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Claudio Canuto
Anita Tabacco

Mathematical Analysis I

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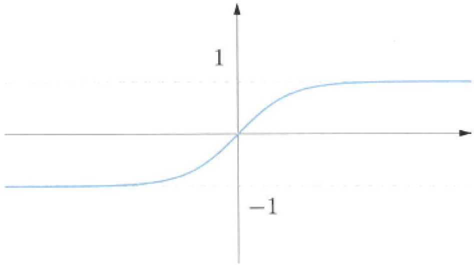

Figure 6.15. Hyperbolic tangent

The inverse map to the hyperbolic sine, appropriately called **inverse hyperbolic sine**, is defined on all of $\mathbb{R}$, and can be made explicit by means of the logarithm (inverse of the exponential)

$$\boxed{\sinh^{-1} x = \log(x + \sqrt{x^2+1}), \qquad x \in \mathbb{R}.} \tag{6.14}$$

There normally is no confusion with the reciprocal $1/\sinh x$, whence the use of notation[1]. The **inverse hyperbolic cosine** is obtained by inversion of the hyperbolic cosine restricted to $[0, +\infty)$

$$\boxed{\cosh^{-1} x = \log(x + \sqrt{x^2-1}), \qquad x \in [1, +\infty).} \tag{6.15}$$

To conclude, the **inverse hyperbolic tangent** inverts the corresponding hyperbolic map on $\mathbb{R}$

$$\boxed{\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}, \qquad x \in (-1, 1).} \tag{6.16}$$

The inverse hyperbolic functions have first derivatives

$$\begin{aligned} \mathrm{D}\sinh^{-1} x &= \frac{1}{\sqrt{x^2+1}}, \qquad \mathrm{D}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \\ \mathrm{D}\tanh^{-1} x &= \frac{1}{1-x^2}. \end{aligned} \tag{6.17}$$

6.11 The Theorem of de l'Hôpital

This final section is entirely devoted to a single result, due to its relevance in computing the limits of indeterminate forms. As always, c is one of x_0, x_0^+, x_0^-, $+\infty$, $-\infty$.

[1] Some authors also like the symbol Arc sinh.

Theorem 6.40 *Let f, g be maps defined on a neighbourhood of c, except possibly at c, and such that*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L,$$

where $L = 0, +\infty$ or $-\infty$. If f and g are differentiable around c, except possibly at c, with $g' \neq 0$, and if

$$\lim_{x\to c} \frac{f'(x)}{g'(x)}$$

exists (finite or not), then also

$$\lim_{x\to c} \frac{f(x)}{g(x)} \tag{6.18}$$

exists and equals the previous limit.

Proof. ⇝ De l'Hôpital's Theorem. □

Under said hypotheses the results states that

$$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.} \tag{6.19}$$

Examples 6.41

i) The limit

$$\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\sin 5x}$$

gives rise to an indeterminate form of type $\frac{0}{0}$. Since numerator and denominator are differentiable functions,

$$\lim_{x\to 0} \frac{2e^{2x} + 2e^{-2x}}{5\cos 5x} = \frac{4}{5}.$$

Therefore

$$\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\sin 5x} = \frac{4}{5}.$$

ii) When the ratio $f'(x)/g'(x)$ is still an indeterminate form, supposing f and g are twice differentiable around c, except maybe at c, we can iterate the recipe of (6.19) by studying the limit of $f''(x)/g''(x)$, and so on.

Consider for instance the indeterminate form $0/0$

$$\lim_{x\to 0} \frac{1+3x-\sqrt{(1+2x)^3}}{x\sin x}.$$

Differentiating numerator and denominator, we are lead to

$$\lim_{x\to 0} \frac{3-3\sqrt{1+2x}}{\sin x + x\cos x},$$

still of the form $0/0$. Thus we differentiate again

$$\lim_{x\to 0} \frac{-\frac{3}{\sqrt{1+2x}}}{2\cos x - x\sin x} = -\frac{3}{2}.$$

Applying (6.19) twice allows to conclude

$$\lim_{x\to 0} \frac{1+3x-\sqrt{(1+2x)^3}}{\sin^2 x} = -\frac{3}{2}.$$ □

Remark 6.42 De l'Hôpital's Theorem is a sufficient condition only, for the existence of (6.18). Otherwise said, it might happen that the limit of the derivatives' difference quotient does not exist, whereas we have the limit of the functions' difference quotient. For example, set $f(x) = x + \sin x$ and $g(x) = 2x + \cos x$. While the ratio f'/g' does not admit limit as $x \to +\infty$ (see Remark 4.19), the limit of f/g exists:

$$\lim_{x\to+\infty} \frac{x+\sin x}{2x+\cos x} = \lim_{x\to+\infty} \frac{x+o(x)}{2x+o(x)} = \frac{1}{2}.$$ □

6.11.1 Applications of de l'Hôpital's theorem

We survey some situations where the result of de l'Hôpital lends a helping hand.

Fundamental limits

By means of Theorem 6.40 we recover the important limits

$$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty, \qquad \lim_{x\to-\infty} |x|^\alpha e^x = 0, \qquad \forall \alpha \in \mathbb{R}, \tag{6.20}$$

$$\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = 0, \qquad \lim_{x\to 0^+} x^\alpha \log x = 0, \qquad \forall \alpha > 0. \tag{6.21}$$

These were presented in (5.6) in the equivalent formulation of the Landau symbols. Let us begin with the first of (6.20) when $\alpha = 1$. From (6.19)

$$\lim_{x\to+\infty} \frac{e^x}{x} = \lim_{x\to+\infty} \frac{e^x}{1} = +\infty.$$

For any other $\alpha > 0$, we have

$$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = \lim_{x\to+\infty} \left(\frac{1}{\alpha}\frac{e^{\frac{x}{\alpha}}}{\frac{x}{\alpha}}\right)^\alpha = \frac{1}{\alpha^\alpha}\left(\lim_{y\to+\infty} \frac{e^y}{y}\right)^\alpha = +\infty.$$

At last, for $\alpha \leq 0$ the result is rather trivial because there is no indeterminacy. As for the second formula of (6.20)

$$\lim_{x\to-\infty} |x|^\alpha e^x = \lim_{x\to-\infty} \frac{|x|^\alpha}{e^{-x}} = \lim_{x\to-\infty} \frac{|x|^\alpha}{e^{|x|}} = \lim_{y\to+\infty} \frac{y^\alpha}{e^y} = 0.$$

Now to (6.21):

$$\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \frac{1}{\alpha} \lim_{x\to+\infty} \frac{1}{x^\alpha} = 0$$

and

$$\lim_{x\to0+} x^\alpha \log x = \lim_{x\to0+} \frac{\log x}{x^{-\alpha}} = \lim_{x\to0+} \frac{\frac{1}{x}}{(-\alpha)x^{-\alpha-1}} = -\frac{1}{\alpha} \lim_{x\to0+} x^\alpha = 0.$$

Proof of Theorem 6.15

We are now in a position to prove this earlier claim.

Proof. By definition only,

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0};$$

but this is an indeterminate form, since

$$\lim_{x\to x_0} \big(f(x) - f(x_0)\big) = \lim_{x\to x_0} (x - x_0) = 0,$$

hence de l'Hôpital implies

$$f'(x_0) = \lim_{x\to x_0} \frac{f'(x)}{1}.$$

□

Computing the order of magnitude of a map

Through examples we explain how de l'Hôpital's result detects the order of magnitude of infinitesimal or infinite functions, and their principal parts.

The function

$$f(x) = e^x - 1 - \sin x$$

is infinitesimal for $x \to 0$. With infinitesimal test function $\varphi(x) = x$ we apply the theorem twice (supposing for a moment this is possible)

$$\lim_{x\to0} \frac{e^x - 1 - \sin x}{x^\alpha} = \lim_{x\to0} \frac{e^x - \cos x}{\alpha x^{\alpha-1}} = \lim_{x\to0} \frac{e^x + \sin x}{\alpha(\alpha-1)x^{\alpha-2}}.$$

When $\alpha = 2$ the right-most limit exists and is in fact $\frac{1}{2}$. This fact alone justifies the use of de l'Hôpital's Theorem. Thus $f(x)$ is infinitesimal of order 2 at the origin with respect to $\varphi(x) = x$; its principal part is $p(x) = \frac{1}{2}x^2$.

Next, consider

$$f(x) = \tan x,$$

an infinite function for $x \to \frac{\pi}{2}^-$. Setting $\varphi(x) = \dfrac{1}{\frac{\pi}{2} - x}$, we have

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\tan x}{\left(\frac{1}{\frac{\pi}{2}-x}\right)^\alpha} = \lim_{x \to \frac{\pi}{2}^-} \sin x \lim_{x \to \frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - x\right)^\alpha}{\cos x}.$$

While the first limit is 1, for the second we apply de l'Hôpital's Theorem

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - x\right)^\alpha}{\cos x} = \lim_{x \to \frac{\pi}{2}^-} \frac{-\alpha(\frac{\pi}{2} - x)^{\alpha-1}}{-\sin x}.$$

The latter equals 1 when $\alpha = 1$, so $\tan x$ is infinite of first order, for $x \to \frac{\pi}{2}^-$, with respect to $\varphi(x) = \dfrac{1}{\frac{\pi}{2} - x}$. The principal part is indeed $\varphi(x)$.

6.12 Exercises

1. *Discuss differentiability at the point x_0 indicated:*

   a) $f(x) = x + |x - 1|, \quad x_0 = 1$ b) $f(x) = \sin |x|, \quad x_0 = 0$

   c) $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}, \quad x_0 = 0$ d) $f(x) = \sqrt{1 + x^3}, \quad x_0 = -1$

2. *Say where the following maps are differentiable and find the derivatives:*

   a) $f(x) = x\sqrt{|x|}$ b) $f(x) = \cos |x|$

   c) $f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0, \\ e^x - x & \text{if } x < 0 \end{cases}$ d) $f(x) = \begin{cases} x^2 + x - 5 & \text{if } x \geq 1, \\ x - 4 & \text{if } x < 1 \end{cases}$

3. *Compute, where defined, the first derivative of:*

   a) $f(x) = 3x\sqrt[3]{1 + x^2}$ b) $f(x) = \log |\sin x|$

   c) $f(x) = \cos\left(e^{x^2+1}\right)$ d) $f(x) = \dfrac{1}{x \log x}$

4. *On the given interval, find maximum and minimum of:*

   a) $f(x) = \sin x + \cos x, \quad [0, 2\pi]$