In this chapter we deal with the kinematics of a particle moving in two dimensions. Knowing the basics of two-dimensional motion will allow us to examine—in future chapters—a wide variety of motions, ranging from the motion of satellites in orbit to the motion of electrons in a uniform electric field. We begin by studying in greater detail the vector nature of displacement, velocity, and acceleration. As in the case of one-dimensional motion, we derive the kinematic equations for two-dimensional motion from the fundamental definitions of these three quantities. We then treat projectile motion and uniform circular motion as special cases of motion in two dimensions. We also discuss the concept of relative motion, which shows why observers in different frames of reference may measure different displacements, velocities, and accelerations for a given particle.

4.1 THE DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

In Chapter 2 we found that the motion of a particle moving along a straight line is completely known if its position is known as a function of time. Now let us extend this idea to motion in the *xy* plane. We begin by describing the position of a particle by its position vector $\mathbf{r}$, drawn from the origin of some coordinate system to the particle located in the *xy* plane, as in Figure 4.1. At time t_i the particle is at point Ⓐ, and at some later time t_f it is at point Ⓑ. The path from Ⓐ to Ⓑ is not necessarily a straight line. As the particle moves from Ⓐ to Ⓑ in the time interval $\Delta t = t_f - t_i$, its position vector changes from $\mathbf{r}_i$ to $\mathbf{r}_f$. As we learned in Chapter 2, displacement is a vector, and the displacement of the particle is the difference between its final position and its initial position. We now formally define the **displacement vector Δr** for the particle of Figure 4.1 as being the difference between its final position vector and its initial position vector:

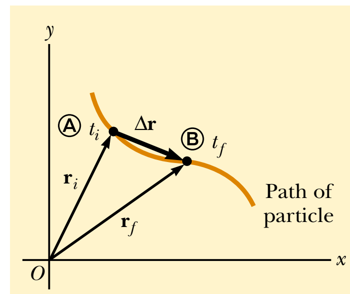

Figure 4.1 A particle moving in the *xy* plane is located with the position vector $\mathbf{r}$ drawn from the origin to the particle. The displacement of the particle as it moves from Ⓐ to Ⓑ in the time interval $\Delta t = t_f - t_i$ is equal to the vector $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$.

Displacement vector

$$\Delta\mathbf{r} \equiv \mathbf{r}_f - \mathbf{r}_i \tag{4.1}$$

The direction of $\Delta\mathbf{r}$ is indicated in Figure 4.1. As we see from the figure, the magnitude of $\Delta\mathbf{r}$ is *less* than the distance traveled along the curved path followed by the particle.

As we saw in Chapter 2, it is often useful to quantify motion by looking at the ratio of a displacement divided by the time interval during which that displacement occurred. In two-dimensional (or three-dimensional) kinematics, everything is the same as in one-dimensional kinematics except that we must now use vectors rather than plus and minus signs to indicate the direction of motion.

We define the **average velocity** of a particle during the time interval Δt as the displacement of the particle divided by that time interval:

Average velocity

$$\bar{\mathbf{v}} \equiv \frac{\Delta\mathbf{r}}{\Delta t} \tag{4.2}$$

Multiplying or dividing a vector quantity by a scalar quantity changes only the magnitude of the vector, not its direction. Because displacement is a vector quantity and the time interval is a scalar quantity, we conclude that the average velocity is a vector quantity directed along $\Delta\mathbf{r}$.

Note that the average velocity between points is *independent of the path* taken. This is because average velocity is proportional to displacement, which depends

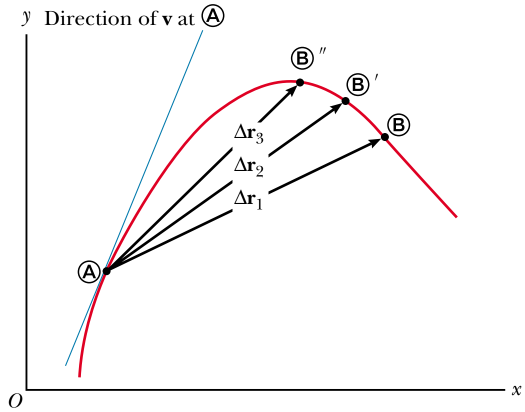

Figure 4.2 As a particle moves between two points, its average velocity is in the direction of the displacement vector $\Delta\mathbf{r}$. As the end point of the path is moved from Ⓑ to Ⓑ′ to Ⓑ″, the respective displacements and corresponding time intervals become smaller and smaller. In the limit that the end point approaches Ⓐ, Δt approaches zero, and the direction of $\Delta\mathbf{r}$ approaches that of the line tangent to the curve at Ⓐ. By definition, the instantaneous velocity at Ⓐ is in the direction of this tangent line.

only on the initial and final position vectors and not on the path taken. As we did with one-dimensional motion, we conclude that if a particle starts its motion at some point and returns to this point via any path, its average velocity is zero for this trip because its displacement is zero.

Consider again the motion of a particle between two points in the *xy* plane, as shown in Figure 4.2. As the time interval over which we observe the motion becomes smaller and smaller, the direction of the displacement approaches that of the line tangent to the path at Ⓐ.

Instantaneous velocity

The **instantaneous velocity v** is defined as the limit of the average velocity $\Delta\mathbf{r}/\Delta t$ as Δt approaches zero:

$$\mathbf{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \tag{4.3}$$

That is, the instantaneous velocity equals the derivative of the position vector with respect to time. The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion (Fig. 4.3).

The magnitude of the instantaneous velocity vector $v = |\mathbf{v}|$ is called the *speed,* which, as you should remember, is a scalar quantity.

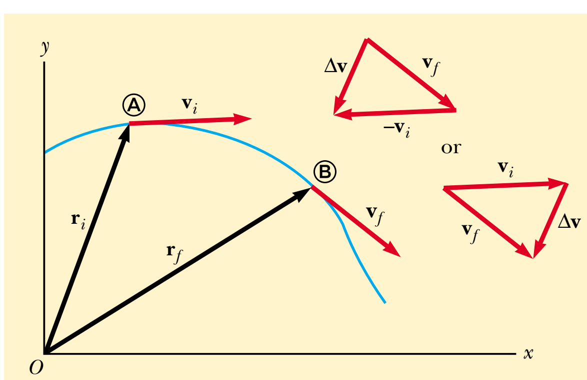

Figure 4.3 A particle moves from position Ⓐ to position Ⓑ. Its velocity vector changes from $\mathbf{v}_i$ to $\mathbf{v}_f$. The vector diagrams at the upper right show two ways of determining the vector $\Delta\mathbf{v}$ from the initial and final velocities.

As a particle moves from one point to another along some path, its instantaneous velocity vector changes from $\mathbf{v}_i$ at time t_i to $\mathbf{v}_f$ at time t_f. Knowing the velocity at these points allows us to determine the average acceleration of the particle:

The **average acceleration** of a particle as it moves from one position to another is defined as the change in the instantaneous velocity vector $\Delta\mathbf{v}$ divided by the time Δt during which that change occurred:

$$\overline{\mathbf{a}} \equiv \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta\mathbf{v}}{\Delta t} \tag{4.4}$$

Average acceleration

Because it is the ratio of a vector quantity $\Delta\mathbf{v}$ and a scalar quantity Δt, we conclude that average acceleration $\overline{\mathbf{a}}$ is a vector quantity directed along $\Delta\mathbf{v}$. As indicated in Figure 4.3, the direction of $\Delta\mathbf{v}$ is found by adding the vector $-\mathbf{v}_i$ (the negative of $\mathbf{v}_i$) to the vector $\mathbf{v}_f$, because by definition $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$.

When the average acceleration of a particle changes during different time intervals, it is useful to define its instantaneous acceleration $\mathbf{a}$:

The **instantaneous acceleration a** is defined as the limiting value of the ratio $\Delta\mathbf{v}/\Delta t$ as Δt approaches zero:

$$\mathbf{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \tag{4.5}$$

Instantaneous acceleration

3.5

In other words, the instantaneous acceleration equals the derivative of the velocity vector with respect to time.

It is important to recognize that various changes can occur when a particle accelerates. First, the magnitude of the velocity vector (the speed) may change with time as in straight-line (one-dimensional) motion. Second, the direction of the velocity vector may change with time even if its magnitude (speed) remains constant, as in curved-path (two-dimensional) motion. Finally, both the magnitude and the direction of the velocity vector may change simultaneously.

Quick Quiz 4.1

The gas pedal in an automobile is called the *accelerator.* (a) Are there any other controls in an automobile that can be considered accelerators? (b) When is the gas pedal not an accelerator?

4.2 TWO-DIMENSIONAL MOTION WITH CONSTANT ACCELERATION

Let us consider two-dimensional motion during which the acceleration remains constant in both magnitude and direction.

The position vector for a particle moving in the xy plane can be written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \tag{4.6}$$

where x, y, and $\mathbf{r}$ change with time as the particle moves while $\mathbf{i}$ and $\mathbf{j}$ remain constant. If the position vector is known, the velocity of the particle can be obtained from Equations 4.3 and 4.6, which give

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} \tag{4.7}$$