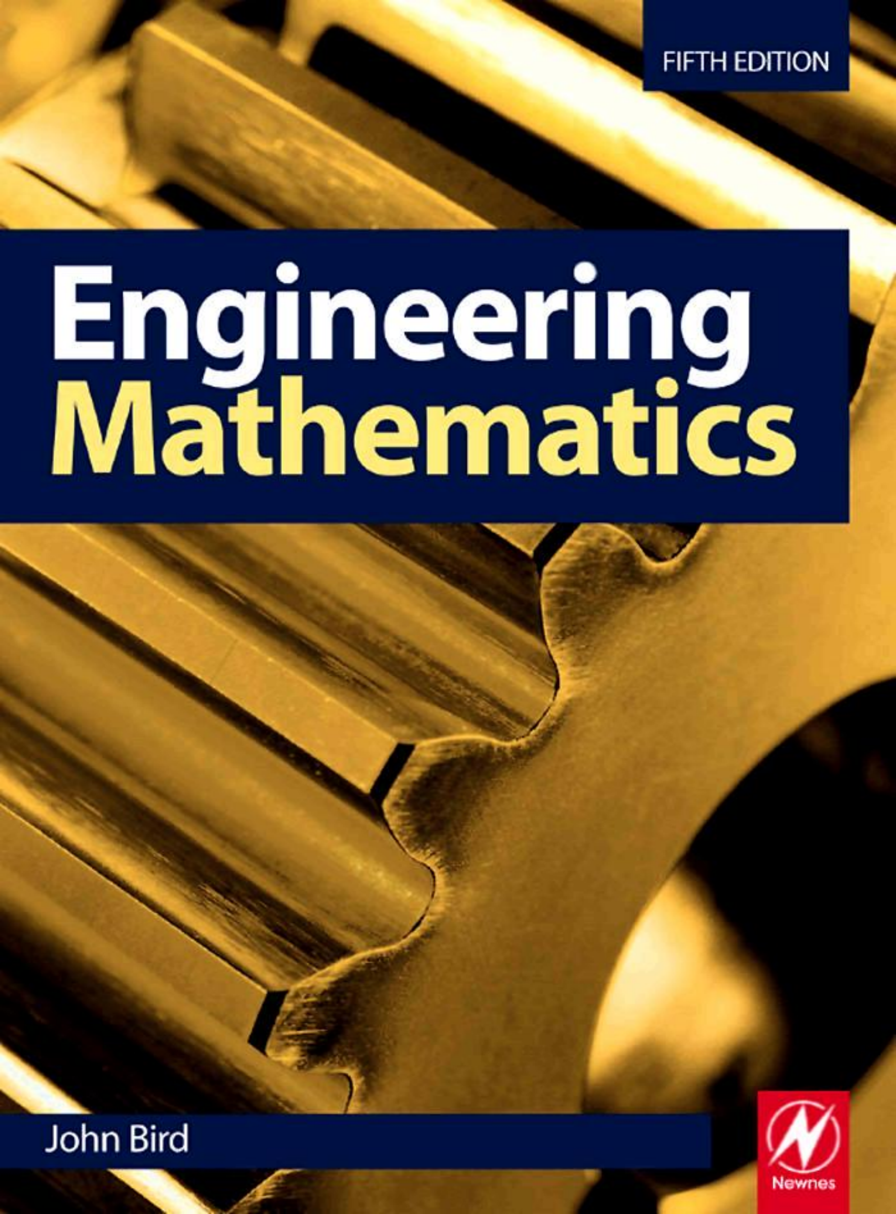

FIFTH EDITION

Engineering Mathematics

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Newnes

Chapter 56

Mean and root mean square values

56.1 Mean or average values

(i) The mean or average value of the curve shown in Fig. 56.1, between $x = a$ and $x = b$, is given by: **mean or average value,**

$$\bar{y} = \frac{\textbf{area under curve}}{\textbf{length of base}}$$

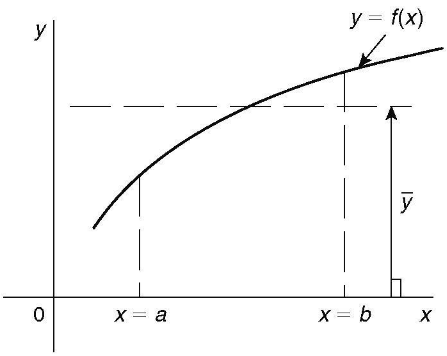

Figure 56.1

(ii) When the area under a curve may be obtained by integration then:
mean or average value,

$$\bar{y} = \frac{\int_a^b y\,dx}{b-a}$$

i.e. $$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(iii) For a periodic function, such as a sine wave, the mean value is assumed to be 'the mean value over half a cycle', since the mean value over a complete cycle is zero.

Problem 1. Determine, using integration, the mean value of $y = 5x^2$ between $x = 1$ and $x = 4$

Mean value,

$$\bar{y} = \frac{1}{4-1}\int_1^4 y\,dx = \frac{1}{3}\int_1^4 5x^2\,dx$$

$$= \frac{1}{3}\left[\frac{5x^3}{3}\right]_1^4 = \frac{5}{9}[x^3]_1^4 = \frac{5}{9}(64-1) = \mathbf{35}$$

Problem 2. A sinusoidal voltage is given by $v = 100 \sin \omega t$ volts. Determine the mean value of the voltage over half a cycle using integration

Half a cycle means the limits are 0 to π radians. Mean value,

$$\bar{v} = \frac{1}{\pi - 0}\int_0^\pi v\,d(\omega t)$$

$$= \frac{1}{\pi}\int_0^\pi 100 \sin \omega t\,d(\omega t) = \frac{100}{\pi}[-\cos \omega t]_0^\pi$$

$$= \frac{100}{\pi}[(-\cos \pi) - (-\cos 0)]$$

$$= \frac{100}{\pi}[(+1) - (-1)] = \frac{200}{\pi}$$

$$= \mathbf{63.66\ volts}$$

[Note that for a sine wave,

$$\textbf{mean value} = \frac{\mathbf{2}}{\boldsymbol{\pi}} \times \textbf{maximum value}$$

In this case, mean value $= \frac{2}{\pi} \times 100 = 63.66\,\text{V}]$

Problem 3. Calculate the mean value of $y = 3x^2 + 2$ in the range $x = 0$ to $x = 3$ by (a) the mid-ordinate rule and (b) integration

(a) A graph of $y = 3x^2$ over the required range is shown in Fig. 56.2 using the following table:

x	0	0.5	1.0	1.5	2.0	2.5	3.0
y	2.0	2.75	5.0	8.75	14.0	20.75	29.0

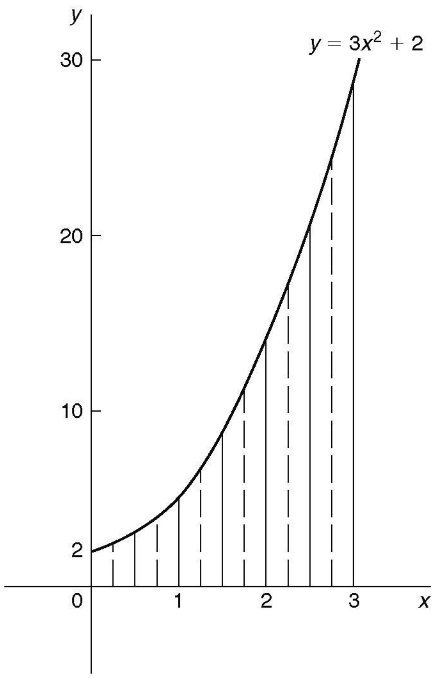

Figure 56.2

Using the mid-ordinate rule, mean value

$$= \frac{\text{area under curve}}{\text{length of base}}$$

$$= \frac{\text{sum of mid-ordinates}}{\text{number of mid-ordinates}}$$

Selecting 6 intervals, each of width 0.5, the mid-ordinates are erected as shown by the broken lines in Fig. 56.2.

$$\text{Mean value} = \frac{2.2 + 3.7 + 6.7 + 11.2 + 17.2 + 24.7}{6}$$

$$= \frac{65.7}{6} = \mathbf{10.95}$$

(b) By integration, mean value

$$= \frac{1}{3-0}\int_0^3 y\,dx = \frac{1}{3}\int_0^3 (3x^2 + 2)dx$$

$$= \frac{1}{3}[x^3 + 2x]_0^3 = \frac{1}{3}[(27 + 6) - (0)]$$

$$= \mathbf{11}$$

The answer obtained by integration is exact; greater accuracy may be obtained by the mid-ordinate rule if a larger number of intervals are selected.

Problem 4. The number of atoms, N, remaining in a mass of material during radioactive decay after time t seconds is given by: $N = N_0 e^{-\lambda t}$, where N_0 and λ are constants. Determine the mean number of atoms in the mass of material for the time period $t = 0$ and $t = \frac{1}{\lambda}$

Mean number of atoms

$$= \frac{1}{\frac{1}{\lambda} - 0}\int_0^{1/\lambda} N\,dt = \frac{1}{\frac{1}{\lambda}}\int_0^{1/\lambda} N_0 e^{-\lambda t}\,dt$$

$$= \lambda N_0 \int_0^{1/\lambda} e^{-\lambda t}\,dt = \lambda N_0 \left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^{1/\lambda}$$

$$= -N_0[e^{-\lambda(1/\lambda)} - e^0] = -N_0[e^{-1} - e^0]$$

$$= +N_0[e^0 - e^{-1}] = N_0[1 - e^{-1}] = \mathbf{0.632\,N_0}$$

Now try the following exercise

Exercise 194 Further problems on mean or average values

1. Determine the mean value of (a) $y = 3\sqrt{x}$ from $x = 0$ to $x = 4$ (b) $y = \sin 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$ (c) $y = 4e^t$ from $t = 1$ to $t = 4$

   $\left[\text{(a) } 4 \quad \text{(b) } \frac{2}{\pi} \text{ or } 0.637 \quad \text{(c) } 69.17\right]$

2. Calculate the mean value of $y = 2x^2 + 5$ in the range $x = 1$ to $x = 4$ by (a) the mid-ordinate rule, and (b) integration. [19]

3. The speed v of a vehicle is given by: $v=(4t+3)$ m/s, where t is the time in seconds. Determine the average value of the speed from $t=0$ to $t=3$ s. [9 m/s]

4. Find the mean value of the curve $y=6+x-x^2$ which lies above the x-axis by using an approximate method. Check the result using integration. [4.17]

5. The vertical height h km of a missile varies with the horizontal distance d km, and is given by $h=4d-d^2$. Determine the mean height of the missile from $d=0$ to $d=4$ km. [2.67 km]

6. The velocity v of a piston moving with simple harmonic motion at any time t is given by: $v=c\sin\omega t$, where c is a constant. Determine the mean velocity between $t=0$ and $t=\frac{\pi}{\omega}$ $\left[\frac{2c}{\pi}\right]$

56.2 Root mean square values

The **root mean square value** of a quantity is 'the sqaure root of the mean value of the squared values of the quantity' taken over an interval. With reference to Fig. 56.1, the r.m.s. value of $y=f(x)$ over the range $x=a$ to $x=b$ is given by:

$$\textbf{r.m.s. value}=\sqrt{\frac{1}{b-a}\int_a^b y^2\,dx}$$

One of the principal applications of r.m.s. values is with alternating currents and voltages. The r.m.s. value of an alternating current is defined as that current which will give the same heating effect as the equivalent direct current.

Problem 5. Determine the r.m.s. value of $y=2x^2$ between $x=1$ and $x=4$

R.m.s. value

$$=\sqrt{\frac{1}{4-1}\int_1^4 y^2\,dx}=\sqrt{\frac{1}{3}\int_1^4 (2x^2)^2\,dx}$$

$$=\sqrt{\frac{1}{3}\int_1^4 4x^4\,dx}=\sqrt{\frac{4}{3}\left[\frac{x^5}{5}\right]_1^4}$$

$$=\sqrt{\frac{4}{15}(1024-1)}=\sqrt{272.8}=\mathbf{16.5}$$

Problem 6. A sinusoidal voltage has a maximum value of 100 V. Calculate its r.m.s. value

A sinusoidal voltage v having a maximum value of 100 V may be written as: $v=100\sin\theta$. Over the range $\theta=0$ to $\theta=\pi$,

r.m.s. value

$$=\sqrt{\frac{1}{\pi-0}\int_0^\pi v^2\,d\theta}$$

$$=\sqrt{\frac{1}{\pi}\int_0^\pi (100\sin\theta)^2\,d\theta}$$

$$=\sqrt{\frac{10\,000}{\pi}\int_0^\pi \sin^2\theta\,d\theta}$$

which is not a 'standard' integral. It is shown in Chapter 27 that $\cos 2A=1-2\sin^2 A$ and this formula is used whenever $\sin^2 A$ needs to be integrated.
Rearranging $\cos 2A=1-2\sin^2 A$ gives
$\sin^2 A=\frac{1}{2}(1-\cos 2A)$

Hence $\sqrt{\frac{10\,000}{\pi}\int_0^\pi \sin^2\theta\,d\theta}$

$$=\sqrt{\frac{10\,000}{\pi}\int_0^\pi \frac{1}{2}(1-\cos 2\theta)\,d\theta}$$

$$=\sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^\pi}$$

$$=\sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\left(\pi-\frac{\sin 2\pi}{2}\right)-\left(0-\frac{\sin 0}{2}\right)\right]}$$

$$=\sqrt{\frac{10\,000}{\pi}\frac{1}{2}[\pi]}=\sqrt{\frac{10\,000}{2}}$$

$$=\frac{100}{\sqrt{2}}=\mathbf{70.71\ volts}$$

[Note that for a sine wave,

$$\textbf{r.m.s. value}=\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\times\textbf{maximum value}.$$

In this case, r.m.s. value $=\frac{1}{\sqrt{2}}\times 100=70.71$ V]

Problem 7. In a frequency distribution the average distance from the mean, y, is related to the variable, x, by the equation $y = 2x^2 - 1$. Determine, correct to 3 significant figures, the r.m.s. deviation from the mean for values of x from -1 to $+4$

R.m.s. deviation

$$= \sqrt{\frac{1}{4 - -1}\int_{-1}^{4} y^2\,dx}$$

$$= \sqrt{\frac{1}{5}\int_{-1}^{4} (2x^2 - 1)^2 dx}$$

$$= \sqrt{\frac{1}{5}\int_{-1}^{4} (4x^4 - 4x^2 + 1)dx}$$

$$= \sqrt{\frac{1}{5}\left[\frac{4x^5}{5} - \frac{4x^3}{3} + x\right]_{-1}^{4}}$$

$$= \sqrt{\frac{1}{5}\left[\left(\frac{4}{5}(4)^5 - \frac{4}{3}(4)^3 + 4\right) - \left(\frac{4}{5}(-1)^5 - \frac{4}{3}(-1)^3 + (-1)\right)\right]}$$

$$= \sqrt{\frac{1}{5}[(737.87) - (-0.467)]}$$

$$= \sqrt{\frac{1}{5}[738.34]}$$

$= \sqrt{147.67} = 12.152 =$ **12.2, correct to 3 significant figures**.

Now try the following exercise

Exercise 195 Further problems on root mean square values

1. Determine the r.m.s. values of:
   (a) $y = 3x$ from $x = 0$ to $x = 4$
   (b) $y = t^2$ from $t = 1$ to $t = 3$
   (c) $y = 25 \sin\theta$ from $\theta = 0$ to $\theta = 2\pi$

   $\left[\text{(a) } 6.928 \quad \text{(b) } 4.919 \quad \text{(c) } \frac{25}{\sqrt{2}} \text{ or } 17.68\right]$

2. Calculate the r.m.s. values of:
   (a) $y = \sin 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$
   (b) $y = 1 + \sin t$ from $t = 0$ to $t = 2\pi$
   (c) $y = 3\cos 2x$ from $x = 0$ to $x = \pi$

   (Note that $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$, from Chapter 27).

   $\left[\text{(a) } \frac{1}{\sqrt{2}} \text{ or } 0.707 \quad \text{(b) } 1.225 \quad \text{(c) } 2.121\right]$

3. The distance, p, of points from the mean value of a frequency distribution are related to the variable, q, by the equation $p = \frac{1}{q} + q$. Determine the standard deviation (i.e. the r.m.s. value), correct to 3 significant figures, for values from $q = 1$ to $q = 3$. [2.58]

4. A current, $i = 30\sin 100\pi t$ amperes is applied across an electric circuit. Determine its mean and r.m.s. values, each correct to 4 significant figures, over the range $t = 0$ to $t = 10$ ms. [19.10 A, 21.21 A]

5. A sinusoidal voltage has a peak value of 340 V. Calculate its mean and r.m.s. values, correct to 3 significant figures. [216 V, 240 V]

6. Determine the form factor, correct to 3 significant figures, of a sinusoidal voltage of maximum value 100 volts, given that form factor $= \frac{\text{r.m.s. value}}{\text{average value}}$ [1.11]

7. A wave is defined by the equation:

   $$v = E_1 \sin\omega t + E_3 \sin 3\omega t$$

   where, E_1, E_3 and ω are constants.

   Determine the r.m.s. value of v over the interval $0 \le t \le \frac{\pi}{\omega}$

   $\left[\sqrt{\frac{E_1^2 + E_3^2}{2}}\right]$