ADVANCED ENGINEERING MATHEMATICS

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PETER V. O'NEIL

The University of Alabama at Birmingham

Now consider the nonhomogeneous linear system of n equations in m unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n. \end{aligned}$$

In matrix form,

$$\mathbf{AX} = \mathbf{B} \tag{7.1}$$

where $\mathbf{A}$ is the coefficient matrix,

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

The system is *nonhomogeneous* if at least one $b_j \neq 0$. Nonhomogeneous systems differ from linear systems in two significant ways.

1. A nonhomogeneous system may have no solution. For example, the system

$$2x_1 - 3x_2 = 6$$
$$4x_1 - 6x_2 = 8$$

can have no solution. If $2x_1 - 3x_2 = 6$, then $4x_1 - 6x_2$ must equal 12, not 8.

Matrix inverses relate to systems of linear equations in the following way.

THEOREM 7.16

Let **A** be $n \times n$.

1. A homogeneous system $\mathbf{AX} = \mathbf{O}$ has a nontrivial solution if and only if **A** is singular.
2. A consistent nonhomogeneous system $\mathbf{AX} = \mathbf{B}$ has a unique solution if and only if **A** is nonsingular. In this case the solution is

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}. \blacklozenge$$

Proof If **A** is singular, then $\mathbf{A}_R \neq \mathbf{I}_n$ by Theorem 7.15, conclusion (5), so the system $\mathbf{AX} = \mathbf{O}$ has a nontrivial solution by Corollary 7.3.

Conversely, suppose the system $\mathbf{AX} = \mathbf{O}$ has a nontrivial solution. Then $\text{rank}(\mathbf{A}) < n$ by Theorem 7.15, conclusion (6), so **A** is singular.

This proves conclusion (1). For conclusion (2), suppose the system is consistent. The general solution has the form $\mathbf{X} = \mathbf{H} + \mathbf{U}_p$, where **H** is the general solution of the associated homogeneous system. Therefore the given system has a unique solution exactly when the homogeneous system has only the trivial solution, which occurs if and only if **A** is nonsingular. ♦

Finding the inverse of a nonsingular matrix is most easily done using a software routine. In the `linalg` package of linear algebra routines of MAPLE, the inverse of a matrix **A** that has been entered can be found using

```
inverse(A);
```

If it happens that **A** is singular, the routine will return this conclusion.

EXAMPLE 7.30

We will solve the system

$$2x_1 - x_2 + 3x_3 = 4$$
$$x_1 + 9x_2 - 2x_3 = -8$$
$$4x_1 - 8x_2 + 11x_3 = 15.$$

The matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 9 & -2 \\ 4 & -8 & 11 \end{pmatrix}.$$

A routine reduction yields

$$[\mathbf{A} \vdots \mathbf{I}_3]_R = \begin{pmatrix} 1 & 0 & 0 & \vdots & 83/53 & -13/53 & -25/53 \\ 0 & 1 & 0 & \vdots & -19/53 & 10/53 & 7/53 \\ 0 & 0 & 1 & \vdots & -44/53 & 12/53 & 19/53 \end{pmatrix}.$$

The first three columns are $\mathbf{I}_3$, hence **A** is nonsingular and the system has a unique solution. The last three columns of the reduced augmented matrix give us

$$\mathbf{A}^{-1} = \frac{1}{53}\begin{pmatrix} 83 & -13 & -25 \\ -19 & 10 & 7 \\ -44 & 12 & 19 \end{pmatrix}.$$

The unique solution of the system is $\mathbf{A}^{-1}\mathbf{B}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{53}\begin{pmatrix} 83 & -13 & -25 \\ -19 & 10 & 7 \\ -44 & 12 & 19 \end{pmatrix}\begin{pmatrix} 4 \\ -8 \\ 15 \end{pmatrix} = \begin{pmatrix} 61/53 \\ -51/53 \\ 13/53 \end{pmatrix}. \blacklozenge$$

7.5 Homogeneous Systems

We want to develop a method for finding all solutions of a *linear homogeneous system* of n equations in m unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = 0.$$

The numbers a_{ij} are called the the *coefficients* of the system and $\mathbf{A} = [a_{ij}]$ is the *matrix of coefficients*. Row i contains the coefficients of equation i and column j contains the coefficients of x_j.

Define

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

and write the $n \times 1$ zero matrix as just $\mathbf{O}$, a column of n zeros. Then the system can be written as the matrix equation

$$\mathbf{AX} = \mathbf{O}.$$

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$$\mathbf{AX} = \mathbf{O}.$$

We will develop the following strategy for solving this system.

1. We will show that $\mathbf{AX} = \mathbf{O}$ has the same solutions as the *reduced system* $\mathbf{A}_R\mathbf{X} = \mathbf{O}$.
2. We will show how to write all solutions of the reduced system directly from the reduced matrix $\mathbf{A}_R$.
3. We will also use facts about vector spaces and rank to derive additional information about solutions.

The remainder of this section consists of the details of carrying out this strategy, and examples. The first two examples give us some feeling for what to look for in solving a homogeneous system.

EXAMPLE 7.18

Consider the simple system

$$x_1 - 3x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - 3x_3 = 0.$$

THEOREM 7.12 *Solution Space of a Homogeneous System*

Let $\mathbf{A}$ be $n \times m$. Then

1. The set of all solutions of $\mathbf{AX} = \mathbf{O}$ forms a subspace of R^m, called the *solution space* of this system.
2. The dimension of this solution space is

$$m - \text{ number of nonzero rows of } \mathbf{A}_R,$$

which is the same as $m -$ rank $(\mathbf{A})$. ♦

Proof Let S be the set of all solutions of the system. Since

$$x_1 = x_2 = \cdots = x_m = 0$$

is a solution, the zero m-vector is in S.

Now suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions, and α and β are numbers. Then

$$\mathbf{A}(\alpha\mathbf{X}_1 + \beta\mathbf{X}_2) = \alpha\mathbf{AX}_1 + \beta\mathbf{AX}_2 = \mathbf{O} + \mathbf{O} = \mathbf{O},$$

so linear combinations of solutions are solutions, and S is a subspace of R^m.

For the dimension of S, use the fact that the system has the same solution space as the reduced system. As the examples suggest, the nonzero rows of $\mathbf{A}_R$ enable us to express the general solution as a linear combination of linearly independent solutions, one for each free variable. Since the number of free variables is the number of columns of $\mathbf{A}_R$, minus the number of nonzero rows, then the dimension of S is $m -$ rank$(\mathbf{A})$. ♦

COROLLARY 7.5

If $\mathbf{A}$ is $n \times n$, then $\mathbf{AX} = \mathbf{O}$ has only the trivial solution if and only if $\mathbf{A}_R = \mathbf{I}_n$. ♦

EXAMPLE 7.22

We will solve the system

$$\begin{aligned} -4x_1 + x_2 - 7x_3 &= 0 \\ 2x_1 + 9x_2 - 13x_3 &= 0 \\ x_1 + x_2 + 10x_3 &= 0. \end{aligned}$$

The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} -4 & 1 & -7 \\ 2 & 9 & -13 \\ 1 & 1 & 10 \end{pmatrix}.$$

We find that $\mathbf{A}_R = \mathbf{I}_3$. Therefore the system has only the trivial solution. This can also be seen from the reduced system, which is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0. \end{aligned}$$ ♦

SECTION 7.5 PROBLEMS