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Numerical solution of nonlinear integro-differential equations

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Abstract. The paper is devoted to the development of a numerical algorithm for solving nonlinear integro-differential equations based on the use of quadrature formulas. The Koltunov-Rzhanitsyn kernel with weakly singular features of the Abel type is used as a kernel. To conduct a computational experiment, a computer program was developed; the results obtained by this program are reflected in the form of tables and graphs. A test example was solved, and the obtained approximate numerical results were compared with exact solutions. The influence of nonlinearity and integral parts on the nature of oscillatory process of a viscoelastic body was investigated.

1. Intorduction

For the modern construction of hydraulic structures, various designs are used, which their materials have pronounced rheological properties. The use of these designs provides cost-effectiveness, longevity, durability and safety of structures. Therefore, taking into account the rheological properties of materials for the design of construction is attracting the increasing interest of specialists.

We remind you that numerous hydraulic problems in mathematical models are described by systems of differential, integral or integro-differential equations. For linear and nonlinear systems of differential equations, there are a number of recommended numerical and analytical methods for solving [1-3,10,11,15,16].

Recently, the materials combining elastic, viscous and plastic properties have been widely used in modern technology. Such materials have relaxation properties — their strain processes depends on time. Such materials include polymers, concretes, alloys, metals (at high temperatures), etc. Some structures under load (for example, shock-absorbing and vibro-protective devices) generally behave as a viscoelastic system. The basic physical equations relating stresses and strains of viscoelastic media contain a time factor. A hereditary theory of viscoelasticity is accepted as a theory describing the processes of strain over time [2,8,9,11]; it is based on the Boltzmann-Volterra principle. As a result, such problems are reduced to solving the systems of integro-differential equation (IDE) of Volterra type. The most common methods for solving the integral equation are the asymptotic methods [4-8]. These methods are applicable in cases where the viscosity of the medium is sufficiently small, and it is possible to introduce a small parameter and construct solutions that are asymptotically exact. Asymptotic methods can be used to solve problems of mechanics, the hereditary properties of the material are insignificant in comparison with elastic ones, i.e., there is a sufficiently small parameter in the integral term of the equation of state. It should be noted that, over time, the error of these methods increases significantly.

A number of methods have been developed for solving linear IDE, one of them is given in [12], where an exact analytical solution of linear IDE of Volterra-type for the Yu.N. Rabotnov kernel was constructed, as an elaboration of the F. Trikomi's method [13]. The addition theorems were proved.

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The use of new composite materials in engineering practice, the design and creation of strong, lightweight and reliable structures requires further improvement of the theories of deformable bodies and the development of methods for their calculation with account for real properties of structure materials. Therefore, the development of effective methods for solving nonlinear IDE is relevant in the hereditary mechanics of a deformable rigid body.

2. Problem statement

Let us find a solution to equation

$$\ddot{x}(t) + \omega^2(1 - R^*)\big[x(t) + \gamma \cdot L[x(t)]\big] = f(t) \qquad (1)$$

at the following values of initial conditions

$$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0, \qquad (2)$$

where *x (t)* are the unknown functions, $L[x(t)]$ is the nonlinear part of the equation, *f (t)* is the given function, γ is the nonlinearity coefficient, $\omega^2, x_0, \dot{x}_0$ are the given numbers, R^* is the integral operator with relaxation kernel $R(t) = \varepsilon t^{\alpha-1} e^{-\beta t}$:

$$R^*\varphi = \int_0^t R(t-\tau)\varphi(\tau)d\tau\,.$$

3. Methods

Integrating equation (1) twice over time in the interval [0; t] and taking into account the initial condition (2), we have:

$$x(t) - x_0 - \dot{x}_0 t + \omega^2 \int_0^t G(t-s)\big[x(s) + L[x(s)]\big]ds = \int_0^t (t-s) f(s) ds\,, \qquad (3)$$

where $G(t-s) = t - s - \int_0^{t-s} (t-s-\tau)\, R(\tau) d\tau$.

Setting $t_n = n \cdot \Delta t, n = 1,2,3, \ldots$ (Δt -time step) in (3), and replacing the integrals with quadrature trapezoid formulas, we have:

$$x_n = x_0 + \dot{x}_0 t_n - \omega^2 \sum_{i=0}^{n-1} A_i G(t_n - t_i)\,[x_i + L(x_i)] + \sum_{i=0}^{n-1} A_i (t_n - t_i) f(t_i)$$

where $x_n = x(t_n),\ A_0 = \frac{\Delta t}{2},\ A_j = \Delta t,\ j = \overline{1, n-1}$.

Test case. Testing the algorithms and corresponding computer program was carried out when solving the following problem:

$$\ddot{x}(t) + \omega^2(1 - R^*)[x(t) + \gamma \cdot x^3(t)] = e^{-\beta t}\left\{\beta^2 + \omega^2\left(1 + \gamma e^{-2\beta t}\right) - \omega^2 \varepsilon \left[t + \frac{\gamma}{2\beta}\left(1 - e^{-2\beta t}\right)\right]\right\}$$

$$x(0) = 1; \ \dot{x}(0) = -\beta$$

which has an exact solution:

$$x(t) = e^{-\beta t}.$$

The numerical results obtained are shown in Table 1. The following initial data were used in calculations: : $\omega^2 = 8$; $\gamma = 0.36$; $\alpha = 1$; $\beta = 0.05$; $\varepsilon = 0.01$; $\Delta t = 0.05$.

The exact solution and the obtained numerical approximate solution for various values of the quadrature formula step at certain time intervals are given in Table 1.

Table 1. Exact solution and the obtained numerical approximate solution

t	Exact solution	Approximate solution		
		$\Delta t = 0.01$	$\Delta t = 0.05$	$\Delta t = 0.1$
1	0.9512294	0.9512296	0.9512346	0.9512496
2	0.9048374	0.9048371	0.9048273	0.9047971
3	0.8607080	0.8607087	0.8607220	0.8607630
4	0.8187308	0.8187304	0.8187146	0.8186652

5	0.7788008	0.7788020	0.7788185	0.7788705
6	0.7408182	0.7408183	0.7408024	0.7407513
7	0.7046881	0.7046896	0.7047032	0.7047477
8	0.6703200	0.6703209	0.6703107	0.6702757
9	0.6376282	0.6376299	0.6376355	0.6376571
10	0.6065307	0.6065325	0.6065318	0.6065246

The table shows that the proposed algorithm for solving nonlinear integro-differential equations has a high accuracy. This gives us the opportunity to apply this approach (the mathematical model is described in (1), (2)) to specific applied mechanical problems.

4. Results and conclusions

Mathematical models of the problem of free and forced oscillation of a viscoelastic body are described in (1) and (2). The effect of viscoelastic properties of the material, and parameters entering (1) and (2) on the oscillatory process of a viscoelastic body is studied. Figure 1 shows the influence of the nonlinearity parameter γ on the behavior of the mode of oscillation of a viscoelastic body under constant external load. The graph shows that the oscillatory process occurs close to the creep curve. With increasing parameter γ, the oscillation frequency increases, and the amplitude of oscillations of a viscoelastic body decreases.

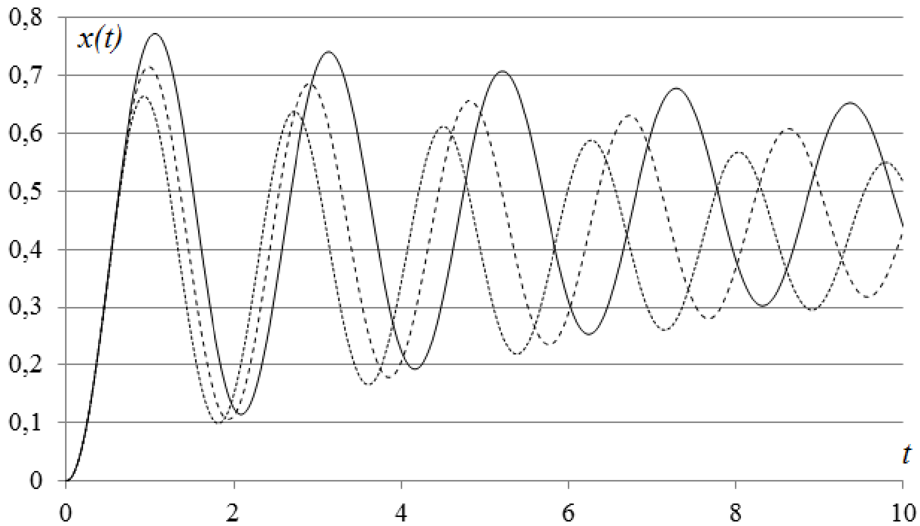

Figure 1. $\omega^2 = 10.45$; $x_0 = \dot{x}_0 = 0$; $f(t) = 3.5$; $\alpha = 0.25$; $\beta = 0.05$; $\varepsilon = 0.05$; $\gamma = 0$ (a solid line); $\gamma = 0.3$ (a dashed line); $\gamma = 0.7$ (a dotted line).

The influence of the parameter ε on the modes of oscillation of a nonlinear viscoelastic body under constant load is studied (Fig. 2). The graph shows that, with an increase in parameter ε, the frequency and amplitude of oscillations decreases.

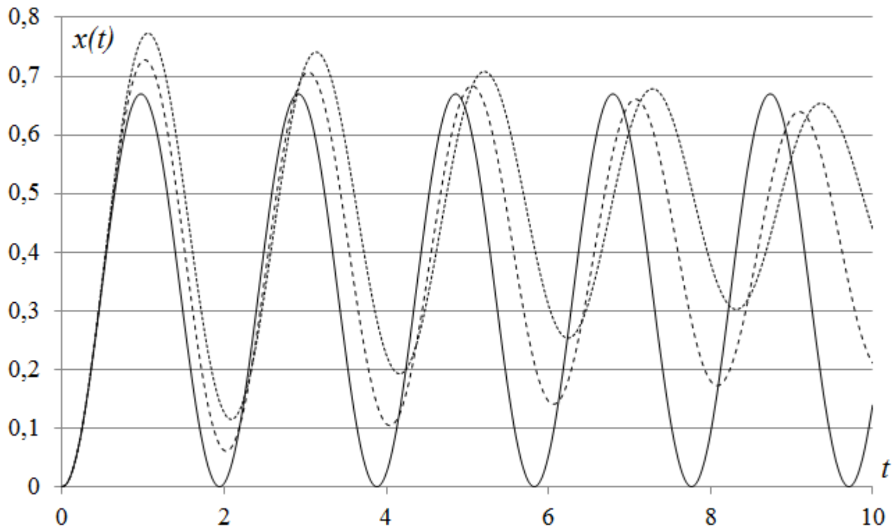

Figure 2. $\omega^2 = 10.45$; $x_0 = \dot{x}_0 = 0$; $f(t) = 3.5$; $\gamma = 0.3$; $\alpha = 0.25$; $\beta = 0.05$; $\varepsilon = 0$ (a solid line); $\varepsilon = 0.03$ (a dashed line); $\varepsilon = 0.05$ (a dotted line)

How do the rheological parameters affect the mode of oscillation of a nonlinear viscoelastic body under constant load? The study of parameter α shows (Fig. 3), that with an increase in α, the oscillation frequency increases as well. The results of the study of parameter β effect on the oscillations process of a viscoelastic body are shown in Fig. 4. As seen from the figure, the parameter β does not significantly affect the amplitude and frequency of oscillations (Sharipov et al., 2019).

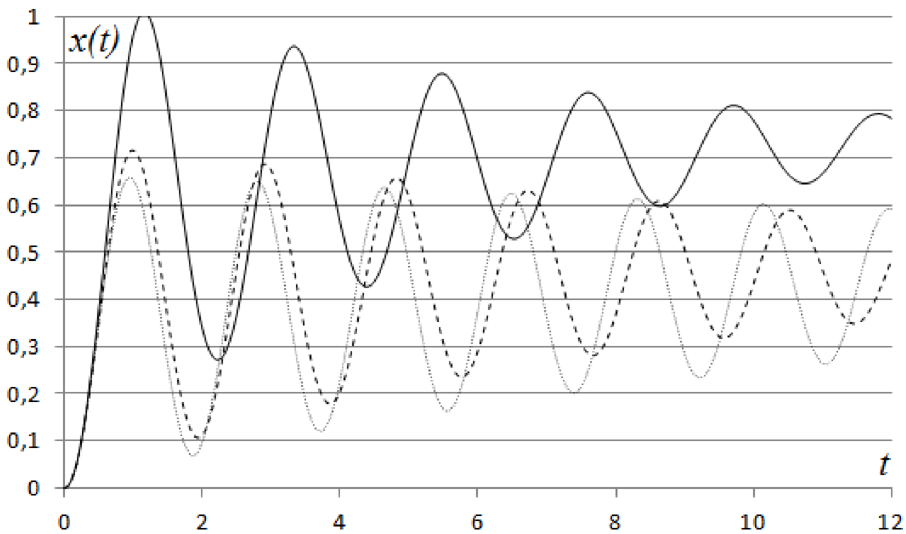

Figure 3. $\omega^2 = 10.45$; $x_0 = \dot{x}_0 = 0$; $f(t) = 3.5$; $\gamma = 0.3$; $\beta = 0.05$; $\varepsilon = 0.05$; $\alpha = 0.1$ (a solid line); $\alpha = 0.25$ (a dashed line); $\alpha = 0.5$ (a dotted line)

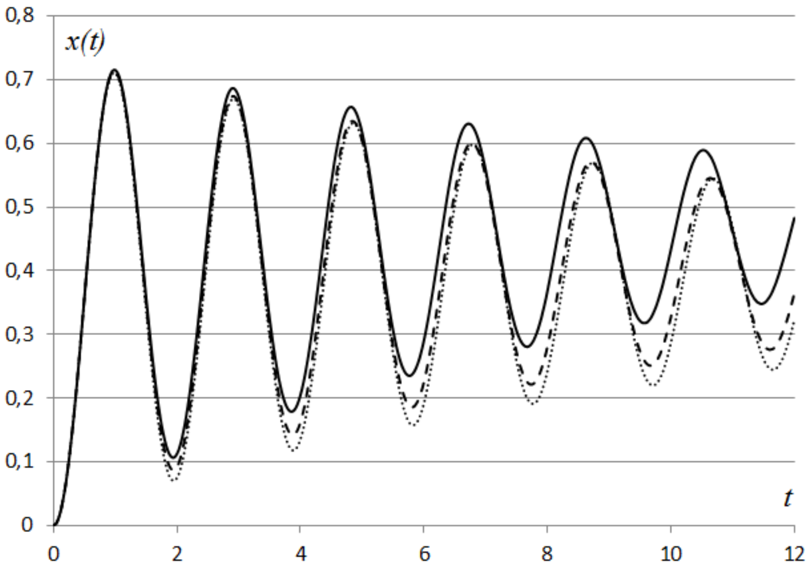

Figure 4. $\omega^2 = 10.45$; $x_0 = \dot{x}_0 = 0$; $f(t) = 3.5$; $\gamma = 0.3$; $\alpha = 0.25$; $\varepsilon = 0.05$; $\beta = 0.05$ (a solid line); $\beta = 0.5$ (a dashed line); $\beta = 1$ (a dotted line)

The effect of parameter γ on the oscillation modes of a viscoelastic body (Fig. 5), under external load acting according to the law $f(t) = 2(\sin\pi t + \cos\pi t)$ is studied. Figure 6 shows the dependences of the oscillation modes of a viscoelastic body on parameter ε.

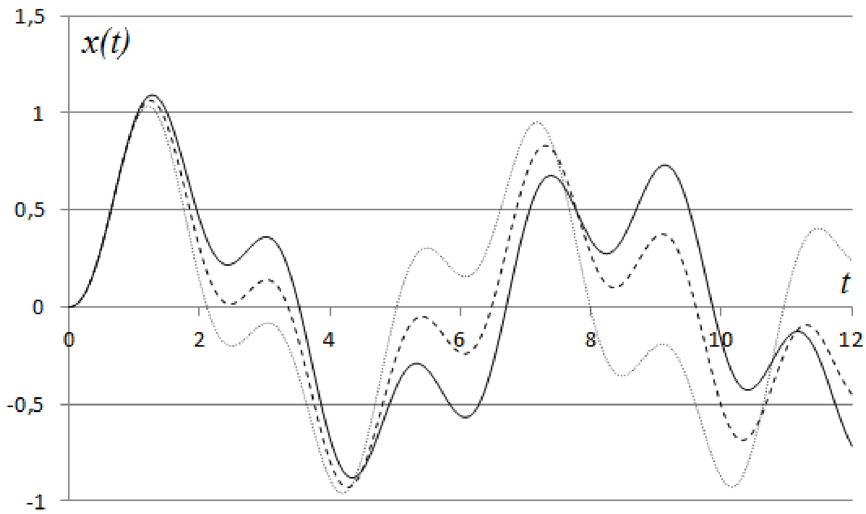

Figure 5. $\omega^2 = 1$; $x_0 = \dot{x}_0 = 0$; $\alpha = 0.25$; $\beta = 0.05$; $\varepsilon = 0.05$; $\gamma = 0$ (a solid line); $\gamma = 0.3$ (a dashed line); $\gamma = 0.7$ (a dotted line)

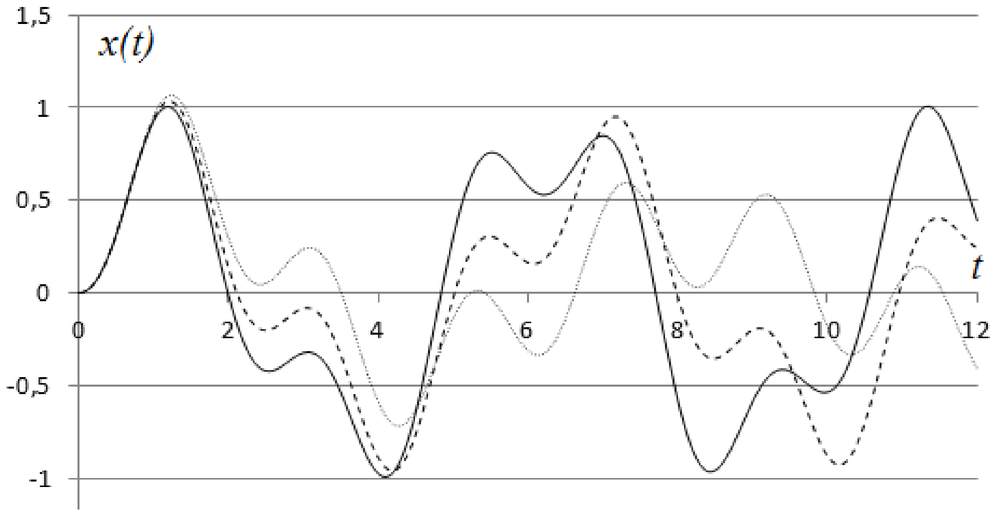

Figure 6. $\omega^2 = 1$; $x_0 = \dot{x}_0 = 0$; $\alpha = 0.25$; $\beta = 0.05$; $\gamma = 0.05$; $\varepsilon = 0$ (a solid line) ; $\varepsilon = 0{,}05$ (a dashed line); $\varepsilon = 0.1$ (a dotted line)

Conclusions

It should be noted that the above-stated methodology, the solutions of nonlinear IMUs, can be used to solve nonlinear problems of oscillations and dynamic stability of a viscoelastic pipe with a fluid flowing through it; tasks for the study of resonance phenomena in especially high-rise structures such as water and television towers; nonlinear problems of oscillations of viscoelastic rods and plates with variable stiffness; dynamic damper of hereditarily deformed systems, both of a finite degree of freedom and with distributed parameters. All these listed tasks have important practical interests in various fields of research.

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