Consider what happens when a golf ball is struck by a club. The ball is given a very large initial velocity as a result of the collision; consequently, it is able to travel more than 100 m through the air. The ball experiences a large acceleration. Furthermore, because the ball experiences this acceleration over a very short time interval, the average force exerted on it during the collision is very great. According to Newton's third law, the ball exerts on the club a reaction force that is equal in magnitude to and opposite in direction to the force exerted by the club on the ball. This reaction force causes the club to accelerate. Because the club is much more massive than the ball, however, the acceleration of the club is much less than the acceleration of the ball.

One of the main objectives of this chapter is to enable you to understand and analyze such events. As a first step, we introduce the concept of *momentum,* which is useful for describing objects in motion and as an alternate and more general means of applying Newton's laws. For example, a very massive football player is often said to have a great deal of momentum as he runs down the field. A much less massive player, such as a halfback, can have equal or greater momentum if his speed is greater than that of the more massive player. This follows from the fact that momentum is defined as the product of mass and velocity. The concept of momentum leads us to a second conservation law, that of conservation of momentum. This law is especially useful for treating problems that involve collisions between objects and for analyzing rocket propulsion. The concept of the center of mass of a system of particles also is introduced, and we shall see that the motion of a system of particles can be described by the motion of one representative particle located at the center of mass.

9.1 LINEAR MOMENTUM AND ITS CONSERVATION

In the preceding two chapters we studied situations too complex to analyze easily with Newton's laws. In fact, Newton himself used a form of his second law slightly different from $\Sigma \mathbf{F} = m\mathbf{a}$ (Eq. 5.2)—a form that is considerably easier to apply in complicated circumstances. Physicists use this form to study everything from subatomic particles to rocket propulsion. In studying situations such as these, it is often useful to know both something about the object and something about its motion. We start by defining a new term that incorporates this information:

Definition of linear momentum of a particle

The **linear momentum** of a particle of mass m moving with a velocity $\mathbf{v}$ is defined to be the product of the mass and velocity:

$$\mathbf{p} \equiv m\mathbf{v} \tag{9.1}$$

6.2

Linear momentum is a vector quantity because it equals the product of a scalar quantity m and a vector quantity $\mathbf{v}$. Its direction is along $\mathbf{v}$, it has dimensions ML/T, and its SI unit is kg·m/s.

If a particle is moving in an arbitrary direction, $\mathbf{p}$ must have three components, and Equation 9.1 is equivalent to the component equations

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \tag{9.2}$$

As you can see from its definition, the concept of momentum provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of a bowling ball moving at 10 m/s is much greater than that of a tennis ball moving at the same speed. Newton called the product $m\mathbf{v}$

quantity of motion; this is perhaps a more graphic description than our present-day word *momentum,* which comes from the Latin word for movement.

Quick Quiz 9.1

Two objects have equal kinetic energies. How do the magnitudes of their momenta compare? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) not enough information to tell.

Newton's second law for a particle

Using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle: **The time rate of change of the linear momentum of a particle is equal to the net force acting on the particle:**

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} \tag{9.3}$$

In addition to situations in which the velocity vector varies with time, we can use Equation 9.3 to study phenomena in which the mass changes. The real value of Equation 9.3 as a tool for analysis, however, stems from the fact that when the net force acting on a particle is zero, the time derivative of the momentum of the particle is zero, and therefore its linear momentum[1] is constant. Of course, if the particle is *isolated,* then by necessity $\Sigma \mathbf{F} = 0$ and $\mathbf{p}$ remains unchanged. This means that $\mathbf{p}$ is conserved. Just as the law of conservation of energy is useful in solving complex motion problems, the law of conservation of momentum can greatly simplify the analysis of other types of complicated motion.

Conservation of Momentum for a Two-Particle System

6.2 Consider two particles 1 and 2 that can interact with each other but are isolated from their surroundings (Fig. 9.1). That is, the particles may exert a force on each other, but no external forces are present. It is important to note the impact of Newton's third law on this analysis. If an internal force from particle 1 (for example, a gravitational force) acts on particle 2, then there must be a second internal force—equal in magnitude but opposite in direction—that particle 2 exerts on particle 1.

Suppose that at some instant, the momentum of particle 1 is $\mathbf{p}_1$ and that of particle 2 is $\mathbf{p}_2$. Applying Newton's second law to each particle, we can write

$$\mathbf{F}_{21} = \frac{d\mathbf{p}_1}{dt} \qquad \text{and} \qquad \mathbf{F}_{12} = \frac{d\mathbf{p}_2}{dt}$$

where $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1 and $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2. Newton's third law tells us that $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are equal in magnitude and opposite in direction. That is, they form an action–reaction pair $\mathbf{F}_{12} = -\mathbf{F}_{21}$. We can express this condition as

$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

or as

$$\frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

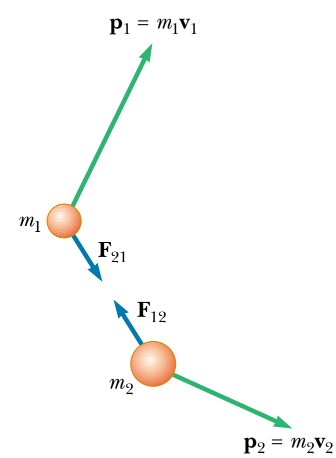


Figure 9.1 At some instant, the momentum of particle 1 is $\mathbf{p}_1 = m_1\mathbf{v}_1$ and the momentum of particle 2 is $\mathbf{p}_2 = m_2\mathbf{v}_2$. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. The total momentum of the system $\mathbf{p}_{\text{tot}}$ is equal to the vector sum $\mathbf{p}_1 + \mathbf{p}_2$.

[1]In this chapter, the terms *momentum* and *linear momentum* have the same meaning. Later, in Chapter 11, we shall use the term *angular momentum* when dealing with rotational motion.

Because the time derivative of the total momentum $\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2$ is *zero,* we conclude that the *total* momentum of the system must remain constant:

$$\mathbf{p}_{\text{tot}} = \sum_{\text{system}} \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \tag{9.4}$$

or, equivalently,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{9.5}$$

where $\mathbf{p}_{1i}$ and $\mathbf{p}_{2i}$ are the initial values and $\mathbf{p}_{1f}$ and $\mathbf{p}_{2f}$ the final values of the momentum during the time interval dt over which the reaction pair interacts. Equation 9.5 in component form demonstrates that the total momenta in the x, y, and z directions are all independently conserved:

$$\sum_{\text{system}} p_{ix} = \sum_{\text{system}} p_{fx} \qquad \sum_{\text{system}} p_{iy} = \sum_{\text{system}} p_{fy} \qquad \sum_{\text{system}} p_{iz} = \sum_{\text{system}} p_{fz} \tag{9.6}$$

This result, known as the **law of conservation of linear momentum,** can be extended to any number of particles in an isolated system. It is considered one of the most important laws of mechanics. We can state it as follows:

Conservation of momentum

Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

This law tells us that **the total momentum of an isolated system at all times equals its initial momentum.**

Notice that we have made no statement concerning the nature of the forces acting on the particles of the system. The only requirement is that the forces must be *internal* to the system.

Quick Quiz 9.2

Your physical education teacher throws a baseball to you at a certain speed, and you catch it. The teacher is next going to throw you a medicine ball whose mass is ten times the mass of the baseball. You are given the following choices: You can have the medicine ball thrown with (a) the same speed as the baseball, (b) the same momentum, or (c) the same kinetic energy. Rank these choices from easiest to hardest to catch.

EXAMPLE 9.1 The Floating Astronaut

A SkyLab astronaut discovered that while concentrating on writing some notes, he had gradually floated to the middle of an open area in the spacecraft. Not wanting to wait until he floated to the opposite side, he asked his colleagues for a push. Laughing at his predicament, they decided not to help, and so he had to take off his uniform and throw it in one direction so that he would be propelled in the opposite direction. Estimate his resulting velocity.

Figure 9.2 A hapless astronaut has discarded his uniform to get somewhere.

Solution We begin by making some reasonable guesses of relevant data. Let us assume we have a 70-kg astronaut who threw his 1-kg uniform at a speed of 20 m/s. For convenience, we set the positive direction of the x axis to be the direction of the throw (Fig. 9.2). Let us also assume that the x axis is tangent to the circular path of the spacecraft.

We take the system to consist of the astronaut and the uniform. Because of the gravitational force (which keeps the astronaut, his uniform, and the entire spacecraft in orbit), the system is not really isolated. However, this force is directed perpendicular to the motion of the system. Therefore, momentum is constant in the x direction because there are no external forces in this direction.

The total momentum of the system before the throw is zero ($m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = 0$). Therefore, the total momentum after the throw must be zero; that is,

$$m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = 0$$

With $m_1 = 70$ kg, $\mathbf{v}_{2f} = 20\mathbf{i}$ m/s, and $m_2 = 1$ kg, solving for $\mathbf{v}_{1f}$, we find the recoil velocity of the astronaut to be

$$\mathbf{v}_{1f} = -\frac{m_2}{m_1}\mathbf{v}_{2f} = -\left(\frac{1 \text{ kg}}{70 \text{ kg}}\right)(20\mathbf{i} \text{ m/s}) = \boxed{-0.3\mathbf{i} \text{ m/s}}$$

The negative sign for $\mathbf{v}_{1f}$ indicates that the astronaut is moving to the left after the throw, in the direction opposite the direction of motion of the uniform, in accordance with Newton's third law. Because the astronaut is much more massive than his uniform, his acceleration and consequent velocity are much smaller than the acceleration and velocity of the uniform.

EXAMPLE 9.2 Breakup of a Kaon at Rest

One type of nuclear particle, called the *neutral kaon* (K^0), breaks up into a pair of other particles called *pions* (π^+ and π^-) that are oppositely charged but equal in mass, as illustrated in Figure 9.3. Assuming the kaon is initially at rest, prove that the two pions must have momenta that are equal in magnitude and opposite in direction.

Solution The breakup of the kaon can be written

$$K^0 \longrightarrow \pi^+ + \pi^-$$

If we let $\mathbf{p}^+$ be the momentum of the positive pion and $\mathbf{p}^-$ the momentum of the negative pion, the final momentum of the system consisting of the two pions can be written

$$\mathbf{p}_f = \mathbf{p}^+ + \mathbf{p}^-$$

Because the kaon is at rest before the breakup, we know that $\mathbf{p}_i = 0$. Because momentum is conserved, $\mathbf{p}_i = \mathbf{p}_f = 0$, so that $\mathbf{p}^+ + \mathbf{p}^- = 0$, or

$$\mathbf{p}^+ = -\mathbf{p}^-$$

The important point behind this problem is that even though it deals with objects that are very different from those in the preceding example, the physics is identical: Linear momentum is conserved in an isolated system.

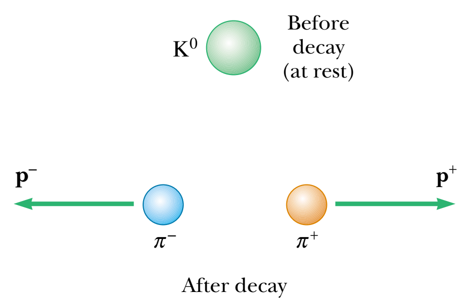


Figure 9.3 A kaon at rest breaks up spontaneously into a pair of oppositely charged pions. The pions move apart with momenta that are equal in magnitude but opposite in direction.

9.2 IMPULSE AND MOMENTUM

6.3 & 6.4 As we have seen, the momentum of a particle changes if a net force acts on the particle. Knowing the change in momentum caused by a force is useful in solving some types of problems. To begin building a better understanding of this important concept, let us assume that a single force $\mathbf{F}$ acts on a particle and that this force may vary with time. According to Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$, or

$$d\mathbf{p} = \mathbf{F}\, dt \qquad \textbf{(9.7)}$$

We can integrate[2] this expression to find the change in the momentum of a particle when the force acts over some time interval. If the momentum of the particle

[2]Note that here we are integrating force with respect to time. Compare this with our efforts in Chapter 7, where we integrated force with respect to position to express the work done by the force.

changes from $\mathbf{p}_i$ at time t_i to $\mathbf{p}_f$ at time t_f, integrating Equation 9.7 gives

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}\, dt \tag{9.8}$$

To evaluate the integral, we need to know how the force varies with time. The quantity on the right side of this equation is called the **impulse** of the force $\mathbf{F}$ acting on a particle over the time interval $\Delta t = t_f - t_i$. Impulse is a vector defined by

Impulse of a force

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt = \Delta\mathbf{p} \tag{9.9}$$

Impulse–momentum theorem

The impulse of the force $\mathbf{F}$ acting on a particle equals the change in the momentum of the particle caused by that force.

This statement, known as the **impulse–momentum theorem,**[3] is equivalent to Newton's second law. From this definition, we see that impulse is a vector quantity having a magnitude equal to the area under the force–time curve, as described in Figure 9.4a. In this figure, it is assumed that the force varies in time in the general manner shown and is nonzero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is the same as the direction of the change in momentum. Impulse has the dimensions of momentum—that is, ML/T. Note that impulse is *not* a property of a particle; rather, it is a measure of the degree to which an external force changes the momentum of the particle. Therefore, when we say that an impulse is given to a particle, we mean that momentum is transferred from an external agent to that particle.

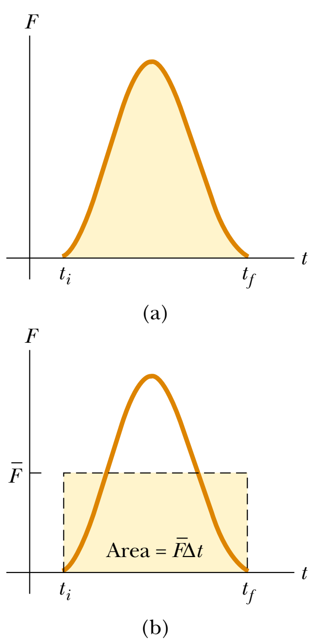


Figure 9.4 (a) A force acting on a particle may vary in time. The impulse imparted to the particle by the force is the area under the force versus time curve. (b) In the time interval Δt, the time-averaged force (horizontal dashed line) gives the same impulse to a particle as does the time-varying force described in part (a).

Because the force imparting an impulse can generally vary in time, it is convenient to define a time-averaged force

$$\overline{\mathbf{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f} \mathbf{F}\, dt \tag{9.10}$$

where $\Delta t = t_f - t_i$. (This is an application of the mean value theorem of calculus.) Therefore, we can express Equation 9.9 as

$$\mathbf{I} \equiv \overline{\mathbf{F}}\, \Delta t \tag{9.11}$$

This time-averaged force, described in Figure 9.4b, can be thought of as the constant force that would give to the particle in the time interval Δt the same impulse that the time-varying force gives over this same interval.

In principle, if $\mathbf{F}$ is known as a function of time, the impulse can be calculated from Equation 9.9. The calculation becomes especially simple if the force acting on the particle is constant. In this case, $\overline{\mathbf{F}} = \mathbf{F}$ and Equation 9.11 becomes

$$\mathbf{I} = \mathbf{F}\, \Delta t \tag{9.12}$$

In many physical situations, we shall use what is called the **impulse approximation, in which we assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present.** This approximation is especially useful in treating collisions in which the duration of the

[3]Although we assumed that only a single force acts on the particle, the impulse–momentum theorem is valid when several forces act; in this case, we replace $\mathbf{F}$ in Equation 9.9 with $\Sigma\mathbf{F}$.

During the brief time the club is in contact with the ball, the ball gains momentum as a result of the collision, and the club loses the same amount of momentum.

collision is very short. When this approximation is made, we refer to the force as an *impulsive force.* For example, when a baseball is struck with a bat, the time of the collision is about 0.01 s and the average force that the bat exerts on the ball in this time is typically several thousand newtons. Because this is much greater than the magnitude of the gravitational force, the impulse approximation justifies our ignoring the weight of the ball and bat. When we use this approximation, it is important to remember that $\mathbf{p}_i$ and $\mathbf{p}_f$ represent the momenta *immediately* before and after the collision, respectively. Therefore, in any situation in which it is proper to use the impulse approximation, the particle moves very little during the collision.

QuickLab

If you can find someone willing, play catch with an egg. What is the best way to move your hands so that the egg does not break when you change its momentum to zero?

Quick Quiz 9.3

Two objects are at rest on a frictionless surface. Object 1 has a greater mass than object 2. When a force is applied to object 1, it accelerates through a distance d. The force is removed from object 1 and is applied to object 2. At the moment when object 2 has accelerated through the same distance d, which statements are true? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) $K_1 < K_2$, (e) $K_1 = K_2$, (f) $K_1 > K_2$.

EXAMPLE 9.3 Teeing Off

A golf ball of mass 50 g is struck with a club (Fig. 9.5). The force exerted on the ball by the club varies from zero, at the instant before contact, up to some maximum value (at which the ball is deformed) and then back to zero when the ball leaves the club. Thus, the force–time curve is qualitatively described by Figure 9.4. Assuming that the ball travels 200 m, estimate the magnitude of the impulse caused by the collision.

Solution Let us use Ⓐ to denote the moment when the club first contacts the ball, Ⓑ to denote the moment when the club loses contact with the ball as the ball starts on its trajectory, and Ⓒ to denote its landing. Neglecting air resistance, we can use Equation 4.14 for the range of a projectile:

$$R = x_{\text{C}} = \frac{{v_{\text{B}}}^2}{g} \sin 2\theta_{\text{B}}$$

Let us assume that the launch angle θ_{B} is 45°, the angle that provides the maximum range for any given launch velocity. This assumption gives $\sin 2\theta_{\text{B}} = 1$, and the launch velocity of the ball is

$$v_B = \sqrt{x_C g} = \sqrt{(200\text{ m})(9.80\text{ m/s}^2)} = 44\text{ m/s}$$

Considering the time interval for the collision, $v_i = v_A = 0$ and $v_f = v_B$ for the ball. Hence, the magnitude of the impulse imparted to the ball is

$$I = \Delta p = mv_B - mv_A = (50 \times 10^{-3}\text{ kg})(44\text{ m/s}) - 0$$
$$= \boxed{2.2\text{ kg}\cdot\text{m/s}}$$

Exercise If the club is in contact with the ball for a time of 4.5×10^{-4} s, estimate the magnitude of the average force exerted by the club on the ball.

Answer 4.9×10^3 N, a value that is extremely large when compared with the weight of the ball, 0.49 N.

Figure 9.5 A golf ball being struck by a club. *(© Harold E. Edgerton/ Courtesy of Palm Press, Inc.)*

EXAMPLE 9.4 How Good Are the Bumpers?

In a particular crash test, an automobile of mass 1 500 kg collides with a wall, as shown in Figure 9.6. The initial and final velocities of the automobile are $\mathbf{v}_i = -15.0\mathbf{i}$ m/s and $\mathbf{v}_f = 2.60\mathbf{i}$ m/s, respectively. If the collision lasts for 0.150 s, find the impulse caused by the collision and the average force exerted on the automobile.

Solution Let us assume that the force exerted on the car by the wall is large compared with other forces on the car so that we can apply the impulse approximation. Furthermore, we note that the force of gravity and the normal force exerted by the road on the car are perpendicular to the motion and therefore do not affect the horizontal momentum.

The initial and final momenta of the automobile are

$$\mathbf{p}_i = m\mathbf{v}_i = (1\,500\text{ kg})(-15.0\mathbf{i}\text{ m/s}) = -2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$
$$\mathbf{p}_f = m\mathbf{v}_f = (1\,500\text{ kg})(2.60\mathbf{i}\text{ m/s}) = 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$

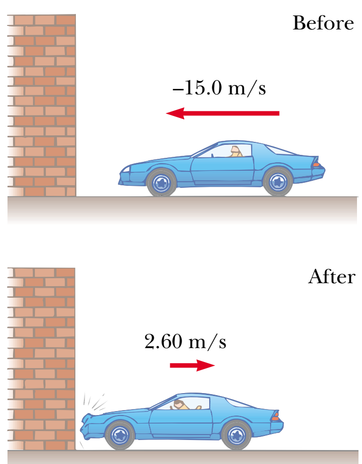
Hence, the impulse is

$$\mathbf{I} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s})$$

$$\mathbf{I} = \boxed{2.64 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}}$$

The average force exerted on the automobile is

$$\overline{\mathbf{F}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{2.64 \times 10^4\,\mathbf{i}\text{ kg}\cdot\text{m/s}}{0.150\text{ s}} = \boxed{1.76 \times 10^5\mathbf{i}\text{ N}}$$



(a)

(b)

Figure 9.6 (a) This car's momentum changes as a result of its collision with the wall. (b) In a crash test, much of the car's initial kinetic energy is transformed into energy used to damage the car.

Note that the magnitude of this force is large compared with the weight of the car ($mg = 1.47 \times 10^4$ N), which justifies our initial assumption. Of note in this problem is how the signs of the velocities indicated the reversal of directions. What would the mathematics be describing if both the initial and final velocities had the same sign?

Quick Quiz 9.4

Rank an automobile dashboard, seatbelt, and airbag in terms of (a) the impulse and (b) the average force they deliver to a front-seat passenger during a collision.

9.3 COLLISIONS

6.5 & 6.6 In this section we use the law of conservation of linear momentum to describe what happens when two particles collide. We use the term **collision** to represent the event of two particles' coming together for a short time and thereby producing impulsive forces on each other. **These forces are assumed to be much greater than any external forces present.**

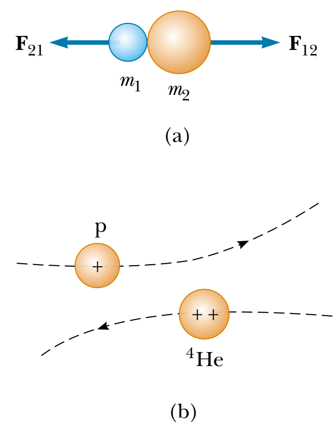


Figure 9.7 (a) The collision between two objects as the result of direct contact. (b) The "collision" between two charged particles.

A collision may entail physical contact between two macroscopic objects, as described in Figure 9.7a, but the notion of what we mean by collision must be generalized because "physical contact" on a submicroscopic scale is ill-defined and hence meaningless. To understand this, consider a collision on an atomic scale (Fig. 9.7b), such as the collision of a proton with an alpha particle (the nucleus of a helium atom). Because the particles are both positively charged, they never come into physical contact with each other; instead, they repel each other because of the strong electrostatic force between them at close separations. When two particles 1 and 2 of masses m_1 and m_2 collide as shown in Figure 9.7, the impulsive forces may vary in time in complicated ways, one of which is described in Figure 9.8. If $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1, and if we assume that no external forces act on the particles, then the change in momentum of particle 1 due to the collision is given by Equation 9.8:

$$\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_{21}\, dt$$

Likewise, if $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2, then the change in momentum of particle 2 is

$$\Delta \mathbf{p}_2 = \int_{t_i}^{t_f} \mathbf{F}_{12}\, dt$$

From Newton's third law, we conclude that

$$\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2$$

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0$$

Because the total momentum of the system is $\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2$, we conclude that the *change* in the momentum of the system due to the collision is zero:

$$\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

This is precisely what we expect because no external forces are acting on the system (see Section 9.2). Because the impulsive forces are internal, they do not change the total momentum of the system (only external forces can do that).

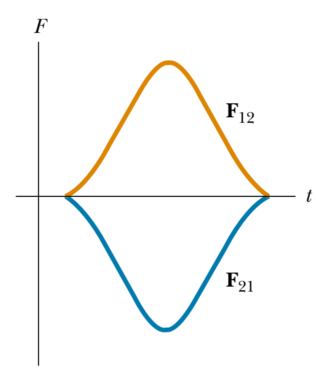


Figure 9.8 The impulse force as a function of time for the two colliding particles described in Figure 9.7a. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.