2 + e^2$ is introduced which includes the contribution of both the electric and the magnetic charges of the BH to the spacetime geometry.

Depending on the values of the parameters $\kappa\lambda$ and ω the CRNK-BH in the RMTG may contain several horizons. The coordinate singularity and a null hypersurface of constant r of the spacetime determine the event horizon (EH). Consequently, the horizons are determined as the zeros of the following algebraic equation

$$r^2 - 2Mr + q^2 + a^2 - l^2 - \alpha r^{\upsilon} = 0 . \tag{5}$$

The horizon coincides to have only outer and inner parts or there does not exist cosmological horizon for the selected values of the parameters $\kappa\lambda$ and ω. Due to this reason, only two zeros for the horizon equation are explored [72] for the specific case selected. When the above horizon equation (5) has more than two zeros, it is impossible to perform analytical calculations, for example, to study the entropy product. Table 1 presents the selected values of parameters $\kappa\lambda$ and ω allowing two analytic roots for the horizon equation from [73].

One can also solve Eq. (5) numerically to explore the dependence of the outer horizon radius on the spacetime parameters as illustrated in Fig. 1. The top-left panel shows that the increase of the parameter α increases the EH radius almost linearly. In the top-right panel, one can see that the effect of the Rastall parameter on the EH radius becomes weak for comparatively smaller values of the parameter but becomes considerable for the values of the parameter very close to zero and this effect is stronger for bigger values of the quintessential intensity, α. The plot in the middle-left panel gives the dependence of the outer horizon on the NUT parameter l. We see that for the bigger values of the NUT parameter l, the outer horizon becomes bigger as well and the change of the Rastall parameter can only shift the lines, not strongly affecting the shape of the horizon. The middle-right panel demonstrates the change of the outer horizon with the change of the spin of the CRNK-BH. One can see the traditional dependence of the EH radius on the spin parameter as in the case of the well-known Kerr BH. The dependence of the EH radius on the electromagnetic charge parameter is given in the bottom-left panel where it is shown that an increase in the charge parameter reduces the EH radius considerably and this reduction in the size of the horizon strongly depends on the parameter α. We show the effect of the parameter of the equation-of-state of the quintessence in the bottom-right panel and it is seen that bigger values of this parameter and the electromagnetic charge make the horizon size smaller.

Table 1
First two columns represent selected values of the BH parameters $\kappa\lambda$ and ω providing exact analytical solutions for the inner and the outer horizons demonstrated in the last column.

ω, $\kappa\lambda$	Horizon ($r_\pm$)
0,0	$\left(M+\frac{\alpha}{2}\right)\pm\sqrt{\left(M+\frac{\alpha}{2}\right)^2+l^2-q^2-a^2}$
−1/3,0	$\frac{M}{1-\alpha}\pm\frac{\sqrt{M^2+(q^2+a^2-l^2)(1-\alpha)}}{\alpha-1}$
0,1/6	$\frac{M}{1-\alpha}\pm\frac{\sqrt{M^2+(q^2+a^2-l^2)(1-\alpha)}}{\alpha-1}$
−1/3,−1/2	$\left(M+\frac{\alpha}{2}\right)\pm\sqrt{\left(M+\frac{\alpha}{2}\right)^2+l^2-q^2-a^2}$
1/3,0	$M\pm\sqrt{M^2+l^2+\alpha-a^2-q^2}$

Table 2
The last column represents exact analytical solutions for ergosphere for the selected values of the BH parameters $\kappa\lambda$ and ω.

ω, $\kappa\lambda$	Ergosphere ($r_{erg\pm}$)
0,0	$\frac{1}{2}\left(2M+\alpha\pm\sqrt{\alpha^2+4l^2+4M^2+4\alpha M-4q^2}\right)$
−1/3,0	$\frac{-2M\pm\sqrt{4M^2-4(-1+\alpha)(l^2-q^2)}}{2(-1+\alpha)}$
0,1/6	$\frac{-2M\pm\sqrt{4M^2-4(-1+\alpha)(l^2-q^2)}}{2(-1+\alpha)}$
−1/3,−1/2	$\frac{1}{2}\left(2M+\alpha\pm\sqrt{\alpha^2+4l^2+4M^2+4\alpha M-4q^2}\right)$
1/3,0	$M\pm\sqrt{\alpha+l^2+M^2-q^2}$

Like in the case of the EH radius, one can show the dependence of the ergosphere radius (Table 2) on the spacetime parameters as presented in Fig. 2. One can see that the behaviour of the ergosphere radius for the spacetime parameters of the CRNK-BH in the RMTG is very similar to the behaviour of the EH radius for all the spacetime parameters except the panel in the middle-right where the dependence on the spin parameter is illustrated. We see that the ergosphere radius depends on the spin very weakly while the dependence of the EH radius on the spin is significant.

We explore the circular geodesic of a massive particle with $r = const$ and $\theta = const$ around the axially symmetric CRNK-BH in the RMTG where $\mathbf{u} \sim \xi_{(t)} + \Omega\xi_{(\phi)}$ in order to restrict the angular velocity $\Omega = d\phi/dt = u^\phi/u^t$. The vector $\mathbf{u}$ is timelike that requires $\Omega_- < \Omega < \Omega_+$, where

$$\Omega_\pm = \frac{-g_{t\phi}\pm\sqrt{(g_{t\phi})^2-g_{tt}\cdot g_{\phi\phi}}}{g_{\phi\phi}}\ . \tag{6}$$

The motion of the photon is responsible for limiting the values of $\Omega = \Omega_\pm$. The $\Omega_+ = 0$ and $\Omega_- = -2g_{t\phi/g_{\phi\phi}}$ at the static limit surface ($g_{tt} = 0$). However, Ω_+ is always positive outside the static limit surface ($g_{tt} < 0$), where Ω_- may be both positive and negative (see Fig. 3).

3. Penrose-process in the ergoregion of the rotating black hole in the Rastall modified theory of gravity

The Penrose-process is one of the classical mechanisms of extraction of rotational energy from rotating BHs. The Penrose-process requires the existence of an ergoregion: the region between the EH and the static limit. In Figs. 4 and 5 we have presented the dependence of the ergoregion on the spacetime parameters of the CRNK-BH in the RMTG. Left and right panels correspond to the case when the spin parameter $a = 0.5$ and $a = 0.8$, respectively. In these figures, the solid lines correspond to the EH, and the dashed lines represent the ergosphere. From the top-left panel of Fig. 4 one can see that increase of the parameter α makes the ergoregion thinner. The increase of the electromagnetic charge causes a decrease in the total size of the ergoregion (see the middle left panel of Fig. 4). One can also see from the top left panel of Fig. 5 that effect of the bigger values of the NUT parameter l is similar to the one due to the quintessential intensity α. Comparing the middle-left and bottom-left panels of Fig. 5 one can notice that the Rastall parameter $\kappa\lambda$ and the parameter ω, have opposite effects on the ergoregion. It can be also noticed from the right panels of Figs. 4 and 5 that increase of the spin parameter a of the BH makes the ergoregion to become bigger (also see the bottom panel of Fig. 4). This is due to the fact that the increase of the spin reduces the EH, however, it does not change the photon sphere as it is demonstrated in Figs. 1 and 2.

For the Penrose-process, we suppose an incident particle (p_0) splits into two particles (p_1) and (p_2) in the BH ergoregion as shown in Fig. 6. Suppose that one of them, say, (p_1) crosses the EH, while the other one, (p_2) escapes to infinity. As a consequence, the escaping particle (p_2) might have an energy higher than the energy of the incident test particle (p_0). Assume that the incident particle (p_0) is falling into the ergoregion of the BH with the energy $E_{(0)} \geq 1$. As the incident particle will be split into two fragments in the ergoregion of the BH, the energies of the particles (p_1) and (p_2) will be $E_{(1)}$ and $E_{(2)}$, respectively. As we mentioned above,

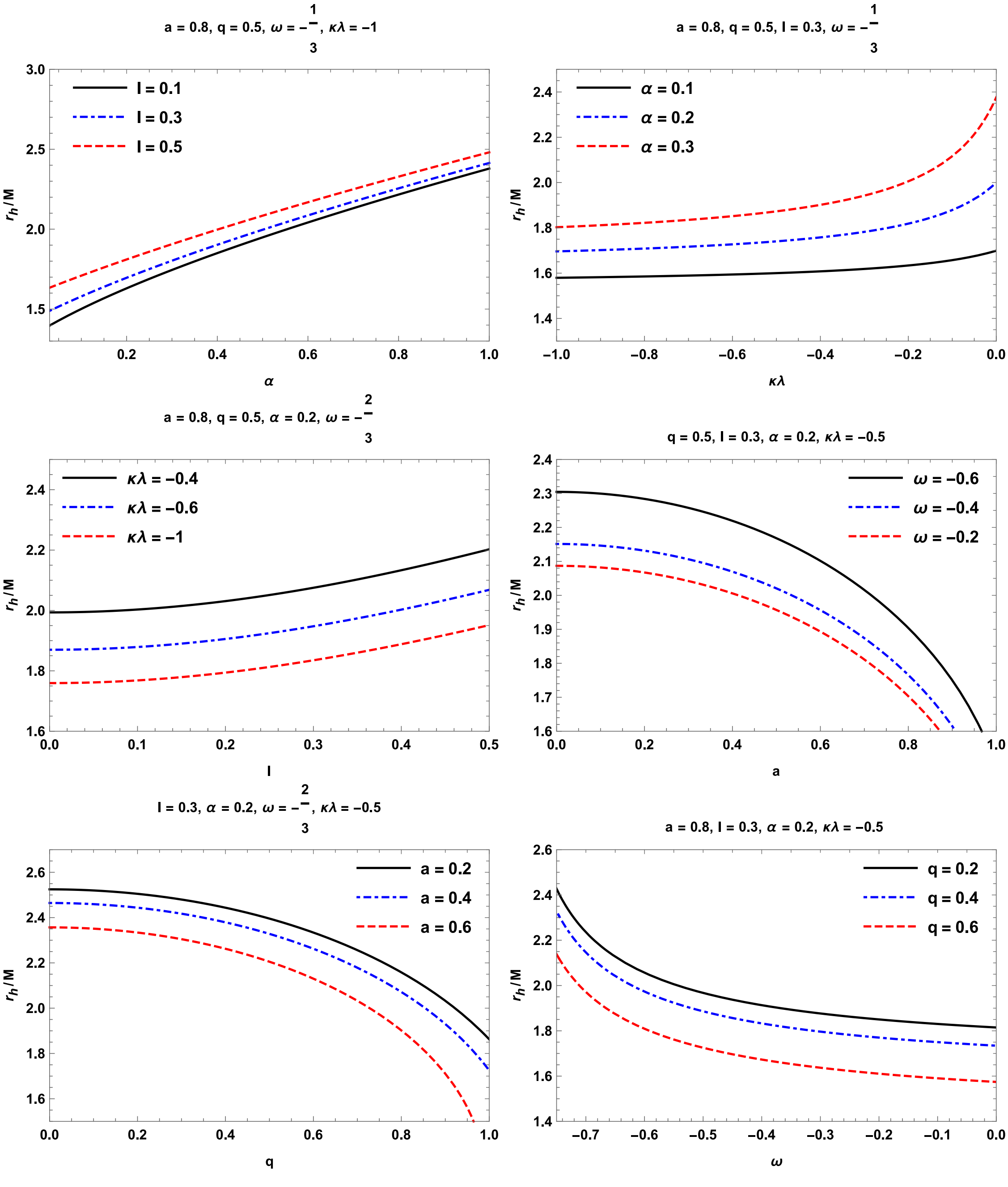

Fig. 1. Effect of the parameters of the CRNK-BH spacetime in the RMTG on the EH radius.

the particle (p_2) escapes to the infinity with the energy $E_{(2)} > 0$, whereas the other particle (p_1) falls into the BH with negative energy $E_{(1)} < 0$. With these assumptions, the conservation laws for different quantities in the ergosphere can be expressed as:

$$\begin{aligned} E_{(0)} &= E_{(1)} + E_{(2)}\ , \\ L_{(0)} &= L_{(1)} + L_{(2)}\ , \\ M_{(0)} &= M_{(1)} + M_{(2)}\ , \end{aligned} \tag{7}$$

where $M_{(i)}$ and $L_{(i)}$ are the mass and the angular momentum of the particle (p_i), respectively, for $i = 0, 1, 2$. In order to extract the energy from the central BH via the Penrose-process, the condition $E_{(1)} < 0$ should be satisfied. In order to estimate the efficiency of

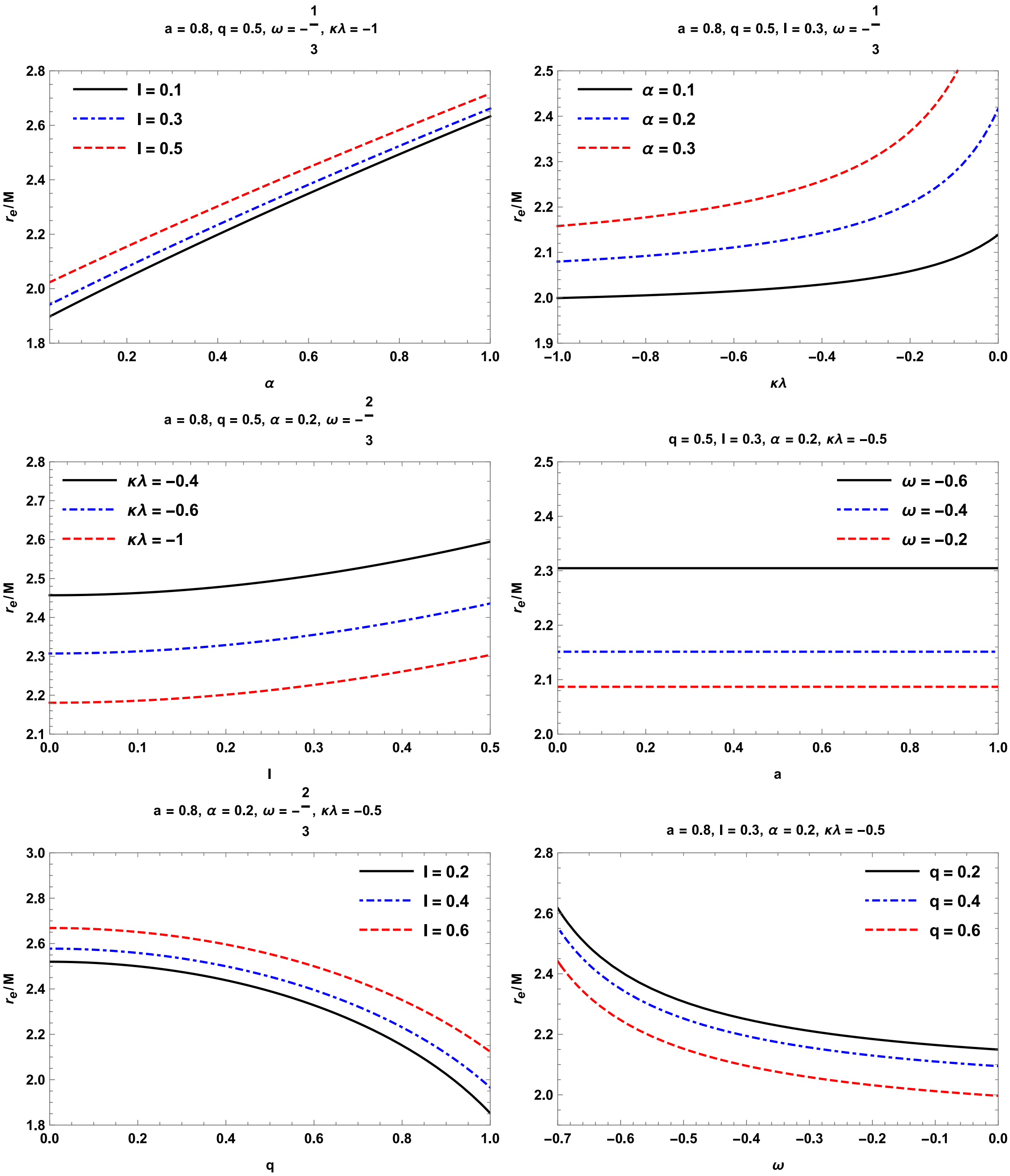

Fig. 2. Effect of the parameters of the CRNK-BH spacetime in the RMTG on the ergosphere.

the energy extraction due to the Penrose-process we consider a simple scenario where the test particles are restricted to move on an equatorial plane ($\theta = \pi/2$). Thus the conserved momentum can be written as:

$$P^{\mu}_{(0)} = P^{\mu}_{(1)} + P^{\mu}_{(2)}. \tag{8}$$

The momenta of the three particles P^{μ}_{j} (j = 0, 1, 2) are timelike by definition and consequently are inside the local light cone. The radial and angular coordinates (r and ϕ) describe the particle's orbit moving on a plane. Then the momentum of a particle travelling along the geodesic line γ takes the form

$$P_{\gamma} = P^{t}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + \Omega\frac{\partial}{\partial \phi}\right) , \tag{9}$$

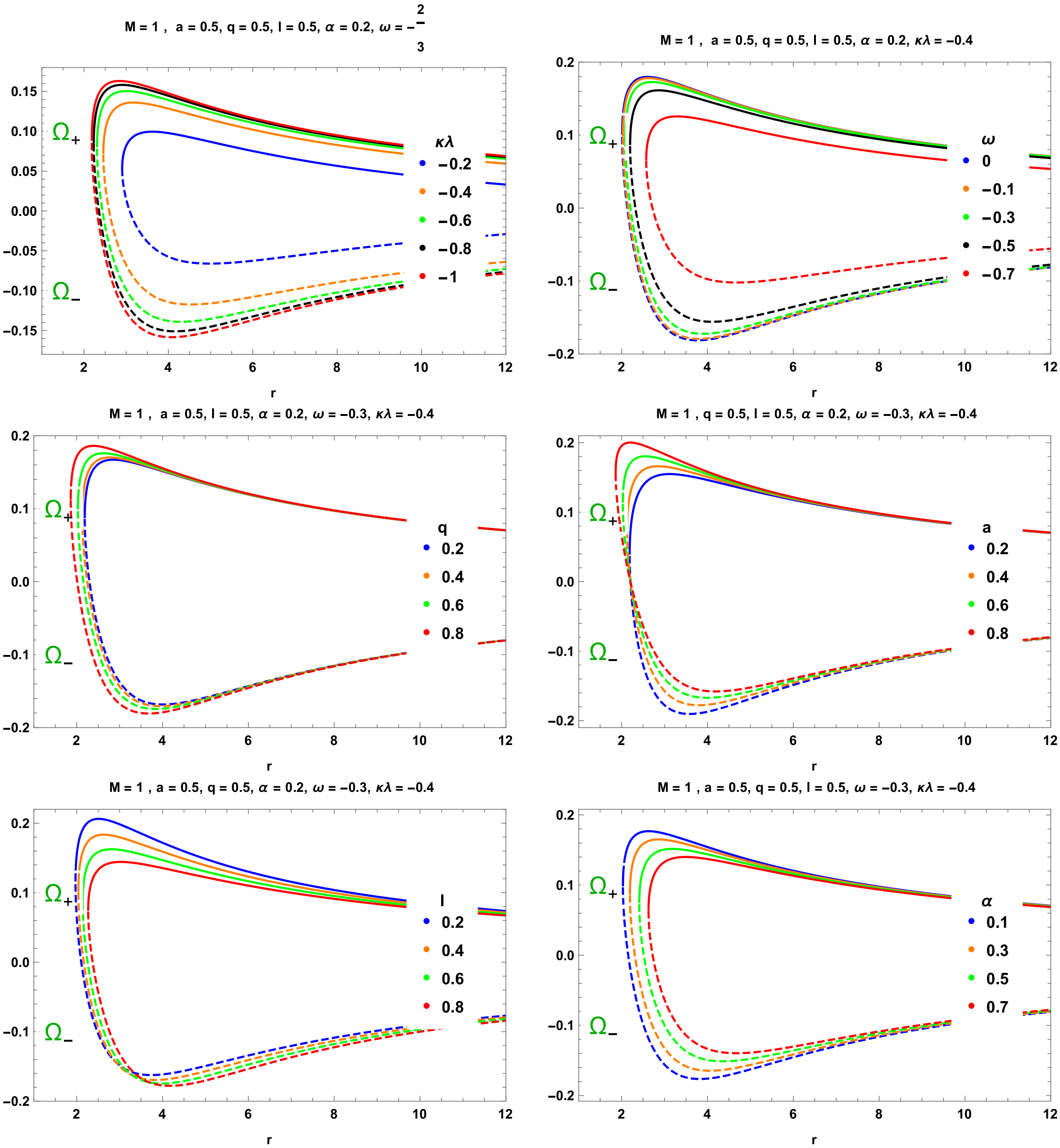

Fig. 3. The graphs show the relation between the angular velocity of the test particle and r for various spacetime parameters of the CRNK-BH in the RMTG. Solid lines represent Ω_+ while the dashed lines correspond to Ω_-.

where $\Omega = d\phi/dt$ and $v = dr/dt$. The conserved energy E is defined as $E = -P_t$ which can be expressed as

$$P^t = -\frac{E}{X} \ , \tag{10}$$

with

$$X = g_{tt} + \Omega g_{t\phi} \ . \tag{11}$$

Using the definition $P^\mu P_\mu = -M^2$, one can easily find the geodesic equation as

$$g_{tt} + v^2 g_{rr} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} = -\left(\frac{MX}{E}\right)^2 \ . \tag{12}$$

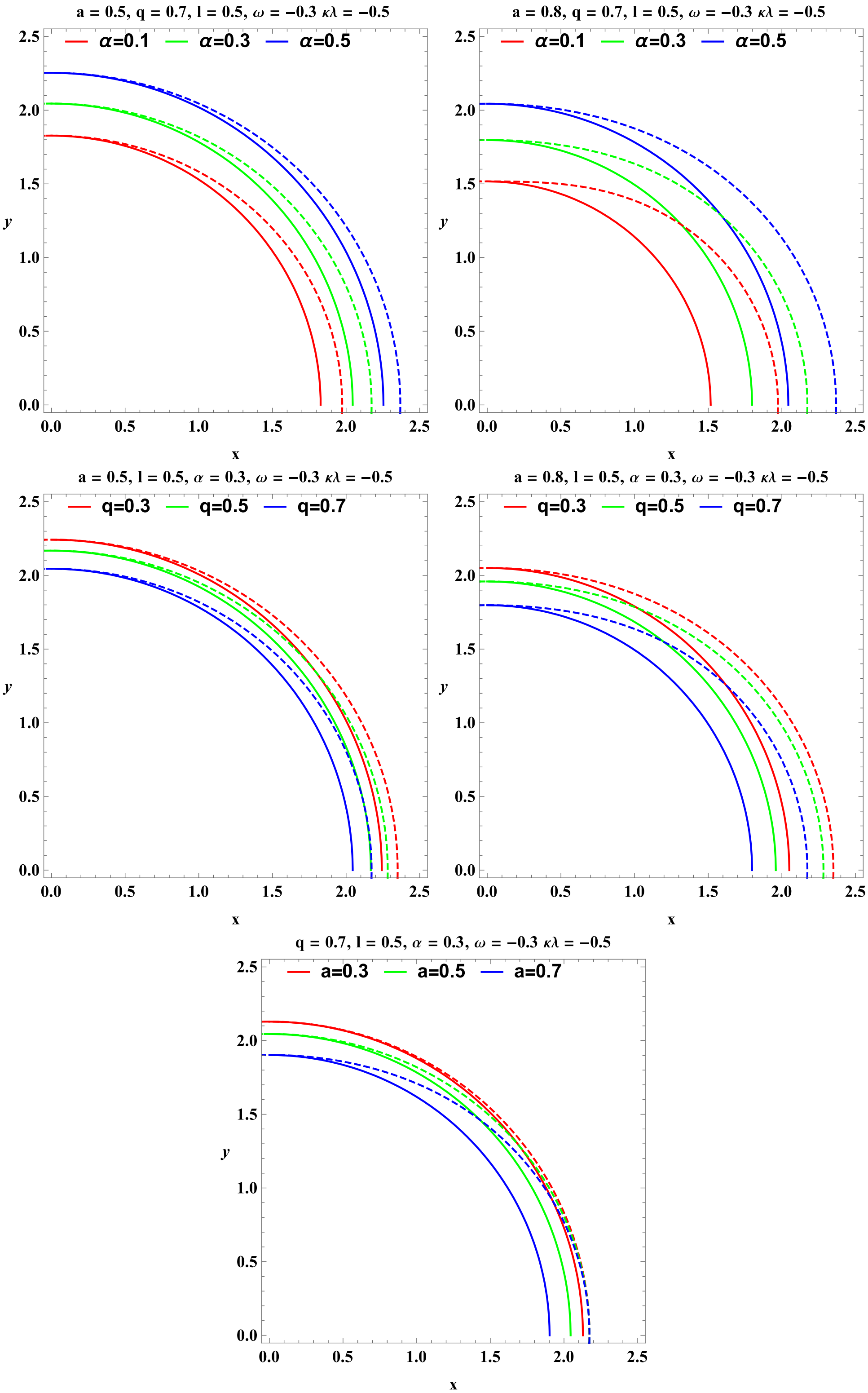

Fig. 4. The graphs illustrate the difference between the EH and the ergosphere which corresponds to the ergoregion.

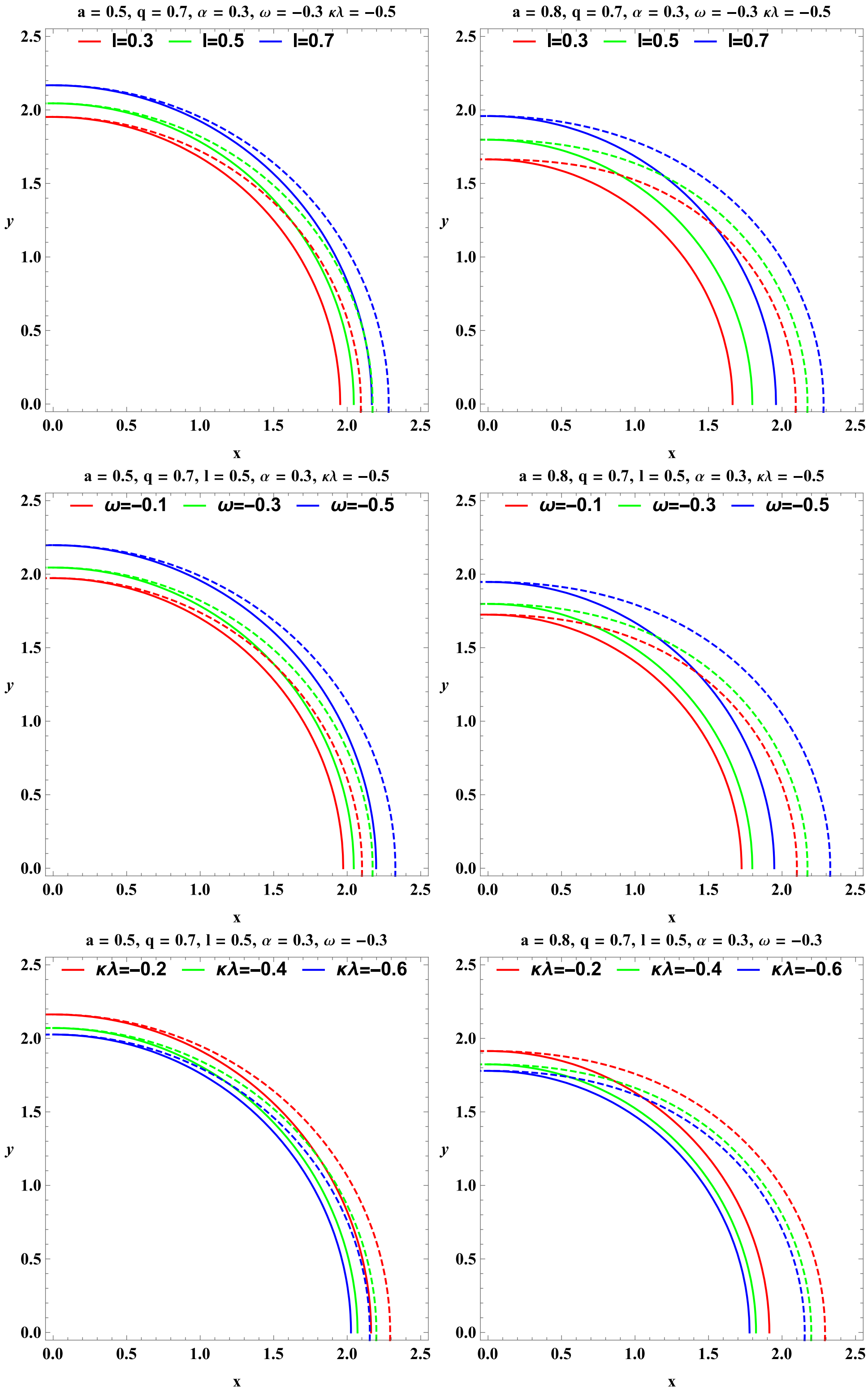

Fig. 5. Continuation of Fig. 4.

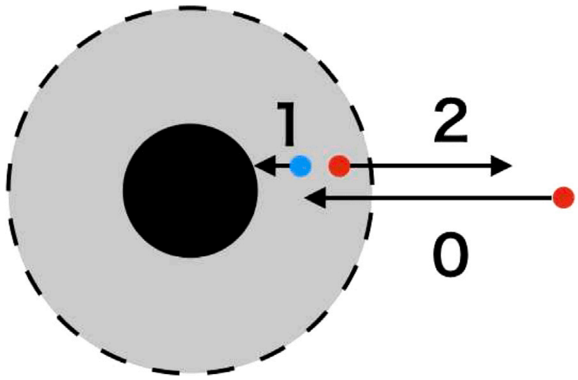

Fig. 6. Schematic illustration of the Penrose-process. The grey shaded region illustrates the ergoregion while the black circle in the centre of the EH is a BH. *Source:* Credited from [4].

A solution of Eq. (12) together with Eq. (6) defines angular velocity Ω. using the expressions for the conserved energy ($E = -P^t X$) and angular momentum ($L = P^t \Omega$), one can get the following equations

$$P^t_{(0)} X_{(0)} = P^t_{(1)} X_{(1)} + P^t_{(2)} X_{(2)} \ , \tag{13}$$

$$P^t_{(0)} \Omega_{(0)} = P^t_{(1)} \Omega_{(2)} + P^t_{(2)} \Omega_{(2)} \ . \tag{14}$$

The Penrose-process provides the efficiency of the energy extraction as

$$\eta = \chi - 1 \ , \tag{15}$$

where

$$\chi = \frac{E_{(2)}}{E_{(0)}} \ . \tag{16}$$

Using Eqs. (13) and (14) one can get expression of ξ as,

$$\chi = \frac{(\Omega_{(0)} - \Omega_{(1)}) X_{(2)}}{(\Omega_{(2)} - \Omega_{(1)}) X_{(0)}} \ . \tag{17}$$

The condition $dr/dt = 0$ in Eq. (9) defines the maximum efficiency. Consequently, one can get expressions of momentum for both particles as

$$P_{(1)} = P^t_{(1)} \left(\frac{\partial}{\partial t} + \Omega_{(1)} \frac{\partial}{\partial \phi} \right) \ , \tag{18}$$

$$P_{(2)} = P^t_{(2)} \left(\frac{\partial}{\partial t} + \Omega_{(2)} \frac{\partial}{\partial \phi} \right) \ . \tag{19}$$

Then one can obtain $\Omega_{(0)}$ as

$$\Omega_{(0)} = \frac{-g_{t\phi}(1 + g_{tt}) + \sqrt{(1 + g_{tt})(g^2_{t\phi} - g_{tt} g_{t\phi})}}{g^2_{t\phi} + g_{\phi\phi}} \ . \tag{20}$$

Using Eqs. (6), (17) and (20) one can get equation for efficiency of Penrose-process as [74]

$$\eta_{max} \leq \frac{g_{\phi\phi}(\sqrt{1 + g_{tt}} + 1) + g^2_{t\phi}}{2 g_{\phi\phi} \sqrt{1 + g_{tt}}} - 1 \ . \tag{21}$$

Table 3 demonstrates the numerical values of the efficiency of the Penrose-process for various values of the electromagnetic charge q, the rotation parameter a, and the quintessential intensity, α. One can see from the table that increase of the spin and electromagnetic charge makes the Penrose-process more efficient while the effect of the quintessential intensity is just opposite.

Due to the complex analytic form of the efficiency here, we present the dependence of it on the various BH spacetime parameters graphically in Fig. 7. The top-left panel demonstrates that for faster rotation of the BH the efficiency of Penrose-process is bigger. The reason for this is the following. The Penrose-process involves extracting energy from the rotational motion of the BH. Therefore, when the BH rotates faster, there is more energy available for extraction, resulting in higher efficiency of the Penrose-process. One can also notice that a change in the value of the Rastall parameter does not change the efficiency of the Penrose-process considerably. The top-right panel shows that the efficiency of the radiation can be bigger for faster rotation of the BH and for smaller values of

Table 3
Efficiency of energy extraction for various a & q parameters (left panel) and α & q parameters (right panel) $\alpha = 0.1$ and $a = 0.8$ respectively. For both case $r = r_h$, $M = 1$, $l = 0.1$, $\omega = -0.1$, $\kappa\lambda = -1$.

		a parameter							α parameter						
		0.2	0.4	0.6	0.8	1.0			0	0.05	0.1	0.15	0.2	0.25	0.3
q parameter	0.2	0.57	1.28	2.67	5.27	11.59	q parameter	0.2	5.81	5.53	5.27	5.05	4.84	4.65	4.47
	0.4	1.87	2.41	3.48	5.45	9.64		0.4	5.85	5.64	5.45	5.28	5.11	4.96	4.82
	0.6	4.02	4.34	5.04	6.30	8.73		0.6	6.64	6.47	6.30	6.15	6.00	5.87	5.74
	0.8	7.02	7.14	7.49	8.17	9.41		0.8	8.57	8.36	8.17	7.99	7.81	7.65	7.49
	1.0	10.87	10.84	10.95	11.21	11.73		1.0	11.78	11.48	11.21	10.95	10.70	10.47	10.24

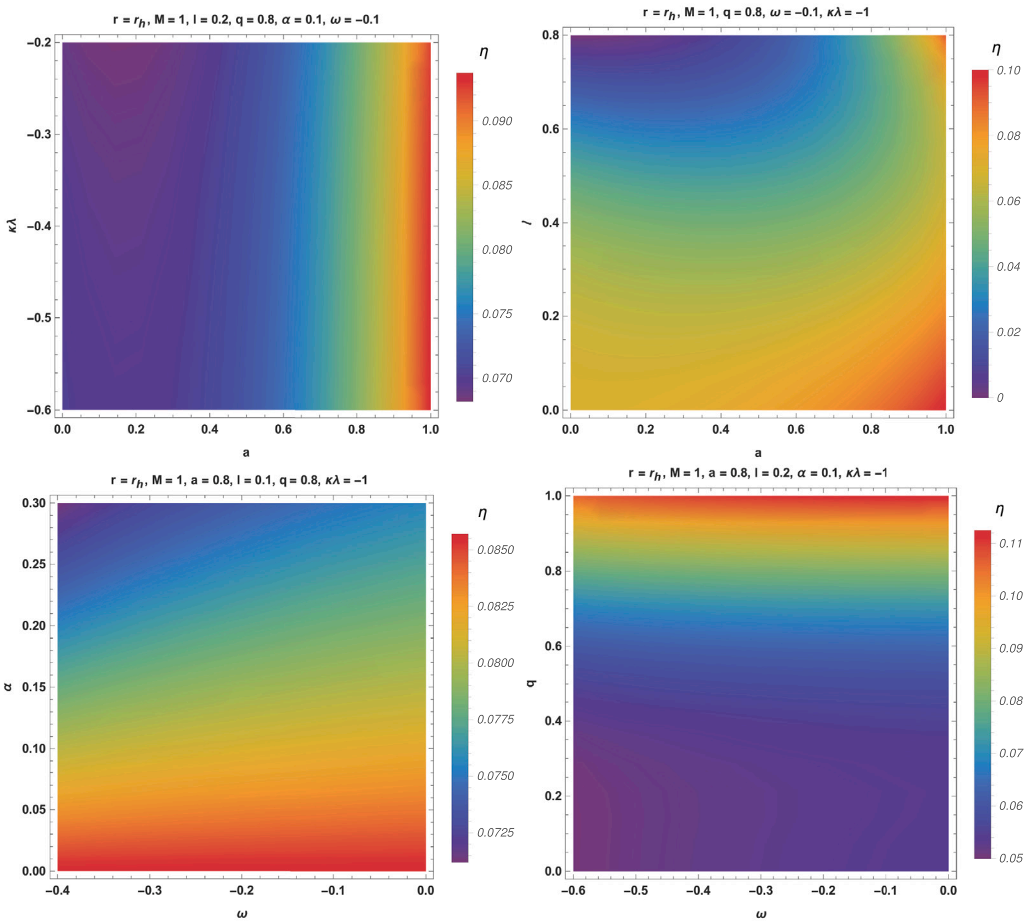

Fig. 7. Effect of the parameters of the CRNK-BH spacetime in the RMTG on the efficiency of the Penrose-process.

the NUT parameter l. It is also noticeable that the efficiency of the Penrose-process is not monotonic with the change of the NUT parameter for bigger values of the spin parameter of the BH. It is clearly seen from the rightmost part of the top-left panel that the efficiency starts decreasing with the increase of the NUT parameter first, and then starting from some point starts going up with the increase of this parameter. In the bottom-left panel, it is shown how the efficiency of the Penrose-process changes with the change of the parameters α and ω. It can be easily noticed that the Penrose-process becomes more efficient for smaller values of the quintessential intensity α. The effect of the equation-of-state parameter of the quintessence ω is relatively weak and becomes considerable for bigger values of the parameter α which shows that bigger values of the former make the efficiency of the Penrose-process stronger. Lastly, in the bottom-right panel, the behaviour of the efficiency of the Penrose-process for different values of the electromagnetic charge q and parameter ω is shown. The effect of ω is still weak for all values of the electromagnetic charge but the changes are visible for smaller values of the charge. It can be seen that bigger values of the electromagnetic charge of the BH make the efficiency stronger (e.g. increases almost twice when changing from 0 to 1). In general, in the presence of an electromagnetic charge, the electromagnetic field can contribute to the creation and acceleration of charged particles, increasing their energy and allowing for more efficient energy extraction.

4. Blandford-Znajek process in the rotating black hole spacetime in the Rastall modified theory of gravity

The stress–energy–momentum tensor, In the force-free approximation, is that of the electromagnetic field,

$$T^{\alpha\beta} = T^{\alpha\beta}_{em} = F^{\alpha\gamma}F^{\beta}_{\gamma} - \frac{1}{4}g^{\alpha\beta}F^{\sigma\tau}F_{\sigma\tau}, \tag{22}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Faraday tensor and A_α denotes the vector potential. Here the equations of motion are

$$\nabla_\alpha T^{\alpha\beta}_{em} = 0. \tag{23}$$

Taking into account the force-free condition, if A_α is independent of the coordinates t and ϕ, the Faraday tensor is then expressed as

$$F_{\alpha\beta} = \sqrt{-g}\begin{pmatrix} 0 & -\omega B^\theta & \omega B^r & 0 \\ \omega B^\theta & 0 & B^\phi & B^\theta \\ -\omega B^r & -B^\phi & 0 & B^r \\ 0 & B^\theta & -B^r & 0 \end{pmatrix}, \tag{24}$$

with

$$\omega = -\frac{\partial_\theta A_t}{\partial_\theta A_\phi} = -\frac{\partial_r A_t}{\partial_r A_\phi}, \tag{25}$$

represents the rotational frequency of the electromagnetic field, g stands for the determinant of the metric tensor of the spacetime i.e $g_{\alpha\beta}$.

The total energy flux from the BH is given as the following integral

$$P_{BZ} = -2\pi\int_0^\pi \sqrt{-g}T^r_t d\theta, \tag{26}$$

where T^r_t represents the radial component of the Poynting vector. The integral in the (26) is evaluated at some surface r = const. Eq. (26) does not depend on the choice of the radial coordinate at which the integral is evaluated on the shape of the magnetic field.

The formula for the BZ-process of the jet power can be written as

$$P_{BZ} = \frac{k}{16\pi}\Phi_B^2\Omega_h^2 + O(\Omega_h^4), \tag{27}$$

where Ω_h stands for the angular velocity of the EH. It is given as $\Omega_h = \left(-\frac{g_{t\phi}}{g_{\phi\phi}}\right)_{r=r_h}$ and the numerical constant k depends on the configuration of the magnetic field (for instance, $k = 0.044$ for a parabolic geometry and $k = 0.053$ for a split monopole geometry) and Φ_B represents the magnetic flux threading the horizon of the BH

$$\Phi_B = \int\int |B^r|\sqrt{-g}d\theta d\phi, \tag{28}$$

we only consider P_{BZ} up to Ω_h^2, neglecting terms $O(\Omega_h^4)$ which gives

$$\log P_{BZ} = \log(M\Omega_h)^2 + \log A. \tag{29}$$

The best-fitting values for the correlation $\log A$ are $2.94 \pm 0.22(90\%)$, for $\Gamma = 2$ and $4.19 \pm 0.22(90\%)$ for $\Gamma = 5$ [75].

In Fig. 8 the dependence of the jet power due to the BZ mechanism on the spacetime parameters of the CRNK-BH in the RMTG is presented. The results presented in the top-left panel clearly indicate that with the growth of spin of the BH, the energy extraction due to the BZ mechanism becomes more efficient being similar to the Penrose-process. However, in the BZ mechanism, the change of the jet power is smoother in comparison to the efficiency of the Penrose-process. One can observe that the Rastall parameter practically does not affect the jet power being similar to the case of the efficiency of the Penrose-process. In the top-right panel, it is shown that jet power due to the BZ mechanism can be highly amplified by rapid rotation of the CRNK-BH in the RMTG, even for small values of the NUT parameter l. One can see that increase of the NUT parameter reduces the jet power of the BZ mechanism monotonically. This effect of the NUT charge on the BZ mechanism can be explained in the following way. In [57] it is shown that the presence of the gravitomagnetic charge (NUT parameter) strengthens the resultant gravitational field around BH causing the photon trajectory to be twisted stronger. Similarly, a stronger gravitational field reduces the energy of radiation emitted due to the redshift of photons. Thus, bigger values of the NUT parameter reduce the efficiency of BZ mechanism in general. The bottom-left panel demonstrates that the bigger values of the equation-of-state parameter of the quintessence even for smaller values of the quintessential intensity can play an important role in the creation of strong jets. The effect of the ω parameter is strengthened for bigger values of α parameter similarly to the bottom-left panel of the preceding figure (see Fig. 7). Finally, the bottom-right panel of Fig. 8 demonstrates the dependence of the jet power associated with the BZ mechanism from the electromagnetic charge and the equation of state parameter of the quintessence. The jet power is strongly amplified with the increase of the electromagnetic charge q and the ω parameter is similar to the behaviour of the efficiency of the Penrose-process discussed in the previous section. In principle, the increased strength of the rotating magnetic field with a larger electromagnetic charge leads to a larger amount of energy being transferred to the jets, thereby increasing their power. Consequently, the jet power generated by the BZ mechanism is stronger in the presence of a larger electromagnetic charge of the BH.

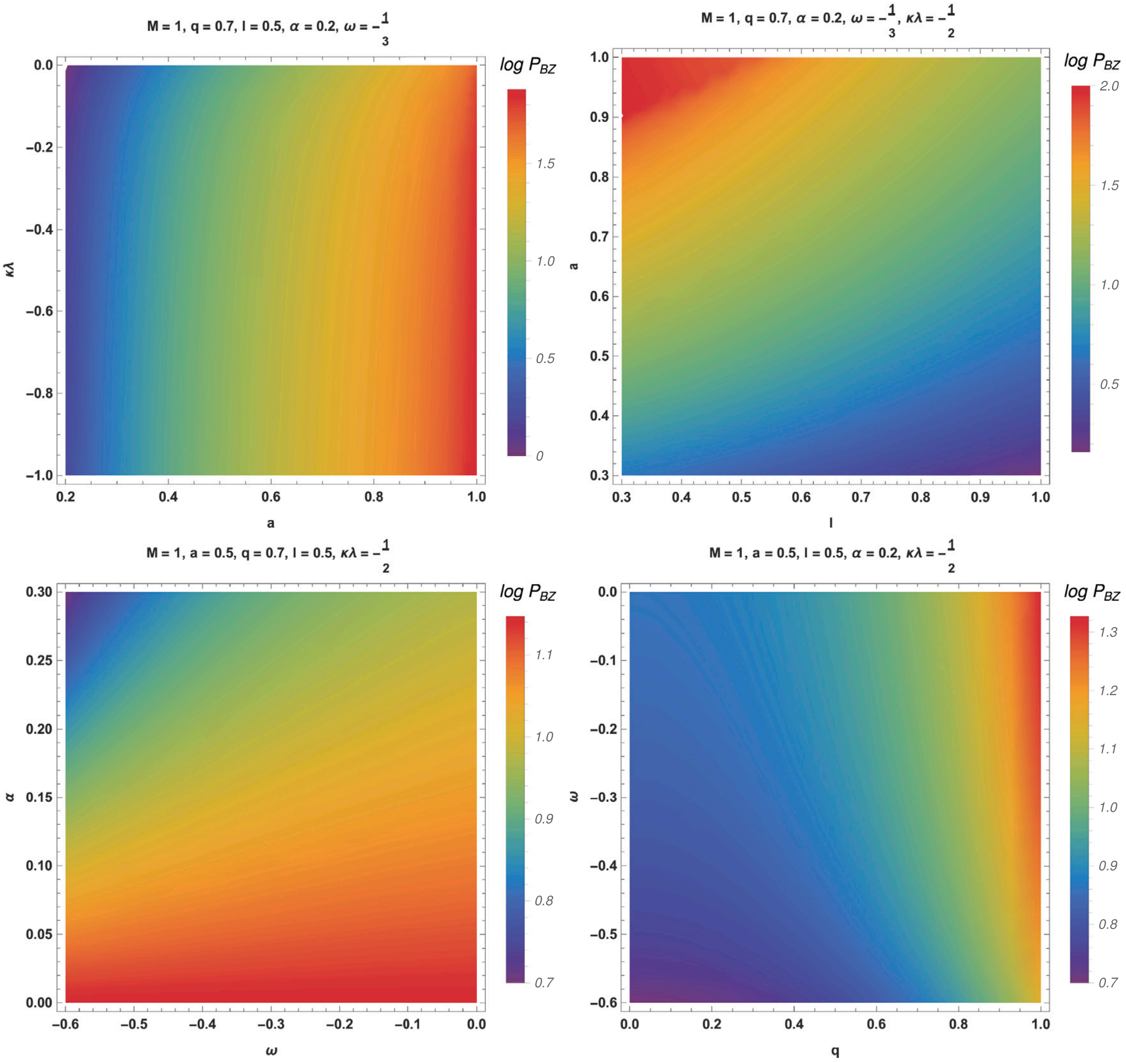

Fig. 8. Effect of the parameters of the CRNK-BH spacetime in the RMTG on the BZ mechanism ($\Gamma = 2$).

Summary

We have studied the energetic properties of the CRNK-BH in the RMTG, which is a significant point to explain the energy extraction process from the BH. First, the behaviour of the EH with a change in the values of the spacetime parameters is explored. We have shown that an increase of the quintessential intensity parameter α, Rastall parameter $\kappa\lambda$, and NUT charge increase the radius of the apparent horizon while increasing the values of the rest of the spacetime parameters reduce the size of the EH. It is also demonstrated that similar effects of the above mentioned spacetime parameters on the size of the ergosphere are observed, besides the effect of the spin parameter. We have observed that the spin of the BH has a negligible influence on the ergosphere radius. In other words we can say that the size of the ergosphere is very weakly sensitive to the spin of the CRNK-BH. Further, we have found that this negligible effect of the spin of the CRNK-BH on the ergosphere radius causes the ergoregion to be larger.

Next, we have investigated the energy extraction from the CRNK-BH in the RMTG due to the Penrose-process. We have found that for smaller values of the parameters α and $\kappa\lambda$ and for the higher spin of the CRNK-BH in the RMTG the efficiency of the energy extraction process becomes stronger. The dependence of the efficiency of the Penrose-process on the NUT parameter l is not monotonic for all the values of this parameter and it takes a minimum point for bigger values of the spin parameter a. We have observed that the increase of the parameter of the equation-of-state of the quintessence dark energy, ω, slightly amplifies the efficiency of the Penrose-process in a monotonic way. Further, we have also noticed that the increase of the electromagnetic charge of the CRNK-BH in the RMTG considerably amplifies the efficiency of the Penrose-process having a very similar effect as in the case of the spin parameter.

Finally, we have studied the BZ mechanism for energy extraction from the CRNK-BH in the RMTG. The energy extraction from the CRNK-BH due to the BZ-process strongly depends on the spin parameter a of the BH. The effect of the electromagnetic charge is also considered in the BZ mechanism and amplifies the jet power. However, an increase of the NUT and quintessential intensity parameters reduces the jet power of the CRNK-BH source in the RMTG. We have also found that the energy extraction through the BZ-process from the CRNK-BH in the RMTG depends on the parameters ω and $\kappa\lambda$. This dependence is similar to the case of the Penrose-process.

Overall, these findings contribute to a comprehensive understanding of the CRNK-BH in the RMTG, shedding light on the influence of various parameters on the EH, ergosphere and energy extraction processes. Future research can focus on exploring the implications of these results in different contexts, investigating the interplay between multiple parameters and further examining the effects of other theoretical frameworks on the behaviour of the CRNK-BH in the RMTG.

The further study of the efficiency of energy extraction processes can be based on: The study indicated that the efficiency of the energy extraction processes (e.g., Penrose-process, BZ mechanism) is influenced by various parameters, including the quintessential intensity, Rastall parameter, NUT parameter, spin parameter, equation-of-state parameter and an electromagnetic charge. Future research can focus on quantifying and analysing the efficiency of these energy extraction processes under different parameter regimes and investigating the underlying physical mechanisms responsible for the observed trends. Investigation of the interplay between different parameters can be based on: The efficiency of the energy extraction processes depends on the interplay between different parameters, such as the spin parameter, electromagnetic charge, NUT parameter, quintessential intensity, and Rastall parameter. Future research can explore the intricate relationships between these parameters and their combined effects on the energy extraction mechanisms, providing a more comprehensive understanding of the underlying physics.

Generalization to other BH solutions and gravity theories can be based on: The study focused on the CRNK-BH in the RMTG. Future research can extend the analysis to other BH solutions and alternative gravity theories to investigate how the observed effects and energy extraction processes are influenced by different spacetime geometries and gravitational theories. This broader exploration can help to assess the robustness and generality of the findings presented in the study.

Astrophysical implications and observational signatures are: Further research can explore the astrophysical implications of the results obtained in this study. Investigating the observational consequences of the energy extraction processes and the interplay between different parameters can provide insights into the potential observational signatures associated with these BH systems. This may involve studying the electromagnetic and gravitational wave signatures of the energy extraction processes in the context of specific astrophysical scenarios.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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