SEVENTH EDITION

ADVANCED ENGINEERING MATHEMATICS

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VECTORS IN THE PLANE AND 3-SPACE THE DOT PRODUCT THE CROSS PRODUCT THE VECTOR SPACE R^N ORTHOGONALIZATION

CHAPTER 6 Vectors and Vector Spaces

6.1 Vectors in the Plane and 3-Space

Some quantities, such as temperature and mass, are completely specified by a number. Such quantities are called *scalars*. By contrast, a *vector* has both a magnitude and a sense of direction. If we push against an object, the effect is determined not only by the strength of the push, but its direction. Velocity and acceleration are also vectors.

We can include both both magnitude and direction in one package by representing a vector as an arrow from the origin to a point (x, y, z) in 3-space, as in Figure 6.1. The choice of the point gives the direction of the vector (when viewed from the origin), and the length is its magnitude. The greater the force, the longer the arrow representation.

To distinguish when we are thinking of a point as a vector (arrow from the origin to the point), we will denote this vector $< x, y, z >$. We call x the *first component of* $< x, y, z >$, y the *second component* and z the *third component*. These components are scalars.

Two vectors are equal exactly when their respective components are equal. That is,

$$< x_1, y_1, z_1 > = < x_2, y_2, z_2 >$$

exactly when $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$.

Since only direction and magnitude are important in specifying a vector, any arrow of the same length and orientation denotes the same vector. The arrows in Figure 6.2 represent the same vector.

The vector $< -x, -y, -z >$ is opposite in direction to $< x, y, z >$, as suggested in Figure 6.3.

It is convenient to denote vectors by bold-face letters (such as **F**, **G**, and **H**) and scalars (real numbers) in ordinary type.

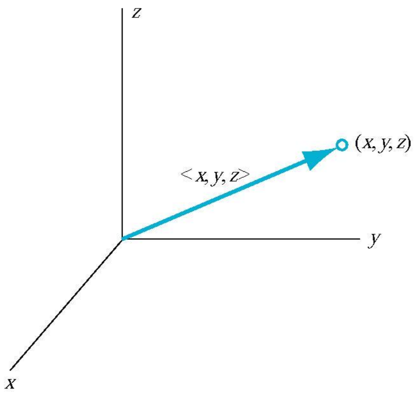

FIGURE 6.1 *Vector* $< x, y, z >$ *from the origin to the point* (x, y, z).

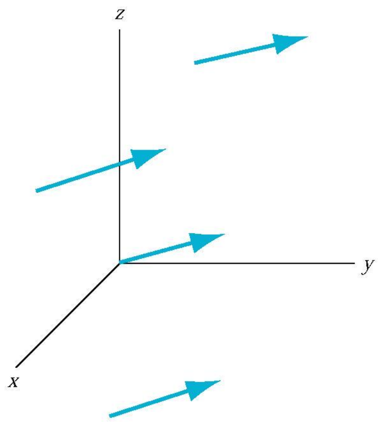

FIGURE 6.2 *Arrow representations of the same vector.*

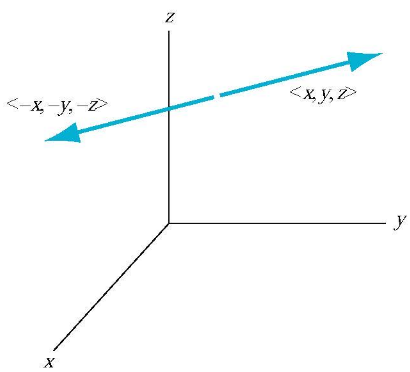

FIGURE 6.3 $< -x, -y, -z >$ *is opposite* $< x, y, z >$.

The *length* (also called the *magnitude* or *norm*) of a vector $\mathbf{F} = < x, y, z >$ is the scalar

$$\| \mathbf{F} \| = \sqrt{x^2 + y^2 + z^2}.$$

This is the distance from the origin to the point (x, y, z) and also the length of any arrow representing the vector $< x, y, z >$. For example, the norm of $\mathbf{G} = < -1, 4, 2 >$ is $\| \mathbf{G} \| = \sqrt{21}$, which is the distance from the origin to the point $(-1, 4, 2)$.

Multiply a vector $\mathbf{F} = < a, b, c >$ by a scalar α by multiplying each component of $\mathbf{F}$ by α. This produces a new vector denoted $\alpha\mathbf{F}$:

$$\alpha\mathbf{F} = < \alpha a, \alpha b, \alpha c > .$$

Then

$$\| \alpha\mathbf{F} \| = |\alpha| \, \| \mathbf{F} \|,$$

because

$$\| \alpha \mathbf{F} \| = \sqrt{(\alpha a)^2 + (\alpha b)^2 + (\alpha c)^2}$$
$$= \sqrt{(\alpha^2)(a^2 + b^2 + c^2)} = |\alpha| \sqrt{a^2 + b^2 + c^2}$$
$$= |\alpha| \, \| \mathbf{F} \| .$$

This means that the length of $\alpha\mathbf{F}$ is $|\alpha|$ times the length of $\mathbf{F}$. We may therefore think of multiplication of a vector by a scalar as a scaling (stretching or shrinking) operation. In particular, take the following cases:

- If $\alpha > 1$, then $\alpha\mathbf{F}$ is longer than $\mathbf{F}$ and in the same direction.
- If $0 < \alpha < 1$, then $\alpha\mathbf{F}$ is shorter than $\mathbf{F}$ and in the same direction.
- If $-1 < \alpha < 0$ then $\alpha\mathbf{F}$ is shorter than $\mathbf{F}$ and in the opposite direction.
- If $\alpha < -1$ then $\alpha\mathbf{F}$ is longer than $\mathbf{F}$ and in the opposite direction.
- If $\alpha = -1$ then $\alpha\mathbf{F}$ has the same length as $\mathbf{F}$, and exactly opposite the direction.
  For example, $\frac{1}{2}\mathbf{F}$ is a vector having the direction of $\mathbf{F}$ and half the length of $\mathbf{F}$, while $2\mathbf{F}$ has the direction of $\mathbf{F}$ and length twice that of $\mathbf{F}$, and $-\frac{1}{2}\mathbf{F}$ has direction opposite that of $\mathbf{F}$ and half the length.
- If $\alpha = 0$, then $\alpha\mathbf{F} = < 0, 0, 0 >$, which we call the *zero vector* and denote $\mathbf{O}$. This is the only vector with zero length and no direction, since it cannot be represented by an arrow.

Consistent with these interpretations of $\alpha\mathbf{F}$, we define two vectors $\mathbf{F}$ and $\mathbf{G}$ to be *parallel* if each is a nonzero scalar multiple of the other. Parallel vectors may differ in length and even be in opposite directions, but the straight lines through arrows representing them are parallel lines in 3-space.

We add two vectors by adding their respective components:
If $\mathbf{F} = < a_1, a_2, a_3 >$ and $\mathbf{G} = < b_1, b_2, b_3 >$, then

$$\mathbf{F} + \mathbf{G} = < a_1 + a_2, b_1 + b_2, c_1 + c_2 > .$$

Vector addition and multiplication by scalars have the following properties:

1. $\mathbf{F} + \mathbf{G} = \mathbf{G} + \mathbf{F}$. (commutativity)
2. $\mathbf{F} + (\mathbf{G} + \mathbf{H}) = (\mathbf{F} + \mathbf{G}) + \mathbf{H}$. (associativity)
3. $\mathbf{F} + \mathbf{O} = \mathbf{F}$.
4. $\alpha(\mathbf{F} + \mathbf{G}) = \alpha\mathbf{F} + \alpha\mathbf{G}$.
5. $(\alpha\beta)\mathbf{F} = \alpha(\beta\mathbf{F})$.
6. $(\alpha + \beta)\mathbf{F} = \alpha\mathbf{F} + \beta\mathbf{F}$.

It is sometimes useful to represent vector addition by the *parallelogram law*. If $\mathbf{F}$ and $\mathbf{G}$ are drawn as arrows from the same point, they form two sides of a parallelogram. The arrow along the diagonal of this parallelogram represents the sum $\mathbf{F} + \mathbf{G}$ (Figure 6.4). Because any arrows having the same lengths and direction represent the same vector, we can also draw the arrows in $\mathbf{F} + \mathbf{G}$ (as in Figure 6.5) with $\mathbf{G}$ drawn from the tip of $\mathbf{F}$. This still puts $\mathbf{F} + \mathbf{G}$ along the diagonal of the parallelogram.

The triangle of Figure 6.5 also suggests an important inequality involving vector sums and lengths. This triangle has sides of length $\| \mathbf{F} \|$, $\| \mathbf{G} \|$, and $\| \mathbf{F} + \mathbf{G} \|$. Because the sum of the

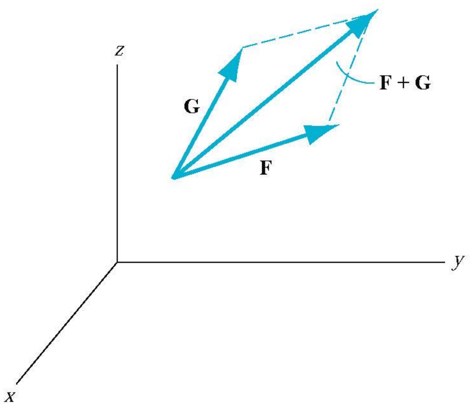

FIGURE 6.4 *Parallelogram law for vector addition.*

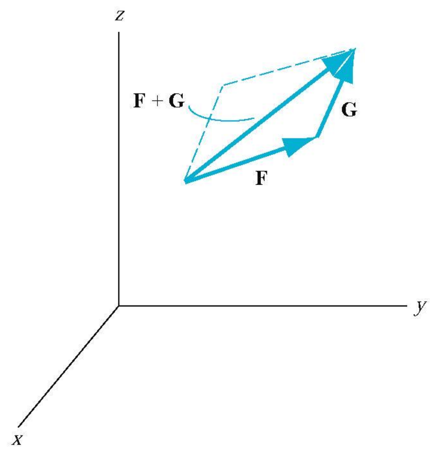

FIGURE 6.5 *Alternative view of the parallelogram law.*

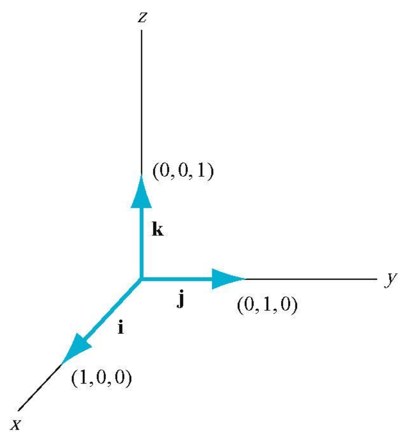

FIGURE 6.6 *Unit vectors* **i**, **j**, *and* **k**.

lengths of any two sides of a triangle must be at least as great as the length of the third side, we have the *triangle inequality*

$$\| \mathbf{F} + \mathbf{G} \| \leq \| \mathbf{F} \| + \| \mathbf{G} \| .$$

A vector of length 1 is called a *unit vector*. The unit vectors along the positive axes are shown in Figure 6.6 and are labeled

$$\mathbf{i} = < 1, 0, 0 >, \ \mathbf{j} = < 0, 1, 0 >, \ \mathbf{k} = < 0, 0, 1 > .$$

We can write any vector $\mathbf{F} = < a, b, c >$ as

$$\begin{aligned} \mathbf{F} = < a, b, c > &= a < 1, 0, 0 > + b < 0, 1, 0 > + c < 0, 0, 1 > \\ &= a\mathbf{i} + b\mathbf{j} + c\mathbf{k}. \end{aligned}$$

We call $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ the *standard representation* of $\mathbf{F}$. When a component of a vector is zero, we usually just omit this term in the standard representation. For example, we would usually write $\mathbf{F}=<-8,0,3>$ as $-8\mathbf{i}+3\mathbf{k}$ instead of $-8\mathbf{i}+0\mathbf{j}+3\mathbf{k}$.

If a vector is represented by an arrow in the x, y-plane, we often omit the third coordinate and use $\mathbf{i}=<1,0>$ and $\mathbf{j}=<0,1>$. For example, the vector $\mathbf{V}$ from the origin to the point $<2,-6,0>$ can be represented as an arrow from the origin to the point $(2,-6)$ in the x, y-plane and can be written in standard form as

$$\mathbf{V}=2\mathbf{i}-6\mathbf{j}$$

where $\mathbf{i}=<1,0>$ and $\mathbf{j}=<0,1>$.

It is often useful use to know the components of the vector $\mathbf{V}$ represented by the arrow from one point to another, say from $P_0=(x_0,y_0,z_0)$ to $P_1:(x_1,y_1,z_1)$. Denote

$$\mathbf{G}=x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k} \text{ and } \mathbf{F}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}.$$

By the parallelogram law in Figure 6.7, the vector $\mathbf{V}$ we want satisfies

$$\mathbf{G}+\mathbf{V}=\mathbf{F}.$$

Therefore,

$$\mathbf{V}=\mathbf{F}-\mathbf{G}=(x_1-x_0)\mathbf{i}+(y_1-y_0)\mathbf{j}+(z_1-z_0)\mathbf{k}.$$

For example, the vector represented by the arrow from $(-1,6,3)$ to $(9,-1,-7)$ if $10\mathbf{i}-7\mathbf{j}-10\mathbf{k}$.

Using this idea, we can find a vector of any length in any given direction. For example, suppose we want a vector of length 7 in the direction from $(-1,6,5)$ to $(-8,4,9)$.

The strategy is to first find a unit vector in the given direction, then multiply it by 7 to obtain a vector of length 7 in that direction. The vector $\mathbf{V}=-7\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ is in the direction from $(-1,6,5)$ to $(-8,4,9)$. Since $\|\mathbf{V}\|=\sqrt{69}$, a unit vector in this direction is

$$\mathbf{F}=\frac{1}{\|\mathbf{V}\|}\mathbf{V}=\frac{1}{\sqrt{69}}\mathbf{V}.$$

Then

$$7\mathbf{F}=\frac{7}{\sqrt{69}}(-7\mathbf{i}-2\mathbf{j}+4\mathbf{k})$$

has length 7 and is in the direction from $(-1,6,5)$ to $(-8,4,9)$.

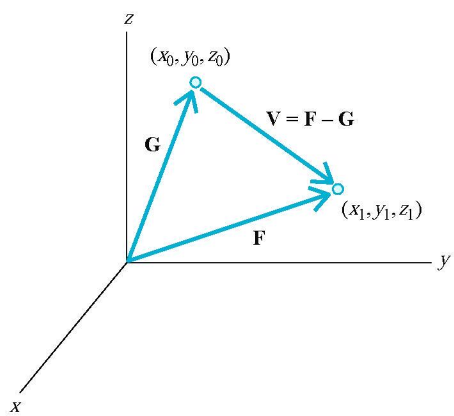

FIGURE 6.7 *Vector from* (x_0,y_0,z_0) *to* (x_1,y_1,z_1).

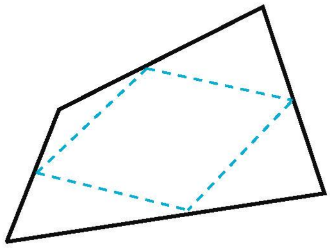

FIGURE 6.8 *Quadrilateral with lines connecting successive midpoints.*

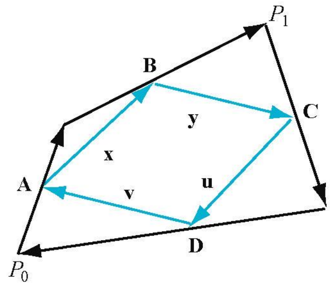

FIGURE 6.9 *Quadrilateral of Figure 6.8 with vectors as sides.*

As an example of the efficiency of vector notation, we will derive a fact about quadrilaterals: the lines formed by connecting successive midpoints of the sides of a quadrilateral form a parallelogram. Figures 6.8 and 6.9 illustrate what we want to show. Draw the quadrilateral again with vectors **A**, **B**, **C**, and **D** as the sides (Figure 6.9). The vectors **x**, **y**, **u**, and **v** connect the midpoints of successive sides. We want to show that **x** and **u** are parallel and of the same length, and the same for **y** and **v**. From the parallelogram law and the choices of these vectors,

$$\mathbf{x} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}$$

and

$$\mathbf{u} = \frac{1}{2}\mathbf{C} + \frac{1}{2}\mathbf{D}.$$

But also by the parallelogram law, $\mathbf{C} + \mathbf{D}$ is the vector from P_1 to P_0, while $\mathbf{A} + \mathbf{B}$ is the vector from P_0 to P_1. These vectors have the same lengths and opposite directions, so

$$\mathbf{A} + \mathbf{B} = -(\mathbf{C} + \mathbf{D}).$$

Then $\mathbf{x} = -\mathbf{u}$, so these vectors are parallel and of the same length (just opposite in direction). Similarly, **y** and **v** are parallel and of the same length.

Equation of a Line in 3-Space

We will show how to find parametric equations of a line L in 3-space containing two given points. This is more subtle than the corresponding problem in the plane, because there is no slope to exploit. To illustrate the idea, suppose the points are $(-2, -4, 7)$ and $(9, 1, -7)$. Form a vector between these two points (in either order). The arrow from the first to the second point represents the vector

$$\mathbf{V} = 11\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}.$$

Because P_0 and P_1 are on L, **V** is parallel to L, hence to any other vector aligned with L. Now suppose (x, y, z) is any point on L. Then the vector $(x+2)\mathbf{i} + (y+4)\mathbf{j} + (z-7)\mathbf{k}$ from $(-2, -4, 7)$ to (x, y, z) is also parallel to L, hence to **V**. This vector must therefore be a scalar multiple of **V**:

$$(x+2)\mathbf{i} + (y+4)\mathbf{j} + (z-7)\mathbf{k} = t\mathbf{V}$$
$$= 11t\mathbf{i} + 5t\mathbf{j} - 14t\mathbf{k}$$

for some scalar t. Since two vectors are equal only when their respective components are equal,

$$x + 2 = 11t,\ y + 4 = 5t,\ z - 7 = -14t.$$

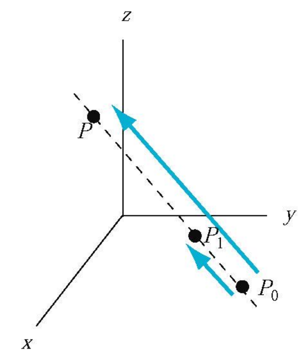

FIGURE 6.10 *Determining parametric equations of a line.*

Usually we write these equations as

$$x = -2 + 11t,\ y = -4 + 5t,\ z = 7 - 14t.$$

These are *parametric equations* of L. As t varies over the real numbers, the point $(-2 + 11t, -4 + 5t, 7 - 14t)$ varies over L. We obtain $(-2, -4, 7)$ when $t = 0$ and $(9, 1, -7)$ when $t = 1$.

The reasoning used in this example can be carried out in general. Suppose we are given points $P_0 : (x_0, y_0, z_0)$ and $P_1 : (x_1, y_1, z_1)$, and we want parametric equations of the line L through P_0 and P_1. The vector

$$(x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}$$

is along L, as is the vector

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

from P_0 to an arbitrary point (x, y, z) on L. These vectors (see Figure 6.10), being both along L, are parallel, hence for some real t,

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$
$$= t[(x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}].$$

Then

$$x - x_0 = t(x_1 - x_0),\ y - y_0 = t(y_1 - y_0),\ z - z_0 = t(z_1 - z_0).$$

Parametric equations of the line are

$$x = x_0 + t(x_1 - x_0),\ y = y_0 + t(y_1 - y_0),\ z = z_0 + t(z_1 - z_0),$$

with t taking on all real values. We get P_0 when $t = 0$ and P_1 when $t = 1$.

EXAMPLE 6.1

Find parametric equations of the line through $(-1, -1, 7)$ and $(7, -1, 4)$.

Let one of these points be P_0 and the other P_1. To be specific, choose $P_0 = (-1, -1, 7) = (x_0, y_0, z_0)$ and $P_1 = (7, -1, 4) = (x_1, y_1, z_1)$. The line through these points has parametric equations

$$x = -1 + (7 - (-1))t,\ y = -1 + (-1 - (-1))t,\ z = 7 + (4 - 7)t$$

for t real. These parametric equations are

$$x = -1 + 8t,\ y = -1,\ z = 7 - 3t$$

for t real. We obtain P_0 when $t = 0$ and P_1 when $t = 1$. In this example, the y-coordinate of every point on the line is -1, so the line is in the plane $y = -1$.

We may also say that this line consists of all points $(-1 + 8t, -1, 7 - 3t)$ for t real. ♦

SECTION 6.1 PROBLEMS

In each of Problems 1 through 5, compute $\mathbf{F} + \mathbf{G}$, $\mathbf{F} - \mathbf{G}$, $2\mathbf{F}$, $3\mathbf{G}$, and $\| \mathbf{F} \|$.

1. $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{G} = \sqrt{2}\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$
2. $\mathbf{F} = \mathbf{i} - 3\mathbf{k}$, $\mathbf{G} = 4\mathbf{j}$
3. $\mathbf{F} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{G} = \mathbf{i} + 5\mathbf{j} - \mathbf{k}$
4. $\mathbf{F} = \sqrt{2}\mathbf{i} - \mathbf{j} - 6\mathbf{k}$, $\mathbf{G} = 8\mathbf{i} + 2\mathbf{k}$
5. $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{G} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

In each of Problems 6 through 9, find a vector having the given length and in the direction from the first point to the second.

6. 5, $(0, 1, 4)$, $(-5, 2, 2)$
7. 9, $(1, 2, 1)$, $(-4, -2, 3)$
8. 12, $(-4, 5, 1)$, $(6, 2, -3)$
9. 4, $(0, 0, 1)$, $(-4, 7, 5)$

In each of Problems 10 through 15, find the parametric equations of the line containing the given points.

10. $(1, 0, 4)$, $(2, 1, 1)$
11. $(3, 0, 0)$, $(-3, 1, 0)$
12. $(2, 1, 1)$, $(2, 1, -2)$
13. $(0, 1, 3)$, $(0, 0, 1)$
14. $(1, 0, -4)$, $(-2, -2, 5)$
15. $(2, -3, 6)$, $(-1, 6, 4)$

6.2 The Dot Product

The *dot product* $\mathbf{F} \cdot \mathbf{G}$ of $\mathbf{F}$ and $\mathbf{G}$ is the real number formed by multiplying the two first components, then the two second components, then the two third components, and adding these three numbers. If $\mathbf{F} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{G} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, then

$$\mathbf{F} \cdot \mathbf{G} = a_1 a_2 + b_1 b_2 + c_1 c_2.$$

Again, this dot product is a number, not a vector. For example,

$$(\sqrt{3}\mathbf{i} + 4\mathbf{j} - \pi\mathbf{k}) \cdot (-2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) = -2\sqrt{3} + 24 - 3\pi.$$

The dot product has the following properties.

1. $\mathbf{F} \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{F}$.
2. $(\mathbf{F} + \mathbf{G}) \cdot \mathbf{H} = \mathbf{F} \cdot \mathbf{H} + \mathbf{G} \cdot \mathbf{H}$.
3. $\alpha(\mathbf{F} \cdot \mathbf{G}) = (\alpha\mathbf{F}) \cdot \mathbf{G} = \mathbf{F} \cdot (\alpha\mathbf{G})$.
4. $\mathbf{F} \cdot \mathbf{F} = \| \mathbf{F} \|^2$.
5. $\mathbf{F} \cdot \mathbf{F} = 0$ if and only if $\mathbf{F} = \mathbf{O}$.
6. $\| \alpha\mathbf{F} + \beta\mathbf{G} \|^2 = \alpha^2 \| \mathbf{F} \|^2 + 2\alpha\beta\mathbf{F} \cdot \mathbf{G} + \beta^2 \| \mathbf{G} \|^2$.

Dot products of vectors can be computed using MAPLE and the `DotProduct` command, which is in the `VectorCalculus` package of subroutines. This command also applies to n-dimensional vectors, which are introduced in Section 6.4.

Conclusions (1), (2), and (3) are routine computations. Conclusion (4) is often used in computations. To verify conclusion (4), suppose

$$\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Then

$$\mathbf{F} \cdot \mathbf{F} = a^2 + b^2 + c^2 = \| \mathbf{F} \|^2.$$

Conclusion (5) follows easily from (4), since $\mathbf{O}$ is the only vector having length 0. For conclusion (6), use conclusions (1) through (4) to write

$$\begin{aligned} \| \alpha\mathbf{F} + \beta\mathbf{G} \|^2 &= (\alpha\mathbf{F} + \beta\mathbf{G}) \cdot (\alpha\mathbf{F} + \beta\mathbf{G}) \\ &= \alpha^2 \mathbf{F} \cdot \mathbf{F} + \alpha\beta\mathbf{F} \cdot \mathbf{G} + \alpha\beta\mathbf{G} \cdot \mathbf{F} + \beta^2 \mathbf{G} \cdot \mathbf{G} \\ &= \alpha^2 \| \mathbf{F} \|^2 + 2\alpha\beta\mathbf{F} \cdot \mathbf{G} + \beta^2 \| \mathbf{G} \|^2. \end{aligned}$$

The dot product can be used to find an angle between two vectors. Recall the law of cosines: For the upper triangle of Figure 6.11 with θ being the angle opposite the side of length c, the law of cosines states that

$$a^2 + b^2 - 2ab\cos(\theta) = c^2.$$

Apply this to the vector triangle of Figure 6.11 (lower), which has sides of length $a = \| \mathbf{G} \|$, $b = \| \mathbf{F} \|$, and $c = \| \mathbf{G} - \mathbf{F} \|$. Using property (6) of the dot product, we obtain

$$\begin{aligned} \| \mathbf{G} \|^2 + \| \mathbf{F} \|^2 - 2 \| \mathbf{F} \| \| \mathbf{G} \| \cos(\theta) &= \| \mathbf{G} - \mathbf{F} \|^2 \\ &= \| \mathbf{G} \|^2 + \| \mathbf{F} \|^2 - 2\mathbf{G} \cdot \mathbf{F}. \end{aligned}$$

Assuming that neither $\mathbf{F}$ nor $\mathbf{G}$ is the zero vector, this gives us

$$\cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{G}}{\| \mathbf{F} \| \| \mathbf{G} \|}. \tag{6.1}$$

Since $|\cos(\theta)| \le 1$ for all θ, equation (6.1) implies the *Cauchy-Schwarz inequality*:

$$|\mathbf{F} \cdot \mathbf{G}| \le \| \mathbf{F} \| \| \mathbf{G} \|.$$

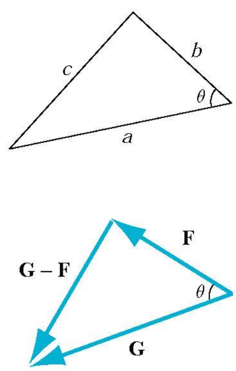

FIGURE 6.11 *The law of cosines and the angle between vectors.*

EXAMPLE 6.2

The angle θ between $\mathbf{F} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{G} = 2\mathbf{j} - 4\mathbf{k}$ is given by

$$\cos(\theta) = \frac{(-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{j} - 4\mathbf{k})}{\| -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \| \| 2\mathbf{j} - 4\mathbf{k} \|}$$

$$= \frac{(-1)(0) + (3)(2) + (1)(-4)}{\sqrt{1^2 + 3^2 + 1^2}\sqrt{2^2 + 4^2}} = \frac{2}{\sqrt{220}}.$$

Then $\theta \approx 1.436$ radians. ♦

EXAMPLE 6.3

Lines L_1 and L_2 have parametric equations

$$L_1 : x = 1 + 6t, y = 2 - 4t, z = -1 + 3t$$

and

$$L_2 : x = 4 - 3p, y = 2p, z = -5 + 4p.$$

The parameters t and p can take on any real values. We want an angle θ between these lines.

The strategy is to take a vector $\mathbf{V}_1$ along L_1 and a vector $\mathbf{V}_2$ along L_2 and find the angle between these vectors. For $\mathbf{V}_1$, find two points on L_1, say $(1, 2, -1)$ when $t = 0$ and $(7, -2, 2)$ when $t = 1$, and form

$$\mathbf{V}_1 = (7 - 1)\mathbf{i} + (-2 - 2)\mathbf{j} + (2 - (-1))\mathbf{k} = 6\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}.$$

On L_2, take $(4, 0, -5)$ with $p = 0$ and $(1, 2, -1)$ with $p = 1$, forming

$$\mathbf{V}_2 = 3\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}.$$

Now compute

$$\cos(\theta) = \frac{6(3) - 4(-2) + 3(-4)}{\sqrt{36 + 16 + 9}\sqrt{9 + 4 + 16}} = \frac{14}{\sqrt{1769}}.$$

An angle between L_1 and L_2 is $\arccos(14/\sqrt{1769})$, which is approximately 1.23 radians. ♦

Two nonzero vectors $\mathbf{F}$ and $\mathbf{G}$ are *orthogonal* (perpendicular) when the angle θ between them is $\pi/2$ radians. This happens exactly when

$$\cos(\theta) = 0 = \frac{\mathbf{F} \cdot \mathbf{G}}{\| \mathbf{F} \| \| \mathbf{G} \|}$$

which occurs when $\mathbf{F} \cdot \mathbf{G} = 0$. It is convenient to also agree that $\mathbf{O}$ is orthogonal to every vector. With this convention, two vectors are orthogonal if and only if their dot product is zero.

EXAMPLE 6.4

Let $\mathbf{F} = -4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{G} = 2\mathbf{i} + 4\mathbf{k}$ and $\mathbf{H} = 6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Then $\mathbf{F} \cdot \mathbf{G} = 0$, so $\mathbf{F}$ and $\mathbf{G}$ are orthogonal. But $\mathbf{F} \cdot \mathbf{H}$ and $\mathbf{G} \cdot \mathbf{H}$ are not zero, so $\mathbf{F}$ and $\mathbf{H}$ are not orthogonal and $\mathbf{G}$ and $\mathbf{H}$ are not orthogonal. ♦

Property (6) of the dot product has a particularly simple form when the vectors are orthogonal. In this case, $\mathbf{F} \cdot \mathbf{G} = 0$, and upon setting $\alpha = \beta = 1$, we have

$$\| \mathbf{F} + \mathbf{G} \|^2 = \| \mathbf{F} \|^2 + \| \mathbf{G} \|^2.$$

This is the familiar *Pythagorean theorem*, because the vectors **F** and **G** form the sides of a right triangle with hypotenuse $\mathbf{F}+\mathbf{G}$ (imagine Figure 6.5 with **F** and **G** forming a right angle).

EXAMPLE 6.5

Suppose two lines are defined parametrically by

$$L_1 : x = 2 - 4t,\ y = 6 + t,\ z = 3t$$

and

$$L_2 : x = -2 + p,\ y = 7 + 2p,\ z = 3 - 4p.$$

We want to know if these lines are orthogonal. Note that the question makes sense even if L_1 and L_2 do not intersect.

The idea is to form a vector along each line and test these vectors for orthogonality. For a vector along L_1, take two points on this line, say $(2, 6, 0)$ when $t = 0$ and $(-2, 7, 3)$ when $t = 1$. Then $\mathbf{V}_1 = -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ is parallel to L_1. Similarly, $(-2, 7, 3)$ is on L_2 when $p = 0$, and $(-1, 9, -1)$ is on L_2 when $p = 1$, so $\mathbf{V}_2 = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ is parallel to L_2. Compute $\mathbf{V}_1 \cdot \mathbf{V}_2 = -14 \neq 0$. Therefore, L_1 and L_2 are not orthogonal. ♦

Orthogonality is also useful for determining the equation of a plane in 3-space. Any plane has an equation of the form

$$ax + by + cz = d.$$

As suggested by Figure 6.12, if we specify a point on the plane and a vector orthogonal to the plane, then the plane is completely determined. Example 6.6 suggests a strategy for finding the equation of this plane.

EXAMPLE 6.6

We will find the equation of the plane Π containing the point $(-6, 1, 1)$ and orthogonal to the vector $\mathbf{N} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. Such a vector **N** is said to be *normal* to Π and is called a *normal vector* to Π.

Here is a strategy. Because $(-6, 1, 1)$ is on Π, a point (x, y, z) is on Π exactly when the vector between $(-6, 1, 1)$ and (x, y, z) lies in Π. But then $(x+6)\mathbf{i} + (y-1)\mathbf{j} + (z-1)\mathbf{k}$ must be orthogonal to **N**, so

$$\mathbf{N} \cdot ((x+6)\mathbf{i} + (y-1)\mathbf{j} + (z-1)\mathbf{k}) = 0.$$

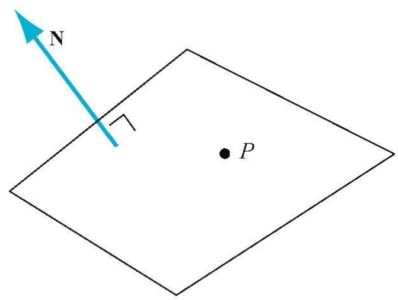

FIGURE 6.12 *A point P and a normal vector* **N** *determine a plane.*

Then

$$-2(x+6)+4(y-1)+(z-1)=0,$$

or

$$-2x+4y+z=17.$$

This is the equation of Π. ♦

Following this reasoning in general, the equation of a plane containing a point $P_0:(x_0, y_0, z_0)$ and having a normal vector $\mathbf{N}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ is

$$\mathbf{N}\cdot[(x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k}]=0$$

or

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0. \quad (6.2)$$

It is also sometimes convenient to notice that the vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ is always a normal vector to a plane $ax+by+cz=d$, for any d. Changing the value of d moves the plane in 3-space but does not change its orientation with respect to the axes, so the normal vector remains the same and is determined by the coefficients a, b, and c only.

Another use for the dot product is in forming vector projections.

Let **u** and **v** be given, nonzero vectors, represented as arrows from a common point (for convenience). The *projection* of **v** onto **u** is a vector $\text{proj}_{\mathbf{u}}\mathbf{v}$ in the direction of **u** having magnitude equal to the length of the perpendicular projection of the arrow representing **v** onto the line along the arrow representing **u**. This projection is done by constructing a perpendicular line from the tip of **v** onto the line through **u**. The base of the right triangle having **v** as hypotenuse is the length d of $\text{proj}_{\mathbf{u}}\mathbf{v}$ (Figure 6.13).

If θ is the angle between **u** and **v**, then

$$\cos(\theta)=\frac{d}{\|\mathbf{v}\|}.$$

Then

$$d=\|\mathbf{v}\|\cos(\theta)=\|\mathbf{v}\|\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}.$$

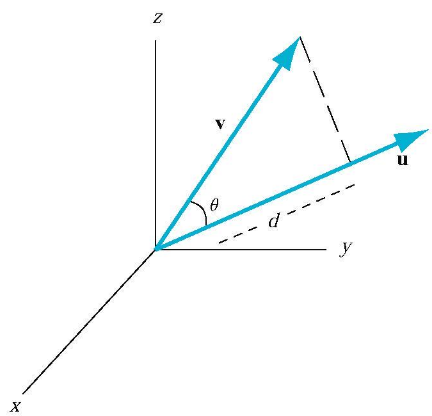

FIGURE 6.13 *Orthogonal projection of* **v** *onto* **u**.

To obtain a vector in the direction of $\mathbf{u}$ and of length d, divide $\mathbf{u}$ by its length to obtain a unit vector, then multiply this vector by d. Therefore,

$$\text{proj}_{\mathbf{u}}\mathbf{v} = d\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

As an example, suppose $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Then

$$\mathbf{u}\cdot\mathbf{v} = 9 \text{ and } \|\mathbf{u}\|^2 = 6,$$

so

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{9}{6}\mathbf{u} = \frac{3}{2}(\mathbf{i} - \mathbf{j} + 2\mathbf{k}).$$

If we think of these vectors as forces, we may interpret $\text{proj}_{\mathbf{u}}\mathbf{v}$ as the effect of $\mathbf{v}$ in the direction of $\mathbf{u}$.

SECTION 6.2 PROBLEMS

In each of Problems 1 through 6, compute the dot product of the vectors and the cosine of the angle between them. Also determine if the vectors are orthogonal.

1. $\mathbf{i}, 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
2. $2\mathbf{i} - 6\mathbf{j} + \mathbf{k}, \mathbf{i} - \mathbf{j}$
3. $-4\mathbf{i} - 2\mathbf{i} + 3\mathbf{k}, 6\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
4. $8\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, -8\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
5. $\mathbf{i} - 3\mathbf{k}, 2\mathbf{j} + 6\mathbf{k}$
6. $\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

In each of Problems 7 through 12, find the equation of the plane containing the given point and orthogonal to the given vector.

7. $(-1, 1, 2), 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
8. $(-1, 0, 0), \mathbf{i} - 2\mathbf{j}$
9. $(2, -3, 4), 8\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$
10. $(-1, -1, -5), -3\mathbf{i} + 2\mathbf{j}$
11. $(0, -1, 4), 7\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$
12. $(-2, 1, -1), 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

In each of Problems 13, 14, and 15, find the projection of $\mathbf{v}$ onto $\mathbf{u}$.

13. $\mathbf{v} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}, \mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
14. $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{u} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
15. $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}, \mathbf{u} = 2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$

6.3 The Cross Product

The dot product produces a scalar from two vectors. The cross product produces a vector from two vectors.

Let $\mathbf{F} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{G} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. The *cross product* of $\mathbf{F}$ with $\mathbf{G}$ is the vector $\mathbf{F} \times \mathbf{G}$ defined by

$$\mathbf{F} \times \mathbf{G} = (b_1c_2 - b_2c_1)\mathbf{i} + (a_2c_1 - a_1c_2)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Here is a simple device for remembering and computing these components. Form the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

having the standard unit vectors in the first row, the components of **F** in the second row, and the components of **G** in the third row. If this determinant is expanded by the first row, we get exactly $\mathbf{F}\times\mathbf{G}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\mathbf{k}$$

$$= (b_1c_2 - b_2c_1)\mathbf{i} + (a_2c_1 - a_1c_2)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$
$$= \mathbf{F}\times\mathbf{G}.$$

The cross product of two 3-vectors can be computed in MAPLE using the `CrossProduct` command, which is part of the `VectorCalculus` package.

We will summarize some properties of the cross product.

1. $\mathbf{F}\times\mathbf{G} = -\mathbf{G}\times\mathbf{F}$.
2. $\mathbf{F}\times\mathbf{G}$ is orthogonal to both **F** and **G**. This is shown in Figure 6.14.
3. $\| \mathbf{F}\times\mathbf{G} \| = \| \mathbf{F} \| \| \mathbf{G} \| \sin(\theta)$ in which θ is the angle between **F** and **G**.
4. If **F** and **G** are nonzero vectors, then $\mathbf{F}\times\mathbf{G} = \mathbf{O}$ if and only if **F** and **G** are parallel.
5. $\mathbf{F}\times(\mathbf{G}+\mathbf{H}) = \mathbf{F}\times\mathbf{G} + \mathbf{F}\times\mathbf{H}$.
6. $\alpha(\mathbf{F}\times\mathbf{G}) = (\alpha\mathbf{F})\times\mathbf{G} = \mathbf{F}\times(\alpha\mathbf{G})$.

Property (1) of the cross product follows from the fact that interchanging two rows of a determinant changes its sign. In computing $\mathbf{F}\times\mathbf{G}$, the components of **F** are in the second row of the determinant, and those of **G** in the third row. These rows are interchanged in computing $\mathbf{G}\times\mathbf{F}$.

For property (2), compute the dot product

$$\mathbf{F}\cdot(\mathbf{F}\times\mathbf{G})$$
$$= a_1[b_1c_2 - b_2c_1] + b_1[a_2c_1 - a_1c_2] + c_1[a_1b_2 - a_2b_1] = 0.$$

Therefore, **F** is orthogonal to $\mathbf{F}\times\mathbf{G}$. Similarly, **G** is orthogonal to $\mathbf{F}\times\mathbf{G}$.

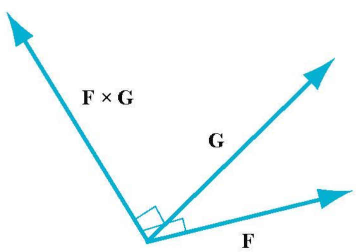

FIGURE 6.14 $\mathbf{F}\times\mathbf{G}$ *is orthogonal to* **F** *and to* **G**.

To derive property (3), suppose both vectors are nonzero and recall that $\cos(\theta) = (\mathbf{F}\cdot\mathbf{G})/\parallel \mathbf{F} \parallel \parallel \mathbf{G} \parallel$, where θ is the angle between **F** and **G**. Now write

$$\begin{aligned}&\parallel \mathbf{F} \parallel^2 \parallel \mathbf{G} \parallel^2 - (\mathbf{F}\cdot\mathbf{G})^2\\ &= \parallel \mathbf{F} \parallel^2 \parallel \mathbf{G} \parallel^2 - \parallel \mathbf{F} \parallel^2 \parallel \mathbf{G} \parallel^2 \cos^2(\theta)\\ &= \parallel \mathbf{F} \parallel^2 \parallel \mathbf{G} \parallel^2 \sin^2(\theta).\end{aligned}$$

It is therefore enough to show that

$$\parallel \mathbf{F}\times\mathbf{G} \parallel^2 = \parallel \mathbf{F} \parallel^2 \parallel \mathbf{G} \parallel^2 - (\mathbf{F}\cdot\mathbf{G})^2,$$

and this is a tedious but routine calculation.

Property (4) follows from (3), since two nonzero vectors are parallel exactly when the angle θ between them is zero, and in this case, $\sin(\theta) = 0$. Properties (5) and (6) are routine computations.

Property (4) provides a test for three points to be *collinear*, that is, to lie on a single line. Let P, Q, and R be the points. These points will be collinear exactly when the vector **F** from P to Q is parallel to the vector **G** from P to R. By property (4), this occurs when $\mathbf{F}\times\mathbf{G} = \mathbf{O}$.

One of the primary uses of the cross product is to produce a vector orthogonal to two given vectors. This can be used to find the equation of a plane containing three given points. The strategy is to pick one of the points and write the vectors from this point to the other two. The cross product of these two vectors is normal to the plane containing the points. Now we know a normal vector and a point (in fact three points) on the plane, so we can use equation (6.2) to write the equation of the plane.

This strategy fails if the cross product is zero. But by property (4), this only occurs if the given points are collinear, hence do not determine a unique plane (there are infinitely many planes through any line in 3-space).

EXAMPLE 6.7

Find the equation of a plane containing the points $P : (-1, 4, 2)$, $Q : (6, -2, 8)$, and R: $(5, -1, -1)$.

Use the three given points to form two vectors in the plane:

$$\mathbf{PQ} = 7\mathbf{i} - 6\mathbf{j} + 6\mathbf{k} \text{ and } \mathbf{PR} = 6\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}.$$

The cross product of these vectors is orthogonal to the plane of these vectors, so

$$\mathbf{N} = \mathbf{PQ}\times\mathbf{PR} = 48\mathbf{i} + 57\mathbf{j} + \mathbf{k}$$

is a normal vector. By equation (6.2), the equation of the plane is

$$48(x+1) + 57(y-4) + (z-2) = 0,$$

or

$$48x + 57y + z = 182. \ \blacklozenge$$

SECTION 6.3 PROBLEMS

In each of Problems 1 through 4, compute $\mathbf{F}\times\mathbf{G}$ and $\mathbf{G}\times\mathbf{F}$ and verify the anticommutativity of the cross product.

1. $\mathbf{F} = -3\mathbf{i} + 6\mathbf{j} + \mathbf{k}$, $\mathbf{G} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
2. $\mathbf{F} = 6\mathbf{i} - \mathbf{k}$, $\mathbf{G} = \mathbf{j} + 2\mathbf{k}$
3. $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{G} = -3\mathbf{i} + 2\mathbf{j}$
4. $\mathbf{F} = 8\mathbf{i} + 6\mathbf{j}$, $\mathbf{G} = 14\mathbf{j}$

In each of Problems 5 through 9, determine whether the points are collinear. If they are not, determine an equation for the plane containing these points.

5. $(-1, 1, 6), (2, 0, 1), (3, 0, 0)$
6. $(4, 1, 1), (-2, -2, 3), (6, 0, 1)$
7. $(1, 0, -2), (0, 0, 0), (5, 1, 1)$
8. $(0, 0, 2), (-4, 1, 0), (2, -1, -1)$
9. $(-4, 2, -6), (1, 1, 3), (-2, 4, 5)$

In each of Problems 10, 11, and 12, find a vector normal to the given plane. There are infinitely many such vectors.

10. $8x - y + z = 12$
11. $x - y + 2z = 0$
12. $x - 3y + 2z = 9$
13. Let **F** and **G** be nonparallel vectors and let R be the parallelogram formed by representing these vectors as arrows from a common point. Show that the area of this parallelogram is $\| \mathbf{F} \times \mathbf{G} \|$.
14. Form a parallelepiped (skewed rectangular box) having as incident sides the vectors **F**, **G**, and **H** drawn as arrows from a common point. Show that the volume of this parallelepiped is

$$|\mathbf{F} \cdot (\mathbf{G} \times \mathbf{H})|.$$

This quantity is called the *scalar triple product* of **F**, **G**, and **H**.

6.4 The Vector Space R^n

For systems involving n variables we may consider n-vectors

$$< x_1, x_2, \cdots, x_n >$$

having n components. The jth component of this n-vector is x_j and this is a real number. The totality of such n-vectors is denoted R^n and is called "n-space". R^1 is the real line, consisting of all real numbers. We can think of numbers as 1-vectors, although we do not usually do this. R^2 is the familiar plane, consisting of vectors with two components. And R^3 is in 3-space. R^n has an algebraic structure which will prove useful when we consider matrices, systems of linear algebraic equations, and systems of linear differential equations.

Two n-vectors are equal exactly when their respective components are equal:

$$< x_1, x_2, \cdots, x_n > = < y_1, y_2, \cdots, y_n >$$

if and only if

$$x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n.$$

Add n-vectors, and multiply them by scalars, in the natural ways:

$$< x_1, x_2, \cdots, x_n > + < y_1, y_2, \cdots, y_n > = < x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n >$$

and

$$\alpha < x_1, x_2, \cdots, x_n > = < \alpha x_1, \alpha x_2, \cdots, \alpha x_n > .$$

These operations have the properties we expect of vector addition and multiplication by scalars. If **F**, **G**, and **H** are in R^n and α an β are real numbers, then

1. $\mathbf{F} + \mathbf{G} = \mathbf{G} + \mathbf{F}$.
2. $\mathbf{F} + (\mathbf{G} + \mathbf{H}) = (\mathbf{F} + \mathbf{G}) + \mathbf{H}$.
3. $\mathbf{F} + \mathbf{O} = \mathbf{F}$,