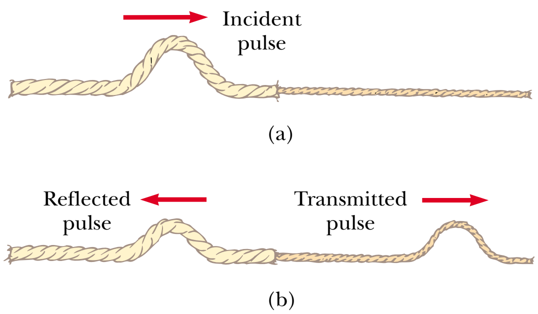

Figure 16.16 (a) A pulse traveling to the right on a heavy string attached to a lighter string. (b) The incident pulse is partially reflected and partially transmitted, and the reflected pulse is not inverted.

When a pulse traveling on a heavy string strikes the boundary between the heavy string and a lighter one, as shown in Figure 16.16, again part is reflected and part is transmitted. In this case, the reflected pulse is not inverted.

In either case, the relative heights of the reflected and transmitted pulses depend on the relative densities of the two strings. If the strings are identical, there is no discontinuity at the boundary and no reflection takes place.

According to Equation 16.4, the speed of a wave on a string increases as the mass per unit length of the string decreases. In other words, a pulse travels more slowly on a heavy string than on a light string if both are under the same tension. The following general rules apply to reflected waves: **When a wave pulse travels from medium A to medium B and $v_A > v_B$ (that is, when B is denser than A), the pulse is inverted upon reflection. When a wave pulse travels from medium A to medium B and $v_A < v_B$ (that is, when A is denser than B), the pulse is not inverted upon reflection.**

16.7 SINUSOIDAL WAVES

In this section, we introduce an important wave function whose shape is shown in Figure 16.17. The wave represented by this curve is called a **sinusoidal wave** because the curve is the same as that of the function sin θ plotted against θ. The sinusoidal wave is the simplest example of a periodic continuous wave and can be used to build more complex waves, as we shall see in Section 18.8. The red curve represents a snapshot of a traveling sinusoidal wave at $t = 0$, and the blue curve represents a snapshot of the wave at some later time t. At $t = 0$, the function describing the positions of the particles of the medium through which the sinusoidal wave is traveling can be written

$$y = A \sin\left(\frac{2\pi}{\lambda} x\right) \tag{16.5}$$

where the constant A represents the wave amplitude and the constant λ is the wavelength. Thus, we see that the position of a particle of the medium is the same whenever x is increased by an integral multiple of λ. If the wave moves to the right with a speed v, then the wave function at some later time t is

$$y = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \tag{16.6}$$

That is, the traveling sinusoidal wave moves to the right a distance vt in the time t, as shown in Figure 16.17. Note that the wave function has the form $f(x - vt)$ and

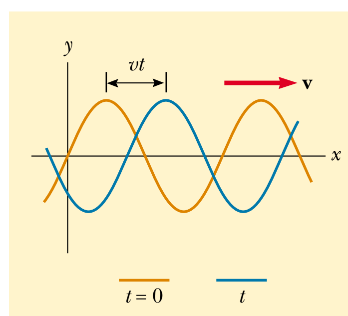

Figure 16.17 A one-dimensional sinusoidal wave traveling to the right with a speed v. The red curve represents a snapshot of the wave at $t = 0$, and the blue curve represents a snapshot at some later time t.

so represents a wave traveling to the right. If the wave were traveling to the left, the quantity $x - vt$ would be replaced by $x + vt$, as we learned when we developed Equations 16.1 and 16.2.

By definition, the wave travels a distance of one wavelength in one period T. Therefore, the wave speed, wavelength, and period are related by the expression

$$v = \frac{\lambda}{T} \tag{16.7}$$

Substituting this expression for v into Equation 16.6, we find that

$$y = A \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \tag{16.8}$$

This form of the wave function clearly shows the *periodic* nature of y. At any given time t (a snapshot of the wave), y has the *same* value at the positions x, $x + \lambda$, $x + 2\lambda$, and so on. Furthermore, at any given position x, the value of y is the same at times t, $t + T$, $t + 2T$, and so on.

We can express the wave function in a convenient form by defining two other quantities, the **angular wave number** k and the **angular frequency** ω:

Angular wave number

$$k \equiv \frac{2\pi}{\lambda} \tag{16.9}$$

Angular frequency

$$\omega \equiv \frac{2\pi}{T} \tag{16.10}$$

Using these definitions, we see that Equation 16.8 can be written in the more compact form

Wave function for a sinusoidal wave

$$y = A \sin(kx - \omega t) \tag{16.11}$$

The frequency of a sinusoidal wave is related to the period by the expression

Frequency

$$f = \frac{1}{T} \tag{16.12}$$

The most common unit for frequency, as we learned in Chapter 13, is second^{-1}, or **hertz** (Hz). The corresponding unit for T is seconds.

Using Equations 16.9, 16.10, and 16.12, we can express the wave speed v originally given in Equation 16.7 in the alternative forms

Speed of a sinusoidal wave

$$v = \frac{\omega}{k} \tag{16.13}$$

$$v = \lambda f \tag{16.14}$$

The wave function given by Equation 16.11 assumes that the vertical displacement y is zero at $x = 0$ and $t = 0$. This need not be the case. If it is not, we generally express the wave function in the form

General expression for a sinusoidal wave

$$y = A \sin(kx - \omega t + \phi) \tag{16.15}$$

where ϕ is the **phase constant,** just as we learned in our study of periodic motion in Chapter 13. This constant can be determined from the initial conditions.

EXAMPLE 16.3 A Traveling Sinusoidal Wave

A sinusoidal wave traveling in the positive x direction has an amplitude of 15.0 cm, a wavelength of 40.0 cm, and a frequency of 8.00 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15.0 cm, as shown in Figure 16.18. (a) Find the angular wave number k, period T, angular frequency ω, and speed v of the wave.

Solution (a) Using Equations 16.9, 16.10, 16.12, and 16.14, we find the following:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{40.0 \text{ cm}} = \boxed{0.157 \text{ rad/cm}}$$

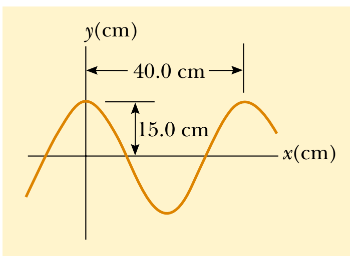

Figure 16.18 A sinusoidal wave of wavelength $\lambda = 40.0$ cm and amplitude $A = 15.0$ cm. The wave function can be written in the form $y = A\cos(kx - \omega t)$.

$$\omega = 2\pi f = 2\pi(8.00 \text{ s}^{-1}) = \boxed{50.3 \text{ rad/s}}$$

$$T = \frac{1}{f} = \frac{1}{8.00 \text{ s}^{-1}} = \boxed{0.125 \text{ s}}$$

$$v = \lambda f = (40.0 \text{ cm})(8.00 \text{ s}^{-1}) = \boxed{320 \text{ cm/s}}$$

(b) Determine the phase constant ϕ, and write a general expression for the wave function.

Solution Because $A = 15.0$ cm and because $y = 15.0$ cm at $x = 0$ and $t = 0$, substitution into Equation 16.15 gives

$$15.0 = (15.0)\sin\phi \quad \text{or} \quad \sin\phi = 1$$

We may take the principal value $\phi = \pi/2$ rad (or 90°). Hence, the wave function is of the form

$$y = A\sin\left(kx - \omega t + \frac{\pi}{2}\right) = A\cos(kx - \omega t)$$

By inspection, we can see that the wave function must have this form, noting that the cosine function has the same shape as the sine function displaced by 90°. Substituting the values for A, k, and ω into this expression, we obtain

$$\boxed{y = (15.0 \text{ cm})\cos(0.157x - 50.3t)}$$

Sinusoidal Waves on Strings

In Figure 16.2, we demonstrated how to create a pulse by jerking a taut string up and down once. To create a train of such pulses, normally referred to as a *wave train,* or just plain *wave,* we can replace the hand with an oscillating blade. If the wave consists of a train of identical cycles, whatever their shape, the relationships $f = 1/T$ and $v = f\lambda$ among speed, frequency, period, and wavelength hold true. We can make more definite statements about the wave function if the source of the waves vibrates in simple harmonic motion. Figure 16.19 represents snapshots of the wave created in this way at intervals of $T/4$. Note that because the end of the blade oscillates in simple harmonic motion, **each particle of the string, such as that at *P*, also oscillates vertically with simple harmonic motion.** This must be the case because each particle follows the simple harmonic motion of the blade. Therefore, every segment of the string can be treated as a simple harmonic oscillator vibrating with a frequency equal to the frequency of oscillation of the blade.[3] Note that although each segment oscillates in the y direction, the wave travels in the x direction with a speed v. Of course, this is the definition of a transverse wave.

[3] In this arrangement, we are assuming that a string segment always oscillates in a vertical line. The tension in the string would vary if a segment were allowed to move sideways. Such motion would make the analysis very complex.

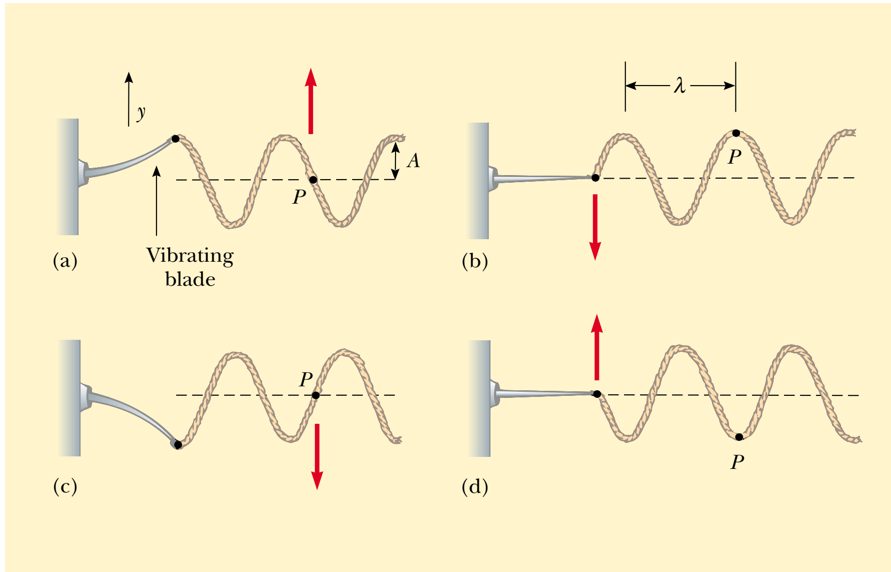

Figure 16.19 One method for producing a train of sinusoidal wave pulses on a string. The left end of the string is connected to a blade that is set into oscillation. Every segment of the string, such as the point *P*, oscillates with simple harmonic motion in the vertical direction.

If the wave at $t = 0$ is as described in Figure 16.19b, then the wave function can be written as

$$y = A\sin(kx - \omega t)$$

We can use this expression to describe the motion of any point on the string. The point *P* (or any other point on the string) moves only vertically, and so its *x* coordinate remains constant. Therefore, the **transverse speed v_y** (not to be confused with the wave speed v) and the **transverse acceleration a_y** are

$$v_y = \left.\frac{dy}{dt}\right]_{x = \text{constant}} = \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t) \qquad \textbf{(16.16)}$$

$$a_y = \left.\frac{dv_y}{dt}\right]_{x = \text{constant}} = \frac{\partial v_y}{\partial t} = -\omega^2 A\sin(kx - \omega t) \qquad \textbf{(16.17)}$$

In these expressions, we must use partial derivatives (see Section 8.6) because *y* depends on both *x* and *t*. In the operation $\partial y/\partial t$, for example, we take a derivative with respect to *t* while holding *x* constant. The maximum values of the transverse speed and transverse acceleration are simply the absolute values of the coefficients of the cosine and sine functions:

$$v_{y,\,\max} = \omega A \qquad \textbf{(16.18)}$$

$$a_{y,\,\max} = \omega^2 A \qquad \textbf{(16.19)}$$

The transverse speed and transverse acceleration do not reach their maximum values simultaneously. The transverse speed reaches its maximum value (ωA) when $y = 0$, whereas the transverse acceleration reaches its maximum value ($\omega^2 A$) when $y = \pm A$. Finally, Equations 16.18 and 16.19 are identical in mathematical form to the corresponding equations for simple harmonic motion, Equations 13.10 and 13.11.

Quick Quiz 16.4

A sinusoidal wave is moving on a string. If you increase the frequency f of the wave, how do the transverse speed, wave speed, and wavelength change?

EXAMPLE 16.4 A Sinusoidally Driven String

The string shown in Figure 16.19 is driven at a frequency of 5.00 Hz. The amplitude of the motion is 12.0 cm, and the wave speed is 20.0 m/s. Determine the angular frequency ω and angular wave number k for this wave, and write an expression for the wave function.

Solution Using Equations 16.10, 16.12, and 16.13, we find that

$$\omega = \frac{2\pi}{T} = 2\pi f = 2\pi(5.00 \text{ Hz}) = \boxed{31.4 \text{ rad/s}}$$

$$k = \frac{\omega}{v} = \frac{31.4 \text{ rad/s}}{20.0 \text{ m/s}} = \boxed{1.57 \text{ rad/m}}$$

Because $A = 12.0$ cm $= 0.120$ m, we have

$$y = A\sin(kx - \omega t) = (0.120 \text{ m})\sin(1.57x - 31.4t)$$

Exercise Calculate the maximum values for the transverse speed and transverse acceleration of any point on the string.

Answer 3.77 m/s; 118 m/s^2.

16.8 RATE OF ENERGY TRANSFER BY SINUSOIDAL WAVES ON STRINGS

As waves propagate through a medium, they transport energy. We can easily demonstrate this by hanging an object on a stretched string and then sending a pulse down the string, as shown in Figure 16.20. When the pulse meets the suspended object, the object is momentarily displaced, as illustrated in Figure 16.20b. In the process, energy is transferred to the object because work must be done for it to move upward. This section examines the rate at which energy is transported along a string. We shall assume a one-dimensional sinusoidal wave in the calculation of the energy transferred.

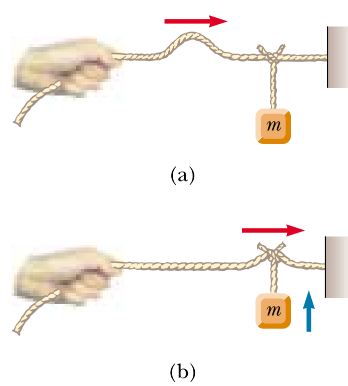

Figure 16.20 (a) A pulse traveling to the right on a stretched string on which an object has been suspended. (b) Energy is transmitted to the suspended object when the pulse arrives.

Consider a sinusoidal wave traveling on a string (Fig. 16.21). The source of the energy being transported by the wave is some external agent at the left end of the string; this agent does work in producing the oscillations. As the external agent performs work on the string, moving it up and down, energy enters the system of the string and propagates along its length. Let us focus our attention on a segment of the string of length Δx and mass Δm. Each such segment moves vertically with simple harmonic motion. Furthermore, all segments have the same angular frequency ω and the same amplitude A. As we found in Chapter 13, the elastic potential energy U associated with a particle in simple harmonic motion is $U = \frac{1}{2}ky^2$, where the simple harmonic motion is in the y direction. Using the relationship $\omega^2 = k/m$ developed in Equations 13.16 and 13.17, we can write this as

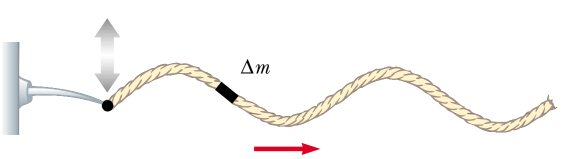

Figure 16.21 A sinusoidal wave traveling along the x axis on a stretched string. Every segment moves vertically, and every segment has the same total energy.

$U = \frac{1}{2}m\omega^2 y^2$. If we apply this equation to the segment of mass Δm, we see that the potential energy of this segment is

$$\Delta U = \tfrac{1}{2}(\Delta m)\,\omega^2 y^2$$

Because the mass per unit length of the string is $\mu = \Delta m/\Delta x$, we can express the potential energy of the segment as

$$\Delta U = \tfrac{1}{2}(\mu \Delta x)\,\omega^2 y^2$$

As the length of the segment shrinks to zero, $\Delta x \rightarrow dx$, and this expression becomes a differential relationship:

$$dU = \tfrac{1}{2}(\mu\, dx)\,\omega^2 y^2$$

We replace the general displacement y of the segment with the wave function for a sinusoidal wave:

$$dU = \tfrac{1}{2}\mu\omega^2[A\sin(kx - \omega t)]^2\, dx = \tfrac{1}{2}\mu\omega^2 A^2 \sin^2(kx - \omega t)\; dx$$

If we take a snapshot of the wave at time $t = 0$, then the potential energy in a given segment is

$$dU = \tfrac{1}{2}\mu\omega^2 A^2 \sin^2 kx\, dx$$

To obtain the total potential energy in one wavelength, we integrate this expression over all the string segments in one wavelength:

$$U_\lambda = \int dU = \int_0^\lambda \tfrac{1}{2}\mu\omega^2 A^2 \sin^2 kx\, dx = \tfrac{1}{2}\mu\omega^2 A^2 \int_0^\lambda \sin^2 kx\, dx$$

$$= \tfrac{1}{2}\mu\omega^2 A^2\left[\tfrac{1}{2}x - \tfrac{1}{4k}\sin 2\,kx\right]_0^\lambda = \tfrac{1}{2}\mu\omega^2 A^2(\tfrac{1}{2}\lambda) = \tfrac{1}{4}\mu\omega^2 A^2 \lambda$$

Because it is in motion, each segment of the string also has kinetic energy. When we use this procedure to analyze the total kinetic energy in one wavelength of the string, we obtain the same result:

$$K_\lambda = \int dK = \tfrac{1}{4}\mu\omega^2 A^2 \lambda$$

The total energy in one wavelength of the wave is the sum of the potential and kinetic energies:

$$E_\lambda = U_\lambda + K_\lambda = \tfrac{1}{2}\mu\omega^2 A^2 \lambda \quad \textbf{(16.20)}$$

As the wave moves along the string, this amount of energy passes by a given point on the string during one period of the oscillation. Thus, the power, or rate of energy transfer, associated with the wave is

Power of a wave

$$\mathcal{P} = \frac{E_\lambda}{\Delta t} = \frac{\frac{1}{2}\mu\omega^2 A^2 \lambda}{T} = \tfrac{1}{2}\mu\omega^2 A^2\left(\frac{\lambda}{T}\right)$$

$$\mathcal{P} = \tfrac{1}{2}\mu\omega^2\, A^2 v \quad \textbf{(16.21)}$$

This shows that the rate of energy transfer by a sinusoidal wave on a string is proportional to (a) the wave speed, (b) the square of the frequency, and (c) the square of the amplitude. In fact: **the rate of energy transfer in any sinusoidal wave is proportional to the square of the angular frequency and to the square of the amplitude.**

EXAMPLE 16.5 Power Supplied to a Vibrating String

A taut string for which $\mu = 5.00 \times 10^{-2}$ kg/m is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?

Solution The wave speed on the string is, from Equation 16.4,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{80.0 \text{ N}}{5.00 \times 10^{-2} \text{ kg/m}}} = 40.0 \text{ m/s}$$

Because $f = 60.0$ Hz, the angular frequency ω of the sinusoidal waves on the string has the value

$$\omega = 2\pi f = 2\pi(60.0 \text{ Hz}) = 377 \text{ s}^{-1}$$

Using these values in Equation 16.21 for the power, with $A = 6.00 \times 10^{-2}$ m, we obtain

$$\begin{aligned} \mathcal{P} &= \tfrac{1}{2}\mu\omega^2 A^2 v \\ &= \tfrac{1}{2}(5.00 \times 10^{-2} \text{ kg/m})(377 \text{ s}^{-1})^2 \\ &= \times (6.00 \times 10^{-2} \text{ m})^2(40.0 \text{ m/s}) \\ &= \boxed{512 \text{ W}} \end{aligned}$$

Optional Section

16.9 THE LINEAR WAVE EQUATION

In Section 16.3 we introduced the concept of the wave function to represent waves traveling on a string. All wave functions $y(x, t)$ represent solutions of an equation called the *linear wave equation*. This equation gives a complete description of the wave motion, and from it one can derive an expression for the wave speed. Furthermore, the linear wave equation is basic to many forms of wave motion. In this section, we derive this equation as applied to waves on strings.

Suppose a traveling wave is propagating along a string that is under a tension T. Let us consider one small string segment of length Δx (Fig. 16.22). The ends of the segment make small angles θ_A and θ_B with the x axis. The net force acting on the segment in the vertical direction is

$$\sum F_y = T \sin \theta_B - T \sin \theta_A = T(\sin \theta_B - \sin \theta_A)$$

Because the angles are small, we can use the small-angle approximation $\sin \theta \approx \tan \theta$ to express the net force as

$$\sum F_y \approx T(\tan \theta_B - \tan \theta_A)$$

However, the tangents of the angles at A and B are defined as the slopes of the string segment at these points. Because the slope of a curve is given by $\partial y/\partial x$, we have

$$\sum F_y \approx T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right] \quad \textbf{(16.22)}$$

We now apply Newton's second law to the segment, with the mass of the segment given by $m = \mu\Delta x$:

$$\sum F_y = ma_y = \mu\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right) \quad \textbf{(16.23)}$$

Combining Equation 16.22 with Equation 16.23, we obtain

$$\mu\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right) = T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{(\partial y/\partial x)_B - (\partial y/\partial x)_A}{\Delta x} \quad \textbf{(16.24)}$$

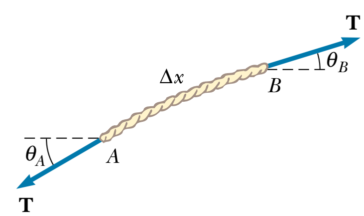

Figure 16.22 A segment of a string under tension T. The slopes at points A and B are given by $\tan \theta_A$ and $\tan \theta_B$, respectively.

The right side of this equation can be expressed in a different form if we note that the partial derivative of any function is defined as

$$\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If we associate $f(x + \Delta x)$ with $(\partial y/\partial x)_B$ and $f(x)$ with $(\partial y/\partial x)_A$, we see that, in the limit $\Delta x \to 0$, Equation 16.24 becomes

Linear wave equation

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \qquad \textbf{(16.25)}$$

This is the linear wave equation as it applies to waves on a string.

We now show that the sinusoidal wave function (Eq. 16.11) represents a solution of the linear wave equation. If we take the sinusoidal wave function to be of the form $y(x, t) = A \sin(kx - \omega t)$, then the appropriate derivatives are

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(kx - \omega t)$$

Substituting these expressions into Equation 16.25, we obtain

$$-\frac{\mu\omega^2}{T}\sin(kx - \omega t) = -k^2 \sin(kx - \omega t)$$

This equation must be true for all values of the variables x and t in order for the sinusoidal wave function to be a solution of the wave equation. Both sides of the equation depend on x and t through the same function $\sin(kx - \omega t)$. Because this function divides out, we do indeed have an identity, provided that

$$k^2 = \frac{\mu\omega^2}{T}$$

Using the relationship $v = \omega/k$ (Eq. 16.13) in this expression, we see that

$$v^2 = \frac{\omega^2}{k^2} = \frac{T}{\mu}$$

$$v = \sqrt{\frac{T}{\mu}}$$

which is Equation 16.4. This derivation represents another proof of the expression for the wave speed on a taut string.

The linear wave equation (Eq. 16.25) is often written in the form

Linear wave equation in general

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad \textbf{(16.26)}$$

This expression applies in general to various types of traveling waves. For waves on strings, y represents the vertical displacement of the string. For sound waves, y corresponds to displacement of air molecules from equilibrium or variations in either the pressure or the density of the gas through which the sound waves are propagating. In the case of electromagnetic waves, y corresponds to electric or magnetic field components.

We have shown that the sinusoidal wave function (Eq. 16.11) is one solution of the linear wave equation (Eq. 16.26). Although we do not prove it here, the linear wave equation is satisfied by *any* wave function having the form $y = f(x \pm vt)$. Furthermore, we have seen that the linear wave equation is a direct consequence of Newton's second law applied to any segment of the string.

SUMMARY

A **transverse wave** is one in which the particles of the medium move in a direction *perpendicular* to the direction of the wave velocity. An example is a wave on a taut string. A **longitudinal wave** is one in which the particles of the medium move in a direction *parallel* to the direction of the wave velocity. Sound waves in fluids are longitudinal. You should be able to identify examples of both types of waves.

Any one-dimensional wave traveling with a speed v in the x direction can be represented by a wave function of the form

$$y = f(x \pm vt) \qquad \textbf{(16.1, 16.2)}$$

where the positive sign applies to a wave traveling in the negative x direction and the negative sign applies to a wave traveling in the positive x direction. The shape of the wave at any instant in time (a snapshot of the wave) is obtained by holding t constant.

The **superposition principle** specifies that when two or more waves move through a medium, the resultant wave function equals the algebraic sum of the individual wave functions. When two waves combine in space, they interfere to produce a resultant wave. The **interference** may be **constructive** (when the individual displacements are in the same direction) or **destructive** (when the displacements are in opposite directions).

The **speed** of a wave traveling on a taut string of mass per unit length μ and tension T is

$$v = \sqrt{\frac{T}{\mu}} \qquad \textbf{(16.4)}$$

A wave is totally or partially reflected when it reaches the end of the medium in which it propagates or when it reaches a boundary where its speed changes discontinuously. If a wave pulse traveling on a string meets a fixed end, the pulse is reflected and inverted. If the pulse reaches a free end, it is reflected but not inverted.

The **wave function** for a one-dimensional sinusoidal wave traveling to the right can be expressed as

$$y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right] = A\sin(kx - \omega t) \qquad \textbf{(16.6, 16.11)}$$

where A is the **amplitude,** λ is the **wavelength,** k is the **angular wave number,** and ω is the **angular frequency.** If T is the **period** and f the **frequency,** v, k and ω can be written

$$v = \frac{\lambda}{T} = \lambda f \qquad \textbf{(16.7, 16.14)}$$

$$k \equiv \frac{2\pi}{\lambda} \qquad \textbf{(16.9)}$$

$$\omega \equiv \frac{2\pi}{T} = 2\pi f \qquad \textbf{(16.10, 16.12)}$$

You should know how to find the equation describing the motion of particles in a wave from a given set of physical parameters.

The **power** transmitted by a sinusoidal wave on a stretched string is

$$\mathcal{P} = \tfrac{1}{2}\mu\omega^2 A^2 v \qquad \textbf{(16.21)}$$