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Integration of differential equations using power series

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Abstract. This article includes some methods and techniques of solving differential equations with using power series, which is effectively used in solving problems of physics and mechanics.

1 Introduction

In general, finding an exact solution of the first-order ordinary differential equation by integrating it is impossible. Moreover, this is not impracticable for a system of ordinary differential equations. This circumstance led to the creation of a large number of approximate methods for solving ordinary differential equations and their systems. Among the approximate methods, there are three groups: analytical, graphical and numerical. Of course, such classification is in some ways conditionally. For example, the graphical method of broken Euler underlies one of the methods for numerically solving a differential equation.

The integration of ordinary differential equations using power series is an approximate analytical method, usually applied, to linear equations of at least second order.

Analytical methods are found in the course of differential equations. For first order equations (with separable variables, homogeneous, linear, etc.) and also for some types of higher order equations (for example, linear with constant coefficients) it is possible to obtain solutions in the form of formulas by analytical transformations.

The aim of the work is to analyze one of the approximate analytical methods, such as integrating ordinary differential equations using power series, and their application in solving differential equations.

By studying differential equations, we get information about relevant processes. These differential equations are a mathematical model of the process under study, and the study of differential equations leads to a complete description of the processes.

An object of the mass m is dropped from the height. If, in addition to force of gravity, resistance force acting on the body, proportional to air velocity (proportionality factor *k*), it is necessary to know by what law the speed of an object changes, that is, you need to find the ratio $\upsilon = f(t)$.

According to Newton's second law,

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$$m\frac{d\vartheta}{dt} = F,$$

where $\frac{d\vartheta}{dt}$ acceleration of a moving body, F - force, acting on the body in the direction of motion, which it consists of gravity mg and air resistance ($-k\,v$). Since the force of air resistance is opposite to the direction of speed, we get it with a negative sign. Thus,

$$m\frac{d\vartheta}{dt} = mg - k\vartheta \qquad (*)$$

We have built a ratio, representing the relationship between the unknown function v and its derivative $\frac{d\vartheta}{dt}$, that is, a differential equation with respect to the unknown function *v*. Solving a differential equation means finding a function $v = f(t)$, which satisfies the given differential equation. There are an infinite number of such functions, satisfying the differential equation. One can check that any function in the form

$$\upsilon = Ce^{-\frac{k}{m}t} + \frac{mg}{k} \qquad (**)$$

in any values of the constant *C* satisfies the equation (*). Which of these functions gives the relation *v*, through the desired *m*. To find it, let us use an additional condition: when the body fell, it was given an initial velocity v_0 (in particular, it can be equal to zero); we assume this initial velocity to be known. But in this case, the required function $v = f(t)$ should be like this, that the condition must be satisfied for it $v = v_0$ at $t = 0$ (at the beginning of the movement). We set the values $t = 0$ and $v = v_0$ the above equation (**):

$$\upsilon_0 = C + \frac{mg}{k}$$

hence

$$C = v_0 - \frac{mg}{k}$$

So, we found the constant C. The desired connection between *v* and *t*

$$v = (v_0 - \frac{mg}{k})\, e^{-\frac{kt}{m}} + \frac{mg}{k}$$

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$$\lim_{k\to 0} v = \lim_{k\to 0}\left[\left(v_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}} + \frac{mg}{k}\right]$$

If in this expression $k=0$ (that is, there is no air resistance or it is negligible), then we get the well-known formula from physics

$$v = v_0 + gt$$

The found function v satisfies the differential equation (*) and initial condition $v = v_0$ at $t=0$.

It can be seen from the above tasks, that one differential equation can be satisfied by several functions, so the main goal of differential equations is to find all solutions of the equation and study their properties [1, 2].

If a function $f(x)$ in some neighborhood $|x-x_0|<\rho$ of the point x_0 can be expanded into a power series, then the function $f(x)$ is called holomorphic at the point x_0.

If the coefficients of the equation are holomorphic at the point x_0, then the point x_0 is called a simple point, otherwise, the point x_0 is called a special point.

Let given a differential equation of the form

$$(py')' + qy = 0 \tag{1}$$

Let the functions $p(x)$ and $q(x)$ in the differential equation be holomorphic at the point $x = x_0$, that is

$$p(x) = \sum_{k=0}^{\infty} p_k (x-x_0)^k, \quad q(x) = \sum_{k=0}^{\infty} q_k (x-x_0)^k$$

or if $x_0 = 0$

$$p(x) = \sum_{k=0}^{\infty} p_k x^k, \quad q(x) = \sum_{k=0}^{\infty} q_k x^k \tag{2}$$

putting (2) in (1) we obtain:

$$y'' + \sum_{k=0}^{\infty} p_k x^k y' + \sum_{k=0}^{\infty} q_k x^k y = 0 \tag{3}$$

We will search a solution of the equation (3) in the form:

$$y=\sum_{k=0}^{\infty} c_k x^k, \quad c_0=y_0,\ c_1=y_0' \tag{4}$$

Then, putting the expression (4) in (3), we get:

$$\sum_{k=0}^{\infty} k(k-1)c_k x^{k-2}+\sum_{k=0}^{\infty} p_k x^k \cdot \sum_{k=0}^{\infty} k c_k x^{k-1}+\sum_{k=0}^{\infty} q_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k=0$$

Performing some elementary transformations, we obtain the equality

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=0}^{\infty} p_k x^k \cdot \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k+\sum_{k=0}^{\infty} q_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k=0$$

where, using the formula

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} a_{k-n} b_n\right)x^k$$

we get the following equality

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} p_{k-n}(n+1)c_{n+1}\right)x^k+\sum_{k=0}^{\infty}\left(\sum_{k=0}^{k} q_{k-n}c_n\right)x^k=0$$

Equating zero, the coefficients at the powers x^k, we get the formula

$$(k+2)(k+1)c_{k+2}+\sum_{n=0}^{k} p_{k-n}(n+1)c_{n+1}+\sum_{n=0}^{k} q_{k-n}c_n=0$$

For the value $k=0, 1, 2, \ldots$

$$\begin{aligned}
&2\cdot 1\cdot c_2+p_0c_1+q_0c_0=0\\
&3\cdot 2\cdot c_3+p_1c_1+p_0 2c_2+q_1c_0+q_0c_1=0\\
&-----------------------\\
&k(k-1)c_k+\sum_{n=0}^{k} p_{k-n-2}(n+1)c_{n+1}+\sum_{n=0}^{k} q_{k-n-2}c_n=0
\end{aligned}$$

From this system we find the coefficients c_k. So, the solution of the second order equation can be represented as an arbitrary initial condition and its derivative. This method is called the method of undetermined coefficients [3, 4].

The Bessel equation has been comprehensively considered in the classes of differential equations, reduced to constant coefficients, that is

$$x^2y''+xy'+\left(x^2-n^2\right)y=0 \tag{5}$$

or

$$y'' + \frac{1}{x}y' + \frac{-n^2 + x^2}{x^2}y = 0$$

The point $x = 0$ is the special point of the equation. In this case, the solution of equation (5) will be sought in the form of a generalized power series:

$$y = x^{\rho}\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{\rho+k} \tag{6}$$

Finding respectively the derivatives y' and y'' of the function y

$$y' = \left(\sum_{k=0}^{\infty} c_k x^{\rho+k}\right)' = \sum_{k=0}^{\infty} c_k (\rho+k) x^{\rho+k-1}$$

$$y'' = \left(\sum_{k=0}^{\infty} c_k (\rho+k) x^{\rho+k-1}\right)' = \sum_{k=0}^{\infty} c_k (\rho+k)(\rho+k-1) x^{\rho+k-2}$$

Then substituting the received expressions y, y' and y'' into the initial equation (5) and after simplifying we obtain

$$\sum_{k=0}^{\infty}(\rho+k)(\rho+k-1)c_k x^{\rho+k} + \sum_{k=0}^{\infty}(\rho+k)c_k x^{\rho+k} + \sum_{k=0}^{\infty}c_k x^{\rho+k+2} - \sum_{k=0}^{\infty}n^2 c_k x^{\rho+k} = 0$$

Hence, reducing this expression by x^{ρ} and simplifying, we obtain

$$\sum_{k=0}^{\infty}\left((\rho+k)^2 - n^2\right)c_k x^k + \sum_{k=0}^{\infty}c_k x^{k+2} = 0$$

Equating to zero the coefficients at the same powers of x^k, we obtain:

$$\begin{aligned} &x^0 : \left(\rho^2 + n^2\right)c_0 = 0;\ c_0 \neq 0,\ \rho^2 - n^2 = 0,\ \rho_1 = n,\ \rho_2 = -n \\ &x^1 : \left((\rho+1)^2 - n^2\right)c_1 = 0 \\ &------------- \\ &x^k : \left((\rho+k)^2 - n^2\right)c_k + c_{k-2} = 0 \end{aligned} \tag{7}$$

From (7) for the root $\rho_1 = n$, we obtain:

$$\left((n+1)^2 - n^2\right)c_1 = 0,\quad (2n+1)c_1 = 0,\quad c_1 = 0$$
$$\left((n+k)^2 - n^2\right)c_k + c_{k-2} = 0,\quad (k \geq 2)$$

or

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$$c_k = -\frac{c_{k-2}}{k(2n+k)} \qquad (k \ge 2)$$

$$c_{2k+1} = -\frac{c_{2k-1}}{(2k+1)(2n+2k+1)} \tag{8}$$

$$c_{2k} = -\frac{c_{2k-2}}{2k(2n+2k)} = \frac{c_{2k-2}}{2^2 k\ (n+2)}$$

Therefore, for $c_1 = 0$ and for all odd k will be $c_{2k+1} = 0$.

From (8) for even values $k = 2, 4, \ldots$ determine the coefficients c_2 , c_4 , c_6 ,…, c_{2k} . Then, putting these values in (6) and let $\rho = n$

$$y = x^n \sum_{k=0}^{\infty} (-1)^k \frac{c_0}{2^{2k} k!(n+1)(n+2)\ldots(n+k)} x^{2k}$$

We express the solution of the Bessel equation. Exactly the same for $\rho = -n$ solution can also be considered. In this case, the coefficient c_0 can be taken as

$$c_0 = \frac{1}{2^n \Gamma(n+1)}$$

using properties, gamma functions, a particular solution of the Bessel equation is determined

$$y_1 = I_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

So, for $\rho = -n$ we obtain the second particular solution of this equation, which looks like

$$y_2 = I_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k}$$

The general solution of the Bessel equation has the form

$$y = c_1 I_n(x) + c_2 I_{-n}(x)$$

for the following conditions $0 < x < \infty,\ \ |y| < \infty,\ \ |y'| < \infty$ [5, 6].

Remark: In fact, the general solution of the Bessel differential equation can be expressed in terms of the Bessel functions of the first and second kind also for the case of non-integer order n .

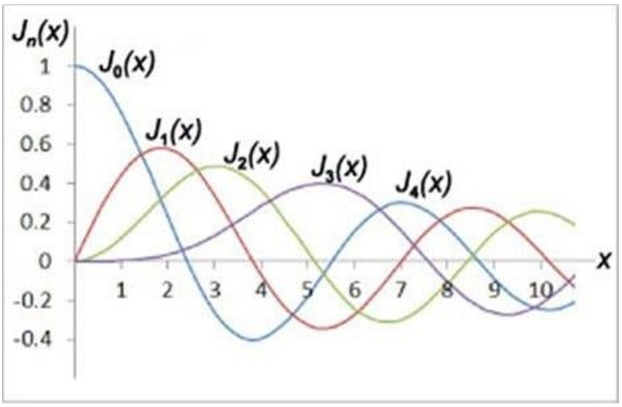

Fig. 1. Bessel functions of the first kind

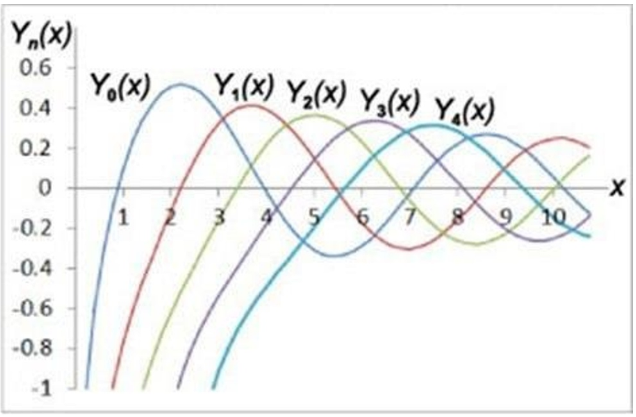

Fig. 2. Bessel functions of the second kind

2 Conclusion

Solving an equation containing unknown functions and their derivatives higher than the first or in some more complicated way is often very difficult.

In recent years, such differential equations have attracted increasing attention. Since the solution of equations is often very complex and difficult to represent with simple formulas, a significant part of modern theory is devoted to a qualitative analysis of their behavior, those, development of methods that allow, without solving equations, to say something significant about the nature of solutions in general: for example, that they are all limited, or have a periodic character, or depend in a certain way on the coefficients.

Thus, was carried out the analysis of the method of integrating differential equations using power and generalized series.

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