Fifth edition

HIGHER ENGINEERING MATHEMATICS

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Newnes

8. (a) $\int \frac{3}{4} \sec^2 3x \, dx$ (b) $\int 2 \operatorname{cosec}^2 4\theta \, d\theta$

$\left[\text{(a)} \frac{1}{4} \tan 3x + c \ \ \text{(b)} -\frac{1}{2} \cot 4\theta + c \right]$

9. (a) $5 \int \cot 2t \operatorname{cosec} 2t \, dt$

(b) $\int \frac{4}{3} \sec 4t \tan 4t \, dt$

$\left[\begin{array}{l} \text{(a)} -\frac{5}{2} \operatorname{cosec} 2t + c \\ \text{(b)} \frac{1}{3} \sec 4t + c \end{array} \right]$

10. (a) $\int \frac{3}{4} e^{2x} \, dx$ (b) $\frac{2}{3} \int \frac{dx}{e^{5x}}$

$\left[\text{(a)} \frac{3}{8} e^{2x} + c \ \ \text{(b)} \frac{-2}{15 e^{5x}} + c \right]$

11. (a) $\int \frac{2}{3x} \, dx$ (b) $\int \left(\frac{u^2 - 1}{u} \right) du$

$\left[\text{(a)} \frac{2}{3} \ln x + c \ \ \text{(b)} \frac{u^2}{2} - \ln u + c \right]$

12. (a) $\int \frac{(2+3x)^2}{\sqrt{x}} \, dx$ (b) $\int \left(\frac{1}{t} + 2t \right)^2 dt$

$\left[\begin{array}{l} \text{(a)} 8\sqrt{x} + 8\sqrt{x^3} + \frac{18}{5}\sqrt{x^5} + c \\ \text{(b)} -\frac{1}{t} + 4t + \frac{4t^3}{3} + c \end{array} \right]$

37.4 Definite integrals

Integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information. **Definite integrals** are those in which limits are applied. If an expression is written as $[x]_a^b$, 'b' is called the upper limit and 'a' the lower limit.
The operation of applying the limits is defined as $[x]_a^b = (b) - (a)$.
The increase in the value of the integral x^2 as x increases from 1 to 3 is written as $\int_1^3 x^2 \, dx$.

Applying the limits gives:

$$\int_1^3 x^2 \, dx = \left[\frac{x^3}{3} + c \right]_1^3 = \left(\frac{3^3}{3} + c \right) - \left(\frac{1^3}{3} + c \right)$$
$$= (9 + c) - \left(\frac{1}{3} + c \right) = \mathbf{8\frac{2}{3}}$$

Note that the 'c' term always cancels out when limits are applied and it need not be shown with definite integrals.

Problem 12. Evaluate
(a) $\int_1^2 3x \, dx$ (b) $\int_{-2}^3 (4 - x^2) \, dx$.

(a) $$\int_1^2 3x \, dx = \left[\frac{3x^2}{2} \right]_1^2 = \left\{ \frac{3}{2}(2)^2 \right\} - \left\{ \frac{3}{2}(1)^2 \right\}$$
$$= 6 - 1\frac{1}{2} = \mathbf{4\frac{1}{2}}$$

(b) $$\int_{-2}^3 (4 - x^2) \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^3$$
$$= \left\{ 4(3) - \frac{(3)^3}{3} \right\} - \left\{ 4(-2) - \frac{(-2)^3}{3} \right\}$$
$$= \{12 - 9\} - \left\{ -8 - \frac{-8}{3} \right\}$$
$$= \{3\} - \left\{ -5\frac{1}{3} \right\} = \mathbf{8\frac{1}{3}}$$

Problem 13. Evaluate $\int_1^4 \left(\frac{\theta + 2}{\sqrt{\theta}} \right) d\theta$, taking positive square roots only.

$$\int_1^4 \left(\frac{\theta + 2}{\sqrt{\theta}} \right) d\theta = \int_1^4 \left(\frac{\theta}{\theta^{\frac{1}{2}}} + \frac{2}{\theta^{\frac{1}{2}}} \right) d\theta$$
$$= \int_1^4 \left(\theta^{\frac{1}{2}} + 2\theta^{\frac{-1}{2}} \right) d\theta$$
$$= \left[\frac{\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2} + 1} + \frac{2\theta^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2} + 1} \right]_1^4$$

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$$= \left[\frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2\theta^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 = \left[\frac{2}{3}\sqrt{\theta^3} + 4\sqrt{\theta} \right]_1^4$$

$$= \left\{ \frac{2}{3}\sqrt{(4)^3} + 4\sqrt{4} \right\} - \left\{ \frac{2}{3}\sqrt{(1)^3} + 4\sqrt{(1)} \right\}$$

$$= \left\{ \frac{16}{3} + 8 \right\} - \left\{ \frac{2}{3} + 4 \right\}$$

$$= 5\frac{1}{3} + 8 - \frac{2}{3} - 4 = \mathbf{8\frac{2}{3}}$$

Problem 14. Evaluate $\int_0^{\frac{\pi}{2}} 3\sin 2x\,dx$.

$$\int_0^{\frac{\pi}{2}} 3\sin 2x\,dx$$

$$= \left[(3)\left(-\frac{1}{2}\right)\cos 2x \right]_0^{\frac{\pi}{2}} = \left[-\frac{3}{2}\cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \left\{ -\frac{3}{2}\cos 2\left(\frac{\pi}{2}\right) \right\} - \left\{ -\frac{3}{2}\cos 2(0) \right\}$$

$$= \left\{ -\frac{3}{2}\cos \pi \right\} - \left\{ -\frac{3}{2}\cos 0 \right\}$$

$$= \left\{ -\frac{3}{2}(-1) \right\} - \left\{ -\frac{3}{2}(1) \right\} = \frac{3}{2} + \frac{3}{2} = \mathbf{3}$$

Problem 15. Evaluate $\int_1^2 4\cos 3t\,dt$.

$$\int_1^2 4\cos 3t\,dt = \left[(4)\left(\frac{1}{3}\right)\sin 3t \right]_1^2 = \left[\frac{4}{3}\sin 3t \right]_1^2$$

$$= \left\{ \frac{4}{3}\sin 6 \right\} - \left\{ \frac{4}{3}\sin 3 \right\}$$

Note that limits of trigonometric functions are always expressed in radians—thus, for example, sin 6 means the sine of 6 radians $= -0.279415\ldots$

Hence $\int_1^2 4\cos 3t\,dt$

$$= \left\{ \frac{4}{3}(-0.279415\ldots) \right\} - \left\{ \frac{4}{3}(0.141120\ldots) \right\}$$

$$= (-0.37255) - (0.18816) = \mathbf{-0.5607}$$

Problem 16. Evaluate

(a) $\int_1^2 4\,e^{2x}\,dx$ (b) $\int_1^4 \frac{3}{4u}\,du$,

each correct to 4 significant figures.

(a) $\int_1^2 4\,e^{2x}\,dx = \left[\frac{4}{2} e^{2x} \right]_1^2 = 2[e^{2x}]_1^2 = 2[e^4 - e^2]$

$$= 2[54.5982 - 7.3891] = \mathbf{94.42}$$

(b) $\int_1^4 \frac{3}{4u}\,du = \left[\frac{3}{4}\ln u \right]_1^4 = \frac{3}{4}[\ln 4 - \ln 1]$

$$= \frac{3}{4}[1.3863 - 0] = \mathbf{1.040}$$

Now try the following exercise.

Exercise 147 Further problems on definite integrals

In problems 1 to 8, evaluate the definite integrals (where necessary, correct to 4 significant figures).

1. (a) $\int_1^4 5x^2\,dx$ (b) $\int_{-1}^1 -\frac{3}{4}t^2\,dt$

$\left[\text{(a) } 105 \quad \text{(b) } -\frac{1}{2}\right]$

2. (a) $\int_{-1}^2 (3 - x^2)\,dx$ (b) $\int_1^3 (x^2 - 4x + 3)\,dx$

$\left[\text{(a) } 6 \quad \text{(b)} -1\frac{1}{3}\right]$

3. (a) $\int_0^{\pi} \frac{3}{2}\cos\theta\,d\theta$ (b) $\int_0^{\frac{\pi}{2}} 4\cos\theta\,d\theta$

[(a) 0 (b) 4]

4. (a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin 2\theta\,d\theta$ (b) $\int_0^2 3\sin t\,dt$

[(a) 1 (b) 4.248]

5. (a) $\int_0^1 5\cos 3x\,dx$ (b) $\int_0^{\frac{\pi}{6}} 3\sec^2 2x\,dx$

[(a) 0.2352 (b) 2.598]

6. (a) $\displaystyle\int_1^2 \operatorname{cosec}^2 4t\,\mathrm{d}t$

   (b) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (3\sin 2x - 2\cos 3x)\,\mathrm{d}x$

   [(a) 0.2527 (b) 2.638]

7. (a) $\displaystyle\int_0^1 3\,\mathrm{e}^{3t}\,\mathrm{d}t$ (b) $\displaystyle\int_{-1}^2 \frac{2}{3\,\mathrm{e}^{2x}}\,\mathrm{d}x$

   [(a) 19.09 (b) 2.457]

8. (a) $\displaystyle\int_2^3 \frac{2}{3x}\,\mathrm{d}x$ (b) $\displaystyle\int_1^3 \frac{2x^2+1}{x}\,\mathrm{d}x$

   [(a) 0.2703 (b) 9.099]

9. The entropy change ΔS, for an ideal gas is given by:

$$\Delta S = \int_{T_1}^{T_2} C_v \frac{\mathrm{d}T}{T} - R\int_{V_1}^{V_2} \frac{\mathrm{d}V}{V}$$

   where T is the thermodynamic temperature, V is the volume and $R = 8.314$. Determine the entropy change when a gas expands from 1 litre to 3 litres for a temperature rise from 100 K to 400 K given that:

$$C_v = 45 + 6\times 10^{-3}T + 8\times 10^{-6}T^2$$

   [55.65]

10. The p.d. between boundaries a and b of an electric field is given by: $V = \displaystyle\int_a^b \frac{Q}{2\pi r\varepsilon_0\varepsilon_r}\,\mathrm{d}r$

    If $a = 10$, $b = 20$, $Q = 2\times 10^{-6}$ coulombs, $\varepsilon_0 = 8.85\times 10^{-12}$ and $\varepsilon_r = 2.77$, show that $V = 9$ kV.

11. The average value of a complex voltage waveform is given by:

$$V_{AV} = \frac{1}{\pi}\int_0^{\pi} (10\sin\omega t + 3\sin 3\omega t + 2\sin 5\omega t)\,\mathrm{d}(\omega t)$$

    Evaluate V_{AV} correct to 2 decimal places.

    [7.26]

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Some applications of integration

38.1 Introduction

There are a number of applications of integral calculus in engineering. The determination of areas, mean and r.m.s. values, volumes, centroids and second moments of area and radius of gyration are included in this chapter.

38.2 Areas under and between curves

In Fig. 38.1,

$$\textbf{total shaded area} = \int_a^b f(x)\mathrm{d}x - \int_b^c f(x)\mathrm{d}x + \int_c^d f(x)\mathrm{d}x$$

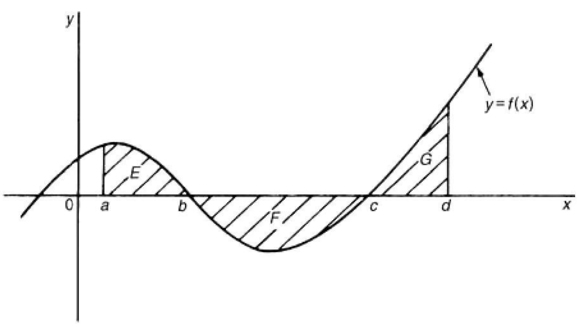

Figure 38.1

Problem 1. Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis.

$y = x^3 - 2x^2 - 8x = x(x^2 - 2x - 8) = x(x+2)(x-4)$

When $y = 0$, $x = 0$ or $(x + 2) = 0$ or $(x - 4) = 0$, i.e. when $y = 0$, $x = 0$ or -2 or 4, which means that the curve crosses the x-axis at 0, -2, and 4. Since the curve is a continuous function, only one other co-ordinate value needs to be calculated before a sketch of the curve can be produced. When $x = 1$, $y = -9$, showing that the part of the curve between $x = 0$ and $x = 4$ is negative. A sketch of $y = x^3 - 2x^2 - 8x$ is shown in Fig. 38.2. (Another method of sketching Fig. 38.2 would have been to draw up a table of values).

Shaded area

$$= \int_{-2}^{0} (x^3 - 2x^2 - 8x)\mathrm{d}x - \int_0^4 (x^3 - 2x^2 - 8x)\mathrm{d}x$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_0^4$$

$$= \left(6\frac{2}{3}\right) - \left(-42\frac{2}{3}\right) = \mathbf{49\frac{1}{3}\ square\ units}$$

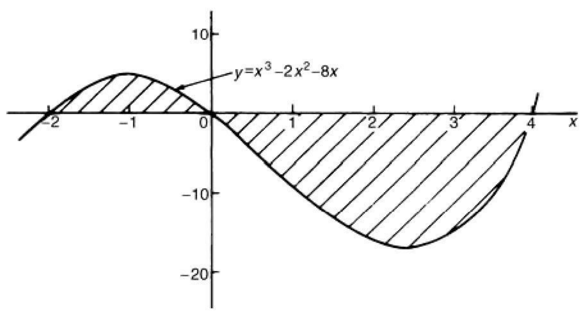

Figure 38.2

Problem 2. Determine the area enclosed between the curves $y = x^2 + 1$ and $y = 7 - x$.

At the points of intersection the curves are equal. Thus, equating the y values of each curve gives:

$$x^2 + 1 = 7 - x$$

from which, $x^2 + x - 6 = 0$

Factorising gives $(x - 2)(x + 3) = 0$

from which $x = 2$ and $x = -3$

By firstly determining the points of intersection the range of x-values has been found. Tables of values are produced as shown below.

x	-3	-2	-1	0	1	2
$y = x^2 + 1$	10	5	2	1	2	5

x	-3	0	2
$y = 7 - x$	10	7	5

A sketch of the two curves is shown in Fig. 38.3.

$$\text{Shaded area} = \int_{-3}^{2} (7 - x)\,dx - \int_{-3}^{2} (x^2 + 1)\,dx$$

$$= \int_{-3}^{2} [(7 - x) - (x^2 + 1)]\,dx$$

$$= \int_{-3}^{2} (6 - x - x^2)\,dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$$

$$= \left(7\frac{1}{3}\right) - \left(-13\frac{1}{2}\right)$$

$$= \mathbf{20\frac{5}{6}\ \textbf{square units}}$$

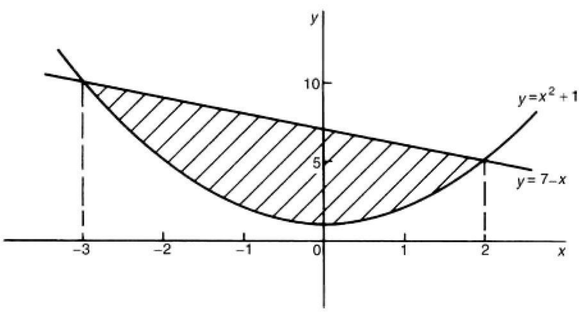

Figure 38.3

Problem 3. Determine by integration the area bounded by the three straight lines $y = 4 - x$, $y = 3x$ and $3y = x$.

Each of the straight lines are shown sketched in Fig. 38.4.

Shaded area

$$= \int_{0}^{1} \left(3x - \frac{x}{3}\right) dx + \int_{1}^{3} \left[(4 - x) - \frac{x}{3}\right] dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^2}{6}\right]_0^1 + \left[4x - \frac{x^2}{2} - \frac{x^2}{6}\right]_1^3$$

$$= \left[\left(\frac{3}{2} - \frac{1}{6}\right) - (0)\right] + \left[\left(12 - \frac{9}{2} - \frac{9}{6}\right) - \left(4 - \frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \left(1\frac{1}{3}\right) + \left(6 - 3\frac{1}{3}\right) = \mathbf{4\ \textbf{square units}}$$

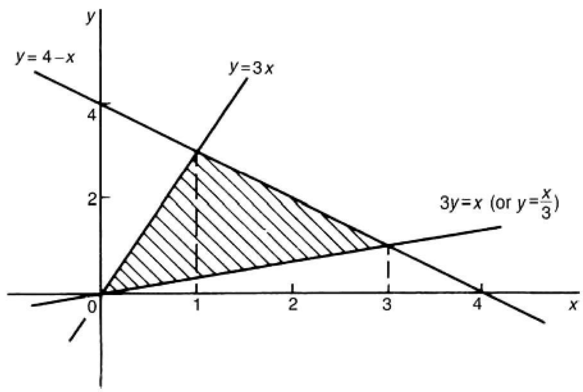

Figure 38.4

Now try the following exercise.

Exercise 148 Further problems on areas under and between curves

1. Find the area enclosed by the curve $y = 4\cos 3x$, the x-axis and ordinates $x = 0$ and $x = \dfrac{\pi}{6}$ [$1\frac{1}{3}$ square units]
2. Sketch the curves $y = x^2 + 3$ and $y = 7 - 3x$ and determine the area enclosed by them. [$20\frac{5}{6}$ square units]
3. Determine the area enclosed by the three straight lines $y = 3x$, $2y = x$ and $y + 2x = 5$. [$2\frac{1}{2}$ square units]

38.3 Mean and r.m.s. values

With reference to Fig. 38.5,

$$\textbf{mean value}, \bar{y} = \frac{1}{b-a}\int_a^b y\,dx$$

and $$\textbf{r.m.s. value} = \sqrt{\left\{\frac{1}{b-a}\int_a^b y^2\,dx\right\}}$$

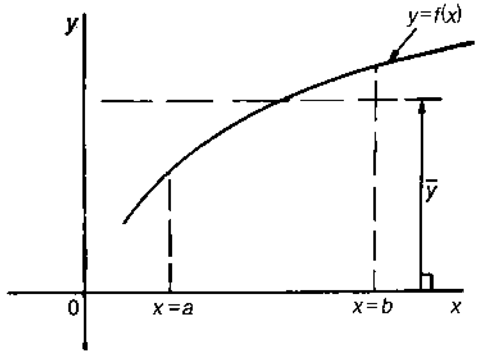

Figure 38.5

> Problem 4. A sinusoidal voltage $v = 100\sin\omega t$ volts. Use integration to determine over half a cycle (a) the mean value, and (b) the r.m.s. value.

(a) Half a cycle means the limits are 0 to π radians.

$$\text{Mean value, } \bar{y} = \frac{1}{\pi - 0}\int_0^\pi v\,d(\omega t)$$
$$= \frac{1}{\pi}\int_0^\pi 100\sin\omega t\,d(\omega t)$$
$$= \frac{100}{\pi}[-\cos\omega t]_0^\pi$$
$$= \frac{100}{\pi}[(-\cos\pi) - (-\cos 0)]$$
$$= \frac{100}{\pi}[(+1) - (-1)] = \frac{200}{\pi}$$
$$= \mathbf{63.66\ volts}$$

[Note that for a sine wave,

$$\textbf{mean value} = \frac{2}{\pi} \times \textbf{maximum value}$$

In this case, mean value $= \frac{2}{\pi} \times 100 = 63.66\,\text{V}$]

(b) r.m.s. value

$$= \sqrt{\left\{\frac{1}{\pi - 0}\int_0^\pi v^2\,d(\omega t)\right\}}$$
$$= \sqrt{\left\{\frac{1}{\pi}\int_0^\pi (100\sin\omega t)^2\,d(\omega t)\right\}}$$
$$= \sqrt{\left\{\frac{10000}{\pi}\int_0^\pi \sin^2\omega t\,d(\omega t)\right\}},$$

which is not a 'standard' integral.

It is shown in Chapter 18 that $\cos 2A = 1 - 2\sin^2 A$ and this formula is used whenever $\sin^2 A$ needs to be integrated.

Rearranging $\cos 2A = 1 - 2\sin^2 A$ gives

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

Hence $\sqrt{\left\{\frac{10000}{\pi}\int_0^\pi \sin^2\omega t\,d(\omega t)\right\}}$

$$= \sqrt{\left\{\frac{10000}{\pi}\int_0^\pi \frac{1}{2}(1 - \cos 2\omega t)\,d(\omega t)\right\}}$$
$$= \sqrt{\left\{\frac{10000}{\pi}\frac{1}{2}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_0^\pi\right\}}$$
$$= \sqrt{\left\{\frac{10000}{\pi}\frac{1}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]\right\}}$$
$$= \sqrt{\left\{\frac{10000}{\pi}\frac{1}{2}[\pi]\right\}}$$
$$= \sqrt{\left\{\frac{10000}{2}\right\}} = \frac{100}{\sqrt{2}} = \mathbf{70.71\ volts}$$

[Note that for a sine wave,

$$\textbf{r.m.s. value} = \frac{1}{\sqrt{2}} \times \textbf{maximum value}.$$

In this case,

$$\text{r.m.s. value} = \frac{1}{\sqrt{2}} \times 100 = 70.71\,\text{V}]$$