FIFTH EDITION

Engineering Mathematics

John Bird

Newnes

Chapter 49

Integration using algebraic substitutions

49.1 Introduction

Functions that require integrating are not always in the 'standard form' shown in Chapter 48. However, it is often possible to change a function into a form which can be integrated by using either:

(i) an algebraic substitution (see Section 49.2),

(ii) trigonometric substitutions (see Chapter 50),

(iii) partial fractions (see Chapter 51),

(iv) the $t = \tan\frac{\theta}{2}$ substitution (see Chapter 52), or

(v) integration by parts (see Chapter 53).

49.2 Algebraic substitutions

With **algebraic substitutions**, the substitution usually made is to let u be equal to $f(x)$ such that $f(u)\,du$ is a standard integral. It is found that integrals of the forms:

$$k\int [f(x)]^n f'(x)dx \quad \text{and} \quad k\int \frac{f'(x)^n}{[f(x)]}dx$$

(where k and n are constants) can both be integrated by substituting u for $f(x)$.

49.3 Worked problems on integration using algebraic substitutions

Problem 1. Determine $\int \cos(3x+7)\,dx$

$\int \cos(3x+7)\,dx$ is not a standard integral of the form shown in Table 48.1, page 436, thus an algebraic substitution is made.

Let $u = 3x+7$ then $\frac{du}{dx} = 3$ and rearranging gives $dx = \frac{du}{3}$

Hence $\int \cos(3x+7)\,dx = \int (\cos u)\frac{du}{3}$

$$= \int \frac{1}{3}\cos u\,du,$$

which is a standard integral

$$= \frac{1}{3}\sin u + c$$

Rewriting u as $(3x+7)$ gives:

$$\int \cos(3x+7)\,dx = \mathbf{\frac{1}{3}\sin(3x+7) + c},$$

which may be checked by differentiating it.

Problem 2. Find: $\int (2x-5)^7 dx$

$(2x-5)$ may be multiplied by itself 7 times and then each term of the result integrated. However, this would be a lengthy process, and thus an algebraic substitution is made.

Let $u = (2x-5)$ then $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$

Hence

$$\int (2x-5)^7 dx = \int u^7\frac{du}{2} = \frac{1}{2}\int u^7 du$$

$$= \frac{1}{2}\left(\frac{u^8}{8}\right) + c = \frac{1}{16}u^8 + c$$

Rewriting u as $(2x-5)$ gives:

$$\int (2x-5)^7 dx = \frac{1}{16}(2x-5)^8 + c$$

Problem 3. Find: $\displaystyle\int \frac{4}{(5x-3)}dx$

Let $u=(5x-3)$ then $\dfrac{du}{dx}=5$ and $dx=\dfrac{du}{5}$

Hence $\displaystyle\int \frac{4}{(5x-3)}dx = \int \frac{4}{u}\frac{du}{5} = \frac{4}{5}\int \frac{1}{u}du$

$$= \frac{4}{5}\ln u + c$$

$$= \frac{4}{5}\ln(5x-3) + c$$

Problem 4. Evaluate $\displaystyle\int_0^1 2e^{6x-1}dx$, correct to 4 significant figures

Let $u=6x-1$ then $\dfrac{du}{dx}=6$ and $dx=\dfrac{du}{6}$

Hence $\displaystyle\int 2e^{6x-1}dx = \int 2e^u\frac{du}{6} = \frac{1}{3}\int e^u du$

$$= \frac{1}{3}e^u + c = \frac{1}{3}e^{6x-1} + c$$

Thus $\displaystyle\int_0^1 2e^{6x-1}dx = \frac{1}{3}\left[e^{6x-1}\right]_0^1$

$$= \frac{1}{3}[e^5 - e^{-1}] = \mathbf{49.35},$$

correct to 4 significant figures.

Problem 5. Determine: $\displaystyle\int 3x(4x^2+3)^5 dx$

Let $u=(4x^2+3)$ then $\dfrac{du}{dx}=8x$ and $dx=\dfrac{du}{8x}$

Hence

$$\int 3x(4x^2+3)^5 dx = \int 3x(u)^5\frac{du}{8x}$$

$$= \frac{3}{8}\int u^5 du, \text{ by cancelling}$$

The original variable 'x' has been completely removed and the integral is now only in terms of u and is a standard integral.

Hence $\displaystyle\frac{3}{8}\int u^5 du = \frac{3}{8}\left(\frac{u^6}{6}\right) + c = \frac{1}{16}u^6 + c$

$$= \frac{1}{16}(4x^2+3)^6 + c$$

Problem 6. Evaluate: $\displaystyle\int_0^{\pi/6} 24\sin^5\theta\cos\theta\, d\theta$

Let $u=\sin\theta$ then $\dfrac{du}{d\theta}=\cos\theta$ and $d\theta=\dfrac{du}{\cos\theta}$

Hence $\displaystyle\int 24\sin^5\theta\cos\theta\, d\theta$

$$= \int 24u^5\cos\theta\frac{du}{\cos\theta}$$

$$= 24\int u^5 du, \text{ by cancelling}$$

$$= 24\frac{u^6}{6} + c = 4u^6 + c = 4(\sin\theta)^6 + c$$

$$= 4\sin^6\theta + c$$

Thus $\displaystyle\int_0^{\pi/6} 24\sin^5\theta\cos\theta\, d\theta$

$$= \left[4\sin^6\theta\right]_0^{\pi/6} = 4\left[\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right]$$

$$= 4\left[\left(\frac{1}{2}\right)^6 - 0\right] = \frac{1}{16} \text{ or } \mathbf{0.0625}$$

Now try the following exercise

Exercise 174 Further problems on integration using algebraic substitutions

In Problems 1 to 6, integrate with respect to the variable.

1. $2\sin(4x+9)$ $\quad\left[-\frac{1}{2}\cos(4x+9)+c\right]$

2. $3\cos(2\theta-5)$ $\quad\left[\frac{3}{2}\sin(2\theta-5)+c\right]$

3. $4\sec^2(3t+1)$ $\left[\frac{4}{3}\tan(3t+1)+c\right]$

4. $\frac{1}{2}(5x-3)^6$ $\left[\frac{1}{70}(5x-3)^7+c\right]$

5. $\frac{-3}{(2x-1)}$ $\left[-\frac{3}{2}\ln(2x-1)+c\right]$

6. $3e^{3\theta+5}$ $[e^{3\theta+5}+c]$

In Problems 7 to 10, evaluate the definite integrals correct to 4 significant figures.

7. $\int_0^1 (3x+1)^5 dx$ [227.5]

8. $\int_0^2 x\sqrt{2x^2+1}\,dx$ [4.333]

9. $\int_0^{\pi/3} 2\sin\left(3t+\frac{\pi}{4}\right) dt$ [0.9428]

10. $\int_0^1 3\cos(4x-3)dx$ [0.7369]

49.4 Further worked problems on integration using algebraic substitutions

Problem 7. Find: $\int \frac{x}{2+3x^2}dx$

Let $u=2+3x^2$ then $\frac{du}{dx}=6x$ and $dx=\frac{du}{6x}$

Hence $\int \frac{x}{2+3x^2}dx$

$$= \int \frac{x}{u}\frac{du}{6x} = \frac{1}{6}\int \frac{1}{u}du, \text{ by cancelling,}$$

$$= \frac{1}{6}\ln u + x$$

$$= \mathbf{\frac{1}{6}\ln(2+3x^2)+c}$$

Problem 8. Determine: $\int \frac{2x}{\sqrt{4x^2-1}}dx$

Let $u=4x^2-1$ then $\frac{du}{dx}=8x$ and $dx=\frac{du}{8x}$

Hence $\int \frac{2x}{\sqrt{4x^2-1}}dx$

$$= \int \frac{2x}{\sqrt{u}}\frac{du}{8x} = \frac{1}{4}\int \frac{1}{\sqrt{u}}du, \text{ by cancelling}$$

$$= \frac{1}{4}\int u^{-1/2}du$$

$$= \frac{1}{4}\left[\frac{u^{(-1/2)+1}}{-\frac{1}{2}+1}\right]+c = \frac{1}{4}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]+c$$

$$= \frac{1}{2}\sqrt{u}+c = \mathbf{\frac{1}{2}\sqrt{4x^2-1}+c}$$

Problem 9. Show that:

$$\int \tan\theta\, d\theta = \ln(\sec\theta)+c$$

$$\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta}d\theta.$$

Let u = $\cos\theta$

then $\frac{du}{d\theta}=-\sin\theta$ and $d\theta=\frac{-du}{\sin\theta}$

Hence

$$\int \frac{\sin\theta}{\cos\theta}d\theta = \int \frac{\sin\theta}{u}\left(\frac{-du}{\sin\theta}\right)$$

$$= -\int \frac{1}{u}du = -\ln u + c$$

$$= -\ln(\cos\theta)+c$$

$$= \ln(\cos\theta)^{-1}+c,$$

by the laws of logarithms

Hence $\int \tan\theta\, d\theta = \ln(\sec\theta)+c$,

since $(\cos\theta)^{-1} = \frac{1}{\cos\theta} = \sec\theta$

49.5 Change of limits

When evaluating definite integrals involving substitutions it is sometimes more convenient to **change the limits** of the integral as shown in Problems 10 and 11.

Problem 10. Evaluate: $\int_1^3 5x\sqrt{2x^2+7}\,dx$, taking positive values of square roots only

Let $u=2x^2+7$, then $\frac{du}{dx}=4x$ and $dx=\frac{du}{4x}$

It is possible in this case to change the limits of integration. Thus when $x=3$, $u=2(3)^2+7=25$ and when $x=1$, $u=2(1)^2+7=9$

Hence $\int_{x=1}^{x=3} 5x\sqrt{2x^2+7}\,dx$

$$= \int_{u=9}^{u=25} 5x\sqrt{u}\,\frac{du}{4x} = \frac{5}{4}\int_9^{25}\sqrt{u}\,du$$

$$= \frac{5}{4}\int_9^{25} u^{1/2}du$$

Thus the limits have been changed, and it is unnecessary to change the integral back in terms of x.

Thus $\int_{x=1}^{x=3} 5x\sqrt{2x^2+7}\,dx$

$$= \frac{5}{4}\left[\frac{u^{3/2}}{3/2}\right]_9^{25} = \frac{5}{6}\left[\sqrt{u^3}\right]_9^{25}$$

$$= \frac{5}{6}[\sqrt{25^3}-\sqrt{9^3}] = \frac{5}{6}(125-27) = \mathbf{81\frac{2}{3}}$$

Problem 11. Evaluate: $\int_0^2 \frac{3x}{\sqrt{2x^2+1}}\,dx$, taking positive values of square roots only

Let $u=2x^2+1$ then $\frac{du}{dx}=4x$ and $dx=\frac{du}{4x}$

Hence $\int_0^2 \frac{3x}{\sqrt{2x^2+1}}\,dx = \int_{x=0}^{x=2}\frac{3x}{\sqrt{u}}\,\frac{du}{4x}$

$$= \frac{3}{4}\int_{x=0}^{x=2} u^{-1/2}du$$

Since $u=2x^2+1$, when $x=2$, $u=9$ and when $x=0$, $u=1$

Thus $\frac{3}{4}\int_{x=0}^{x=2} u^{-1/2}\,du = \frac{3}{4}\int_{u=1}^{u=9} u^{-1/2}\,du$,

i.e. the limits have been changed

$$= \frac{3}{4}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]_1^9 = \frac{3}{2}[\sqrt{9}-\sqrt{1}] = \mathbf{3},$$

taking positive values of square roots only.

Now try the following exercise

Exercise 175 Further problems on integration using algebraic substitutions

In Problems 1 to 7, integrate with respect to the variable.

1. $2x(2x^2-3)^5$ $\left[\frac{1}{12}(2x^2-3)^6+c\right]$

2. $5\cos^5 t\sin t$ $\left[-\frac{5}{6}\cos^6 t+c\right]$

3. $3\sec^2 3x\tan 3x$
$\left[\frac{1}{2}\sec^2 3x+c \text{ or } \frac{1}{2}\tan^2 3x+c\right]$

4. $2t\sqrt{3t^2-1}$ $\left[\frac{2}{9}\sqrt{(3t^2-1)^3}+c\right]$

5. $\frac{\ln\theta}{\theta}$ $\left[\frac{1}{2}(\ln\theta)^2+c\right]$

6. $3\tan 2t$ $\left[\frac{3}{2}\ln(\sec 2t)+c\right]$

7. $\frac{2e^t}{\sqrt{e^t+4}}$ $[4\sqrt{e^t+4}+c]$

In Problems 8 to 10, evaluate the definite integrals correct to 4 significant figures.

8. $\int_0^1 3xe^{(2x^2-1)}dx$ [1.763]

9. $\int_0^{\pi/2} 3\sin^4\theta\cos\theta\,d\theta$ [0.6000]

10. $\int_0^1 \frac{3x}{(4x^2-1)^5}dx$ [0.09259]

11. The electrostatic potential on all parts of a conducting circular disc of radius r is given by the equation:

$$V = 2\pi\sigma \int_0^9 \frac{R}{\sqrt{R^2 + r^2}} dR$$

Solve the equation by determining the integral. $\left[V = 2\pi\sigma\left\{\sqrt{(9^2 + r^2)} - r\right\}\right]$

12. In the study of a rigid rotor the following integration occurs:

$$Z_r = \int_0^\infty (2J+1)\, e^{\frac{-J(J+1)\,h^2}{8\pi^2 IkT}}\, dJ$$

Determine Z_r for constant temperature T assuming h, I and k are constants. $\left[\frac{8\pi^2 IkT}{h^2}\right]$

13. In electrostatics,

$$E = \int_0^\pi \left\{ \frac{a^2\sigma \sin\theta}{2\,\varepsilon\sqrt{(a^2 - x^2 - 2ax\cos\theta)}} d\theta \right\}$$

where a, σ and ε are constants, x is greater than a, and x is independent of θ. Show that $E = \frac{a^2\sigma}{\varepsilon x}$