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CHAPTER 8

Analytic Geometry in Two and Three Dimensions

The oval-shaped lawn behind the White House in Washington, D.C. is called *the Ellipse*. It has views of the Washington Monument, the Jefferson Memorial, the Department of Commerce, and the Old Post Office Building. The Ellipse is 616 ft long, 528 ft wide, and is in the shape of a conic section. Its shape can be modeled using the methods of this chapter. See page 652.

8.4 Translation and Rotation of Axes

What you'll learn about

- Second-Degree Equations in Two Variables
- Translating Axes versus Translating Graphs
- Rotation of Axes
- Discriminant Test

. . . and why

You will see ellipses, hyperbolas, and parabolas as members of the family of conic sections rather than as separate types of curves.

OBJECTIVE

Students will be able to determine equations for translated and rotated axes for conic sections.

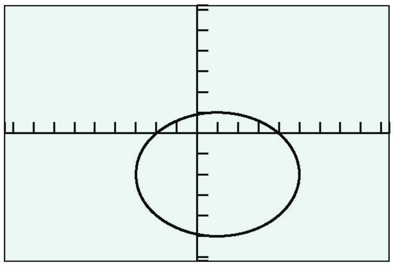
MOTIVATE

Have students solve $y^2 - 2xy + 2x^2 = 4$ for y and graph the result using a function grapher. Describe the graph.

LESSON GUIDE

Day 1: Second-Degree Equations in Two Variables; Translating Axes versus Translating Graphs

Day 2: Rotation of Axes; Discriminant Test



[–9.4, 9.4] by [–6.2, 6.2]

FIGURE 8.32 The graph of $9x^2 + 16y^2 - 18x + 64y - 71 = 0$. (Example 1)

Second-Degree Equations in Two Variables

In Section 8.1, we began with a unified approach to conic sections, learning that parabolas, ellipses, and hyperbolas are all cross sections of a right circular cone. In Sections 8.1–8.3, we gave separate plane-geometry definitions for parabolas, ellipses, and hyperbolas that led to separate kinds of equations for each type of curve. In this section and the next, we once again consider parabolas, ellipses, and hyperbolas as a unified family of interrelated curves.

In Section 8.1, we claimed that the conic sections can be defined algebraically in the Cartesian plane as the graphs of *second-degree equations in two variables*, that is, equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not all zero. In this section, we investigate equations of this type, which are really just *quadratic equations in x and y*. Because they are quadratic equations, we can adapt familiar methods to this unfamiliar setting. That is exactly what we do in Examples 1–3.

EXAMPLE 1 Graphing a Second-Degree Equation

Solve for y, and use a function grapher to graph

$$9x^2 + 16y^2 - 18x + 64y - 71 = 0.$$

SOLUTION Rearranging terms yields the equation:

$$16y^2 + 64y + (9x^2 - 18x - 71) = 0.$$

The quadratic formula gives us

$$y = \frac{-64 \pm \sqrt{64^2 - 4(16)(9x^2 - 18x - 71)}}{2(16)}$$

$$= \frac{-8 \pm 3\sqrt{-x^2 + 2x + 15}}{4}$$

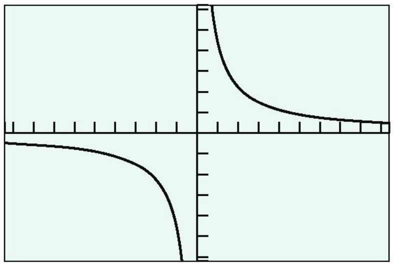
$$= -2 \pm \frac{3}{4}\sqrt{-x^2 + 2x + 15}$$

Let

$$\text{Y1} = -2 + 0.75\sqrt{-x^2 + 2x + 15} \text{ and Y2} = -2 - 0.75\sqrt{-x^2 + 2x + 15},$$

and graph the two equations in the same viewing window, as shown in Figure 8.32. The combined figure appears to be an ellipse.

Now try Exercise 1.



[–9.4, 9.4] by [–6.2, 6.2]

FIGURE 8.33 The graph of $2xy - 9 = 0$. (Example 2)

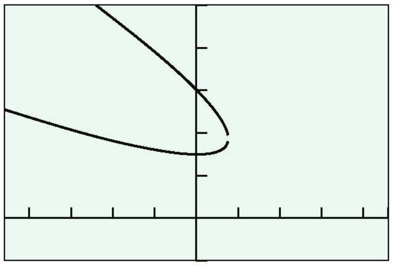
In the equation in Example 1, there was no Bxy term. None of the examples in Sections 8.1–8.3 included such a *cross-product* term. A cross-product term in the equation causes the graph to tilt relative to the coordinate axes, as illustrated in Examples 2 and 3.

EXAMPLE 2 Graphing a Second-Degree Equation

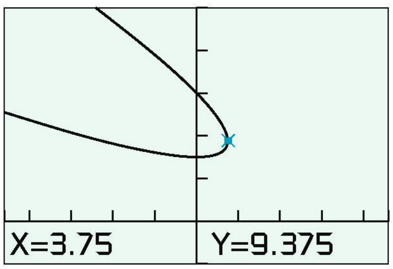
Solve for y, and use a function grapher to graph

$$2xy - 9 = 0.$$

SOLUTION This equation can be rewritten as $2xy = 9$ or as $y = 9/(2x)$. The graph of this equation is shown in Figure 8.33. It appears to be a hyperbola with a slant focal axis. *Now try Exercise 5.*



[–23, 23] by [–5, 25]
(a)



[–23, 23] by [–5, 25]
(b)

FIGURE 8.34 The graph of $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$ (a) with a gap and (b) with the trace feature activated at the connecting point. (Example 3)

EXAMPLE 3 Graphing a Second-Degree Equation

Solve for y, and use a function grapher to graph

$$x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0.$$

SOLUTION We rearrange the terms as a quadratic equation in y:

$$4y^2 + (4x - 90)y + (x^2 - 30x + 450) = 0.$$

The quadratic formula gives us

$$y = \frac{-(4x - 90) \pm \sqrt{(4x - 90)^2 - 4(4)(x^2 - 30x + 450)}}{2(4)}$$

$$= \frac{45 - 2x \pm \sqrt{225 - 60x}}{4}$$

Let

$$y_1 = \frac{45 - 2x + \sqrt{225 - 60x}}{4} \text{ and } y_2 = \frac{45 - 2x - \sqrt{225 - 60x}}{4},$$

and graph the two equations in the same viewing window, as shown in Figure 8.34a. The combined figure appears to be a parabola, with a slight gap due to grapher failure. The combined graph should connect at a point for which the radicand $225 - 60x = 0$, that is, when $x = 225/60 = 15/4 = 3.75$. Figure 8.34b supports this analysis. *Now try Exercise 9.*

The graphs obtained in Examples 1–3 all *appear* to be conic sections, but how can we be sure? If they are conics, then we probably have classified Examples 1 and 2 correctly, but couldn't the graph in Example 3 (Figure 8.34) be part of an ellipse or one branch of a hyperbola? We now set out to answer these questions and to develop methods for simplifying and classifying second-degree equations in two variables.

Translating Axes versus Translating Graphs

The coordinate axes are often viewed as a permanent fixture of the plane, but this just isn't so. We can shift the position of axes just as we have been shifting the position of graphs since Chapter 1. Such a **translation of axes** produces a new set of axes parallel to the original axes, as shown in Figure 8.35 on the next page.

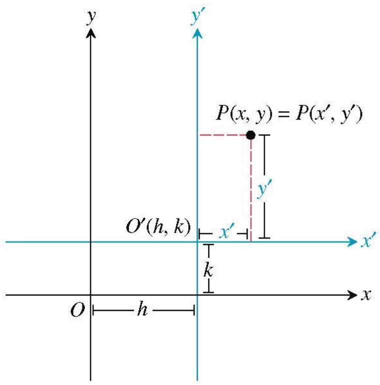


FIGURE 8.35 A translation of Cartesian coordinate axes.

Figure 8.35 shows a plane containing a point P that is named in two ways: using the coordinates (x, y) and the coordinates (x', y'). The coordinates (x, y) are based on the original x- and y-axes and the original origin O, while (x', y') are based on the translated x'- and y'-axes and the corresponding origin O'.

Translation-of-Axes Formulas

The coordinates (x, y) and (x', y') based on parallel sets of axes are related by either of the following **translation formulas**:

$$x = x' + h \quad \text{and} \quad y = y' + k$$

or

$$x' = x - h \quad \text{and} \quad y' = y - k.$$

We use the second pair of translation formulas in Example 4.

EXAMPLE 4 Revisiting Example 1

Prove that $9x^2 + 16y^2 - 18x + 64y - 71 = 0$ is the equation of an ellipse. Translate the coordinate axes so that the origin is at the center of this ellipse.

SOLUTION We complete the square of both x and y:

$$9x^2 - 18x + 16y^2 + 64y = 71$$
$$9(x^2 - 2x + 1) + 16(y^2 + 4y + 4) = 71 + 9(1) + 16(4)$$
$$9(x - 1)^2 + 16(y + 2)^2 = 144$$
$$\frac{(x - 1)^2}{16} + \frac{(y + 2)^2}{9} = 1$$

This is a standard equation of an ellipse. If we let $x' = x - 1$ and $y' = y + 2$, then the equation of the ellipse becomes

$$\frac{(x')^2}{16} + \frac{(y')^2}{9} = 1.$$

Figure 8.36 shows the graph of this final equation in the new $x'y'$ coordinate system, with the original xy-axes overlaid. Compare Figures 8.32 and 8.36.

Now try Exercise 21.

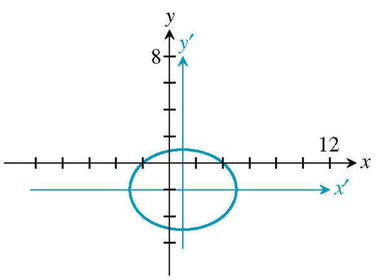


FIGURE 8.36 The graph of $(x')^2/16 + (y')^2/9 = 1$. (Example 4)

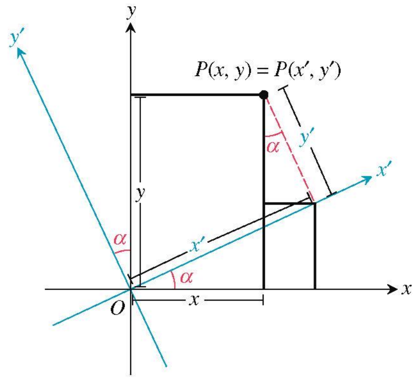


FIGURE 8.37 A rotation of Cartesian coordinate axes.

Rotation of Axes

To show that the equation in Example 2 or 3 is the equation of a conic section, we need to rotate the coordinate axes so that one axis aligns with the (focal) axis of the conic. In such a **rotation of axes**, the origin stays fixed, and we rotate the x- and y-axes through an angle α to obtain the x'- and y'-axes. (See Figure 8.37.)

Figure 8.37 shows a plane containing a point P named in two ways: as (x, y) and as (x', y'). The coordinates (x, y) are based on the original x- and y-axes, while (x', y') are based on the rotated x'- and y'-axes.

Rotation-of-Axes Formulas

The coordinates (x, y) and (x', y') based on rotated sets of axes are related by either of the following **rotation formulas**:

$$x' = x\cos\alpha + y\sin\alpha \quad \text{and} \quad y' = -x\sin\alpha + y\cos\alpha,$$

or

$$x = x'\cos\alpha - y'\sin\alpha \quad \text{and} \quad y = x'\sin\alpha + y'\cos\alpha.$$

where α, $0 < \alpha < \pi/2$, is the *angle of rotation*.

The first pair of equations was established in Example 10 of Section 7.2. The second pair can be derived directly from the geometry of Figure 8.37 (see Exercise 55) and is used in Example 5.

EXAMPLE 5 Revisiting Example 2

Prove that $2xy - 9 = 0$ is the equation of a hyperbola by rotating the coordinate axes through an angle $\alpha = \pi/4$.

SOLUTION Because $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, the rotation equations become

$$x = \frac{x' - y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x' + y'}{\sqrt{2}}.$$

So by rotating the axes, the equation $2xy - 9 = 0$ becomes

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 9 = 0$$

$$(x')^2 - (y')^2 - 9 = 0$$

To see that this is the equation of a hyperbola, we put it in standard form:

$$(x')^2 - (y')^2 = 9$$

$$\frac{(x')^2}{9} - \frac{(y')^2}{9} = 1$$

Figure 8.38 shows the graph of the original equation in the original xy system with the $x'y'$-axes overlaid. *Now try Exercise 37.*

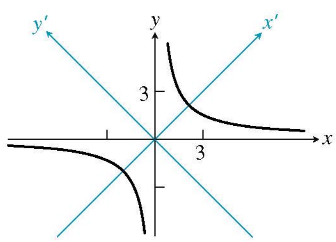


FIGURE 8.38 The graph of $2xy - 9 = 0$. (Example 5)

In Example 5 we converted a second-degree equation in x and y into a second-degree equation in x' and y' using the rotation formulas. By choosing the angle of rotation appropriately, there was no $x'y'$ cross-product term in the final equation, which allowed us to put it in standard form. We now generalize this process.

Coefficients for a Conic in a Rotated System

If we apply the rotation formulas to the general second-degree equation in x and y, we obtain a second-degree equation in x' and y' of the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$$

where the coefficients are

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$
$$B' = B\cos 2\alpha + (C - A)\sin 2\alpha$$
$$C' = C\cos^2\alpha - B\cos\alpha\sin\alpha + A\sin^2\alpha$$
$$D' = D\cos\alpha + E\sin\alpha$$
$$E' = E\cos\alpha - D\sin\alpha$$
$$F' = F$$

In order to eliminate the cross-product term and thus align the coordinate axes with the focal axis of the conic, we rotate the coordinate axes through an angle α that causes B' to equal 0. Setting $B' = B\cos 2\alpha + (C - A)\sin 2\alpha = 0$ leads to the following useful result.

Angle of Rotation to Eliminate the Cross-Product Term

If $B \neq 0$, an angle of rotation α such that

$$\cot 2\alpha = \frac{A - C}{B} \text{ and } 0 < \alpha < \frac{\pi}{2}$$

will eliminate the term $B'x'y'$ from the second-degree equation in the rotated $x'y'$ coordinate system.

EXAMPLE 6 Revisiting Example 3

Prove that $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$ is the equation of a parabola by rotating the coordinate axes through a suitable angle α.

SOLUTION The angle of rotation α must satisfy the equation

$$\cot 2\alpha = \frac{A - C}{B} = \frac{1 - 4}{4} = -\frac{3}{4}.$$

So

$$\cos 2\alpha = -\frac{3}{5},$$

continued

FOLLOW-UP

Ask students to explain why it was okay to assume $\cos \alpha > 0$ and $\sin \alpha > 0$ in Example 6.

and thus

$$\cos \alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}} = \sqrt{\frac{1 + (-3/5)}{2}} = \frac{1}{\sqrt{5}},$$

$$\sin \alpha = \sqrt{\frac{1 - \cos 2\alpha}{2}} = \sqrt{\frac{1 - (-3/5)}{2}} = \frac{2}{\sqrt{5}}.$$

Therefore the coefficients of the transformed equation are

$$A' = 1 \cdot \frac{1}{5} + 4 \cdot \frac{2}{5} + 4 \cdot \frac{4}{5} = \frac{25}{5} = 5$$

$$B' = 0$$

$$C' = 4 \cdot \frac{1}{5} - 4 \cdot \frac{2}{5} + 1 \cdot \frac{4}{5} = 0$$

$$D' = -30 \cdot \frac{1}{\sqrt{5}} - 90 \cdot \frac{2}{\sqrt{5}} = -\frac{210}{\sqrt{5}} = -42\sqrt{5}$$

$$E' = -90 \cdot \frac{1}{\sqrt{5}} + 30 \cdot \frac{2}{\sqrt{5}} = -\frac{30}{\sqrt{5}} = -6\sqrt{5}$$

$$F' = 450$$

So the equation $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$ becomes

$$5x'^2 - 42\sqrt{5}x' - 6\sqrt{5}y' + 450 = 0.$$

After completing the square of the x-terms, the equation becomes

$$\left(x' - \frac{21}{\sqrt{5}}\right)^2 = \frac{6}{\sqrt{5}}\left(y' - \frac{3\sqrt{5}}{10}\right).$$

If we translate using $h = 21/\sqrt{5}$ and $k = 3\sqrt{5}/10$, then the equation becomes

$$(x'')^2 = \frac{6}{\sqrt{5}}(y''),$$

a standard equation of a parabola.

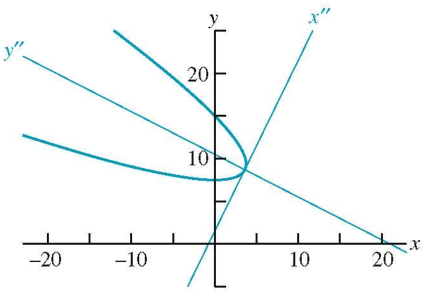


FIGURE 8.39 The graph of $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$. (Example 6)

Figure 8.39 shows the graph of the original equation in the original xy coordinate system, with the $x''y''$-axes overlaid. ***Now try Exercise 39.***

Discriminant Test

Example 6 demonstrates that the algebra of rotation can get ugly. Fortunately, we can determine which type of conic a second-degree equation represents by looking at the sign of the **discriminant** $B^2 - 4AC$.

Discriminant Test

The second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ graphs as

- a hyperbola if $B^2 - 4AC > 0$,
- a parabola if $B^2 - 4AC = 0$,
- an ellipse if $B^2 - 4AC < 0$,

except for degenerate cases.

ASSIGNMENT GUIDE

Day 1: Ex. 3–30, multiples of 3
Day 2: Ex. 33–54, multiples of 3

COOPERATIVE LEARNING

Group Activity: Ex. 65

NOTES ON EXERCISES

Ex. 31–32 ask students to prove the translation formulas.
Ex. 33–52 are basic problems involving rotation of axes.
Ex. 55–56 ask students to prove the rotation formulas.
Ex. 57–62 provide practice for standardized tests.

ONGOING ASSESSMENT

Self-Assessment: Ex. 1, 5, 9, 21, 37, 39, 43
Embedded Assessment: Ex. 31

This test hinges on the fact that the discriminant $B^2 - 4AC$ is **invariant under rotation**; in other words, even though A, B, and C do change when we rotate the coordinate axes, the combination $B^2 - 4AC$ maintains its value.

EXAMPLE 7 Revisiting Examples 5 and 6

(a) In Example 5, before the rotation $B^2 - 4AC = (2)^2 - 4(0)(0) = 4$, and after the rotation $B'^2 - 4A'C' = (0)^2 - 4(1)(-1) = 4$. The positive discriminant tells us the conic is a hyperbola.

(b) In Example 6, before the rotation $B^2 - 4AC = (4)^2 - 4(1)(4) = 0$, and after the rotation $B'^2 - 4A'C' = (0)^2 - 4(5)(0) = 0$. The zero discriminant tells us the conic is a parabola. ***Now try Exercise 43.***

Not only is the discriminant $B^2 - 4AC$ invariant under rotation, but also its *sign* is invariant under translation and under algebraic manipulations that preserve the equivalence of the equation, such as multiplying both sides of the equation by a nonzero constant.

The discriminant test can be applied to degenerate conics. Table 8.2 displays the three basic types of conic sections grouped with their associated degenerate conics. Each conic or degenerate conic is shown with a sample equation and the sign of its discriminant.

Table 8.2 Conics and the Equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Conic	Sample Equation	A	B	C	D	E	F	Sign of Discriminant
Hyperbola	$x^2 - 2y^2 = 1$	1		−2			−1	Positive
Intersecting lines	$x^2 + xy = 0$	1	1					Positive
Parabola	$x^2 = 2y$	1				−2		Zero
Parallel lines	$x^2 = 4$	1					−4	Zero
One line	$y^2 = 0$			1				Zero
No graph	$x^2 = -1$	1					1	Zero
Ellipse	$x^2 + 2y^2 = 1$	1		2			−1	Negative
Circle	$x^2 + y^2 = 9$	1		1			−9	Negative
Point	$x^2 + y^2 = 0$	1		1				Negative
No graph	$x^2 + y^2 = -1$	1		1			1	Negative

QUICK REVIEW 8.4 *(For help, go to Sections 4.7 and 5.4.)*

In Exercises 1–10, assume $0 \le \alpha < \pi/2$.

1. Given that $\cot 2\alpha = 5/12$, find $\cos 2\alpha$. $\cos 2\alpha = 5/13$

2. Given that $\cot 2\alpha = 8/15$, find $\cos 2\alpha$. $\cos 2\alpha = 8/17$

3. Given that $\cot 2\alpha = 1/\sqrt{3}$, find $\cos 2\alpha$. $\cos 2\alpha = 1/2$

4. Given that $\cot 2\alpha = 2/\sqrt{5}$, find $\cos 2\alpha$. $\cos 2\alpha = 2/3$

5. Given that $\cot 2\alpha = 0$, find α. $\alpha = \pi/4$

6. Given that $\cot 2\alpha = \sqrt{3}$, find α. $\alpha = \pi/12$

7. Given that $\cot 2\alpha = 3/4$, find $\cos \alpha$. $\cos \alpha = 2/\sqrt{5}$

8. Given that $\cot 2\alpha = 3/\sqrt{7}$, find $\cos \alpha$. $\cos \alpha = \sqrt{14}/4$

9. Given that $\cot 2\alpha = 5/\sqrt{11}$, find $\sin \alpha$. $\sin \alpha = 1/\sqrt{12}$

10. Given that $\cot 2\alpha = 45/28$, find $\sin \alpha$. $\sin \alpha = 2/\sqrt{53}$