CHAPTER 8

Analytic Geometry in Two and Three Dimensions

The oval-shaped lawn behind the White House in Washington, D.C. is called *the Ellipse*. It has views of the Washington Monument, the Jefferson Memorial, the Department of Commerce, and the Old Post Office Building. The Ellipse is 616 ft long, 528 ft wide, and is in the shape of a conic section. Its shape can be modeled using the methods of this chapter. See page 652.

8.2 Ellipses

What you'll learn about

- Geometry of an Ellipse
- Translations of Ellipses
- Orbits and Eccentricity
- Reflective Property of an Ellipse

. . . and why

Ellipses are the paths of planets and comets around the Sun, or of moons around planets.

OBJECTIVE

Students will be able to find the equation, vertices, and foci of an ellipse.

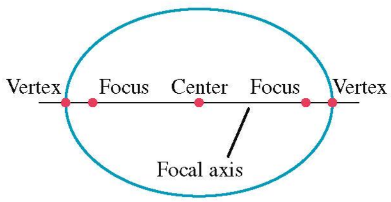


FIGURE 8.11 Key points on the focal axis of an ellipse.

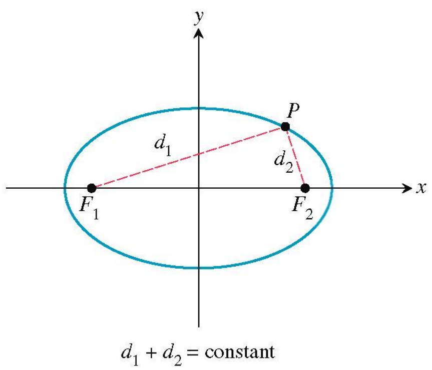


FIGURE 8.12 *Structure of an Ellipse.* The sum of the distances from the foci to each point on the ellipse is a constant.

Geometry of an Ellipse

When a plane intersects one nappe of a right circular cylinder and forms a simple closed curve, the curve is an ellipse.

DEFINITION Ellipse

An **ellipse** is the set of all points in a plane whose distances from two fixed points in the plane have a constant sum. The fixed points are the **foci** (plural of focus) of the ellipse. The line through the foci is the **focal axis**. The point on the focal axis midway between the foci is the **center**. The points where the ellipse intersects its axis are the **vertices** of the ellipse. (See Figure 8.11.)

Figure 8.12 shows a point $P(x, y)$ of an ellipse. The fixed points F_1 and F_2 are the foci of the ellipse, and the distances whose sum is constant are d_1 and d_2. We can construct an ellipse using a pencil, a loop of string, and two pushpins. Put the loop around the two pins placed at F_1 and F_2, pull the string taut with a pencil point P, and move the pencil around to trace out the ellipse (Figure 8.13).

We now use the definition to derive an equation for an ellipse. For some constants a and c with $a > c \geq 0$, let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci (Figure 8.14). Then an ellipse is defined by the set of points $P(x, y)$ such that

$$PF_1 + PF_2 = 2a.$$

Using the distance formula, the equation becomes

$$\sqrt{(x + c)^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = 2a.$$

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \quad \text{Square.}$$

$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx \quad \text{Simplify.}$$

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2 \quad \text{Square.}$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \quad \text{Simplify.}$$

Letting $b^2 = a^2 - c^2$, we have

$$b^2x^2 + a^2y^2 = a^2b^2,$$

which is usually written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

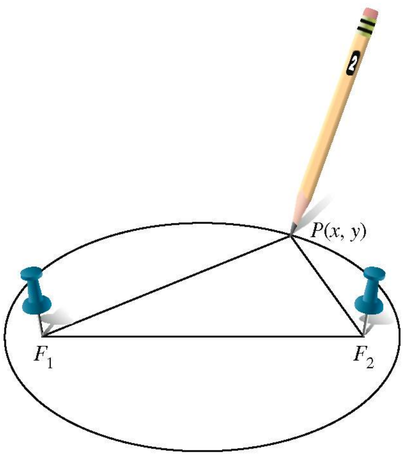


FIGURE 8.13 How to draw an ellipse.

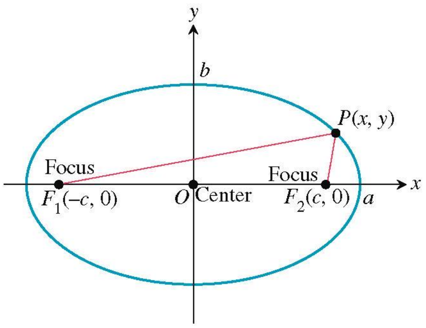


FIGURE 8.14 The ellipse defined by $PF_1 + PF_2 = 2a$ is the graph of the equation $x^2/a^2 + y^2/b^2 = 1$, where $b^2 = a^2 - c^2$.

AXIS ALERT

For an ellipse, the word *axis* is used in several ways. The focal axis is a *line*. The major and minor axes are *line segments*. The semimajor and semiminor axes are *numbers*.

MOTIVATE

Ask students how the graph of $x^2 + (y/2)^2 = 1$ is related to the graph of $x^2 + y^2 = 1$.

LESSON GUIDE

Day 1: Geometry of an Ellipse; Translations of Ellipses
Day 2: Orbits and Eccentricity; Reflective Property of an Ellipse

Because these steps can be reversed, a point $P(x, y)$ satisfies this last equation if and only if the point lies on the ellipse defined by $PF_1 + PF_2 = 2a$, provided that $a > c \geq 0$ and $b^2 = a^2 - c^2$. The *Pythagorean relation* $b^2 = a^2 - c^2$ can be written many ways, including $c^2 = a^2 - b^2$ and $a^2 = b^2 + c^2$.

The equation $x^2/a^2 + y^2/b^2 = 1$ is the **standard form** of the equation of an ellipse centered at the origin with the x-axis as its focal axis. An ellipse centered at the origin with the y-axis as its focal axis is the *inverse* of $x^2/a^2 + y^2/b^2 = 1$, and thus has an equation of the form

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

As with circles and parabolas, a line segment with endpoints on an ellipse is a **chord** of the ellipse. The chord lying on the focal axis is the **major axis** of the ellipse. The chord through the center perpendicular to the focal axis is the **minor axis** of the ellipse. The length of the major axis is $2a$, and of the minor axis is $2b$. The number a is the **semimajor axis**, and b is the **semiminor axis**.

Ellipses with Center (0, 0)

• **Standard equation**	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
• **Focal axis**	x-axis	y-axis
• **Foci**	$(\pm c, 0)$	$(0, \pm c)$
• **Vertices**	$(\pm a, 0)$	$(0, \pm a)$
• **Semimajor axis**	a	a
• **Semiminor axis**	b	b
• **Pythagorean relation**	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$

See Figure 8.15.

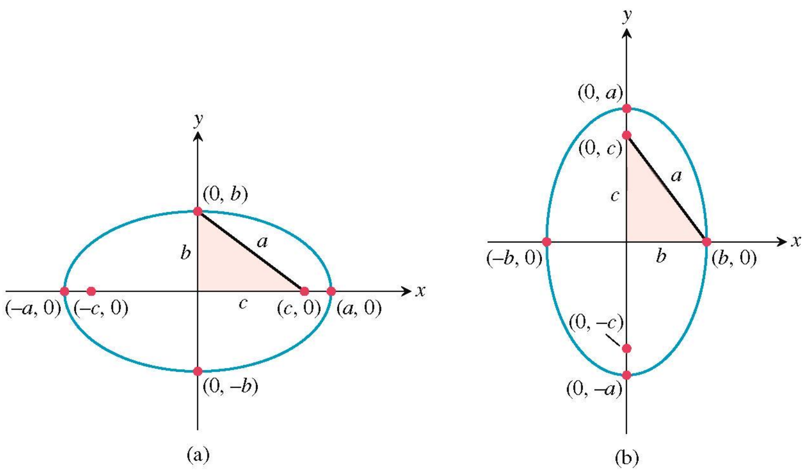


FIGURE 8.15 Ellipses centered at the origin with foci on (a) the x-axis and (b) the y-axis. In each case, a right triangle illustrating the Pythagorean relation is shown.

EXAMPLE 1 Finding the Vertices and Foci of an Ellipse

Find the vertices and the foci of the ellipse $4x^2 + 9y^2 = 36$.

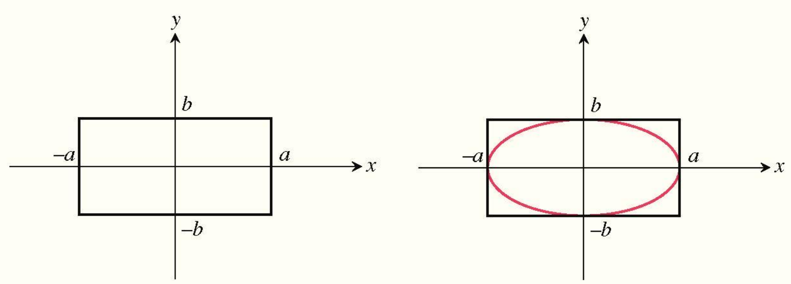
SOLUTION Dividing both sides of the equation by 36 yields the standard form $x^2/9 + y^2/4 = 1$. Because the larger number is the denominator of x^2, the focal axis is the x-axis. So $a^2 = 9$, $b^2 = 4$, and $c^2 = a^2 - b^2 = 9 - 4 = 5$. Thus the vertices are $(\pm 3, 0)$, and the foci are $(\pm\sqrt{5}, 0)$. *Now try Exercise 1.*

An ellipse centered at the origin with its focal axis on a coordinate axis is symmetric with respect to the origin and both coordinate axes. Such an ellipse can be sketched by first drawing a *guiding rectangle* centered at the origin with sides parallel to the coordinate axes and then sketching the ellipse inside the rectangle, as shown in the Drawing Lesson.

Drawing Lesson

How to Sketch the Ellipse $x^2/a^2 + y^2/b^2 = 1$

1. Sketch line segments at $x = \pm a$ and $y = \pm b$ and complete the rectangle they determine.
2. Inscribe an ellipse that is tangent to the rectangle at $(\pm a, 0)$ and $(0, \pm b)$.



If we wish to graph an ellipse using a function grapher, we need to solve the equation of the ellipse for y, as illustrated in Example 2.

EXAMPLE 2 Finding an Equation and Graphing an Ellipse

Find an equation of the ellipse with foci $(0, -3)$ and $(0, 3)$ whose minor axis has length 4. Sketch the ellipse and support your sketch with a grapher.

SOLUTION The center is $(0, 0)$. The foci are on the y-axis with $c = 3$. The semiminor axis is $b = 4/2 = 2$. Using $a^2 = b^2 + c^2$, we have $a^2 = 2^2 + 3^2 = 13$. So the standard form of the equation for the ellipse is

$$\frac{y^2}{13} + \frac{x^2}{4} = 1.$$

Using $a = \sqrt{13} \approx 3.61$ and $b = 2$, we can sketch a guiding rectangle and then the ellipse itself, as explained in the Drawing Lesson. (Try doing this.) To graph the ellipse using a function grapher, we solve for y in terms of x.

$$\frac{y^2}{13} = 1 - \frac{x^2}{4}$$

$$y^2 = 13(1 - x^2/4)$$

$$y = \pm\sqrt{13(1 - x^2/4)}$$

continued

NOTES ON EXAMPLES

Notice that we chose square viewing windows in Figure 8.16. A nonsquare window would give a distorted view of an ellipse.

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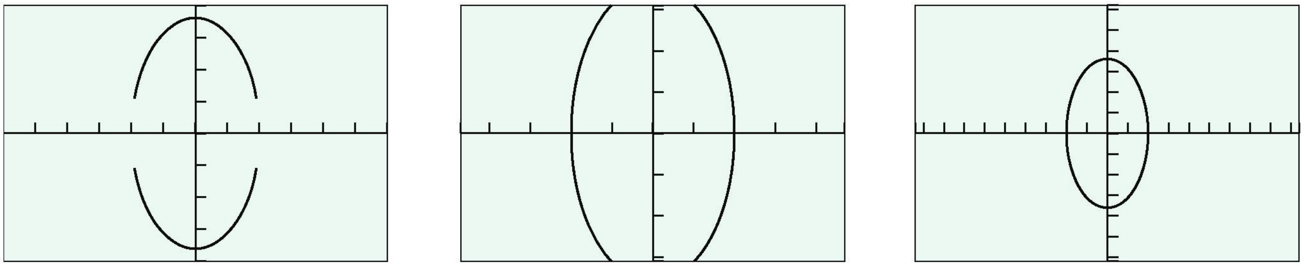
Figure 8.16 shows three views of the graphs of

$$Y1 = \sqrt{13(1 - x^2/4)} \quad \text{and} \quad Y2 = -\sqrt{13(1 - x^2/4)}.$$

We must select the viewing window carefully to avoid grapher failure.

Now try Exercise 17.



[–6, 6] by [–4, 4]
(a)

[–4.7, 4.7] by [–3.1, 3.1]
(b)

[–9.4, 9.4] by [–6.2, 6.2]
(c)

FIGURE 8.16 Three views of the ellipse $y^2/13 + x^2/4 = 1$. All of the windows are square or approximately square viewing windows so we can see the true shape. Notice that the gaps between the upper and lower function branches do not show when the grapher window includes columns of pixels whose x-coordinates are ± 2 as in (b) and (c). (Example 2)

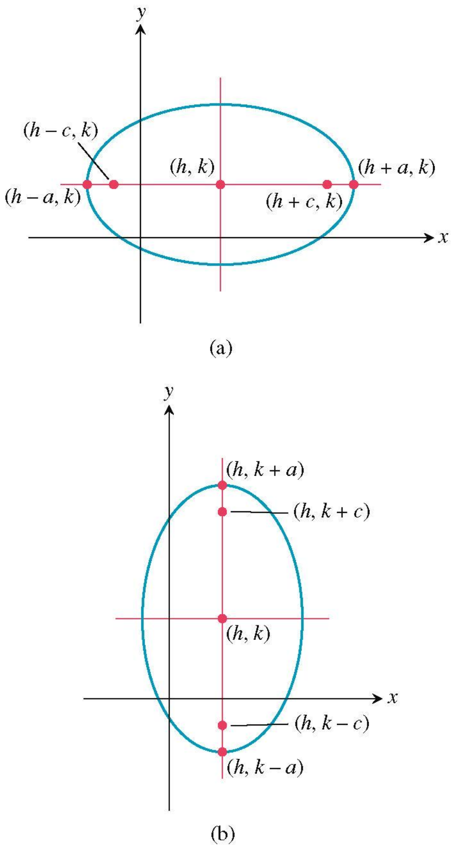


FIGURE 8.17 Ellipses with center (h, k) and foci on (a) $y = k$ and (b) $x = h$.

Translations of Ellipses

When an ellipse with center (0, 0) is translated horizontally by h units and vertically by k units, the center of the ellipse moves from (0, 0) to (h, k), as shown in Figure 8.17. Such a translation does not change the length of the major or minor axis or the Pythagorean relation.

Ellipses with Center (h, k)

• **Standard equation**	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
• **Focal axis**	$y = k$	$x = h$
• **Foci**	$(h \pm c, k)$	$(h, k \pm c)$
• **Vertices**	$(h \pm a, k)$	$(h, k \pm a)$
• **Semimajor axis**	a	a
• **Semiminor axis**	b	b
• **Pythagorean relation**	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$

See Figure 8.17.

NOTES ON EXAMPLES

When solving a problem such as Example 3, students should be strongly encouraged to draw a sketch of the given information.

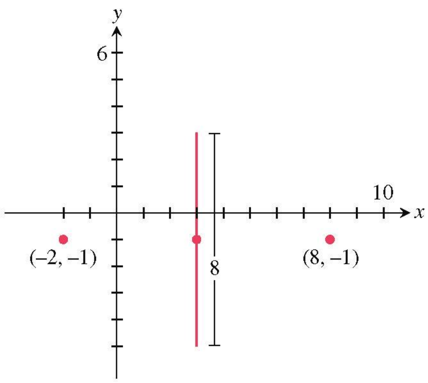


FIGURE 8.18 Given information for Example 3.

EXAMPLE 3 Finding an Equation of an Ellipse

Find the standard form of the equation for the ellipse whose major axis has endpoints $(-2, -1)$ and $(8, -1)$, and whose minor axis has length 8.

SOLUTION Figure 8.18 shows the major-axis endpoints, the minor axis, and the center of the ellipse. The standard equation of this ellipse has the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

where the center (h, k) is the midpoint $(3, -1)$ of the major axis. The semimajor axis and semiminor axis are

$$a = \frac{8-(-2)}{2} = 5 \quad \text{and} \quad b = \frac{8}{2} = 4.$$

So the equation we seek is

$$\frac{(x-3)^2}{5^2} + \frac{(y-(-1))^2}{4^2} = 1,$$

$$\frac{(x-3)^2}{25} + \frac{(y+1)^2}{16} = 1.$$

Now try Exercise 31.

EXAMPLE 4 Locating Key Points of an Ellipse

Find the center, vertices, and foci of the ellipse

$$\frac{(x+2)^2}{9} + \frac{(y-5)^2}{49} = 1.$$

SOLUTION The standard equation of this ellipse has the form

$$\frac{(y-5)^2}{49} + \frac{(x+2)^2}{9} = 1.$$

The center (h, k) is $(-2, 5)$. Because the semimajor axis $a = \sqrt{49} = 7$, the vertices $(h, k \pm a)$ are

$$(h, k+a) = (-2, 5+7) = (-2, 12) \quad \text{and}$$

$$(h, k-a) = (-2, 5-7) = (-2, -2).$$

Because

$$c = \sqrt{a^2 - b^2} = \sqrt{49-9} = \sqrt{40}$$

the foci $(h, k \pm c)$ are $(-2, 5 \pm \sqrt{40})$, or approximately $(-2, 11.32)$ and $(-2, -1.32)$.

Now try Exercise 37.

With the information found about the ellipse in Example 4 and knowing that its semiminor axis $b = \sqrt{9} = 3$, we could easily sketch the ellipse. Obtaining an accurate graph of the ellipse using a function grapher is another matter. Generally, the best way to graph an ellipse using a graphing utility is to use parametric equations.

EXPLORATION 1 **Graphing an Ellipse Using Its Parametric Equations**

1. Use the Pythagorean trigonometry identity $\cos^2 t + \sin^2 t = 1$ to prove that the parameterization $x = -2 + 3 \cos t$, $y = 5 + 7 \sin t$, $0 \le t \le 2\pi$ will produce a graph of the ellipse $(x + 2)^2/9 + (y - 5)^2/49 = 1$.
2. Graph $x = -2 + 3 \cos t$, $y = 5 + 7 \sin t$, $0 \le t \le 2\pi$ in a square viewing window to support part 1 graphically.
3. Create parameterizations for the ellipses in Examples 1, 2, and 3.
4. Graph each of your parameterizations in part 3 and check the features of the obtained graph to see whether they match the expected geometric features of the ellipse. Revise your parameterization and regraph until all features match.
5. Prove that each of your parameterizations is valid.

EXPLORATION EXTENSIONS

Give an equation in standard form for the ellipse defined by the parametric equations $x = -3 + \cos t$, $y = 4 + 5 \sin t$, $0 \le t < 2\pi$.

Orbits and Eccentricity

Kepler's first law of planetary motion, published in 1609, states that the path of a planet's orbit is an ellipse with the Sun at one of the foci. Asteroids, comets, and other bodies that orbit the Sun follow elliptical paths. The closest point to the Sun in such an orbit is the *perihelion*, and the farthest point is the *aphelion* (Figure 8.19). The shape of an ellipse is related to its *eccentricity*.

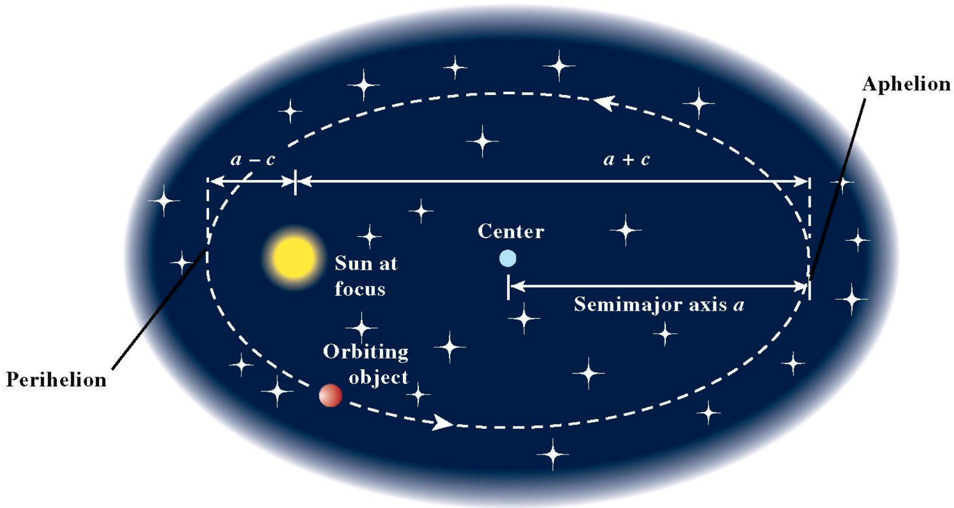


FIGURE 8.19 Many celestial objects have elliptical orbits around the Sun.

DEFINITION Eccentricity of an Ellipse

The **eccentricity** of an ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a},$$

where a is the semimajor axis, b is the semiminor axis, and c is the distance from the center of the ellipse to either focus.

A NEW *e*

Try not to confuse the eccentricity e with the natural base e used in exponential and logarithmic functions. The context should clarify which meaning is intended.

FOLLOW-UP

Ask your students if they agree or disagree with the following statement: "The sound you hear in a whispering gallery should be muddy, because the sound waves will take varying lengths of time to travel from one focus to the other, depending on where they reflect off the wall." (Disagree)

ASSIGNMENT GUIDE

Day 1: Ex. 3–42, multiples of 3
Day 2: Ex. 19, 31, 37, 41, 47, 52, 53, 55, 58

COOPERATIVE LEARNING

Group Activity: Ex. 61–63

NOTES ON EXERCISES

Ex. 21–36 and 49–50 require students to find equations of ellipses.
Ex. 52–60 are application problems involving astronomy or lithotripters.
Ex. 65–70 provide practice for standardized tests.

ONGOING ASSESSMENT

Self-Assessment: Ex. 1, 17, 31, 37, 53, 59
Embedded Assessment: Ex. 51, 52, 60

The noun *eccentricity* comes from the adjective *eccentric*, which means off-center. Mathematically, the eccentricity is the ratio of c to a. The larger c is, compared to a, the more off-center the foci are.

In any ellipse, $a > c \geq 0$. Dividing this inequality by a shows that $0 \leq e < 1$. So the eccentricity of an ellipse is between 0 and 1.

Ellipses with highly off-center foci are elongated and have eccentricities close to 1; for example, the orbit of Halley's comet has eccentricity $e \approx 0.97$. Ellipses with foci near the center are almost circular and have eccentricities close to 0; for instance, Venus's orbit has an eccentricity of 0.0068.

What happens when the eccentricity $e = 0$? In an ellipse, because a is positive, $e = c/a = 0$ implies that $c = 0$ and thus $a = b$. In this case, the ellipse *degenerates* into a circle. Because the ellipse is a circle when $a = b$, it is customary to denote this common value as r and call it the radius of the circle.

Surprising things happen when an ellipse is nearly but not quite a circle, as in the orbit of our planet, Earth.

EXAMPLE 5 Analyzing Earth's Orbit

Earth's orbit has a semimajor axis $a \approx 149.598$ Gm (gigameters) and an eccentricity of $e \approx 0.0167$. Calculate and interpret b and c.

SOLUTION Because $e = c/a$, $c = ea \approx 0.0167 \times 149.598 = 2.4982866$ and

$$b = \sqrt{a^2 - c^2} \approx \sqrt{149.598^2 - 2.4982866^2} \approx 149.577.$$

The semiminor axis $b \approx 149.577$ Gm is only 0.014% shorter than the semimajor axis $a \approx 149.598$ Gm. The aphelion distance of Earth from the Sun is $a + c \approx 149.598 + 2.498 = 152.096$ Gm, and the perihelion distance is $a - c \approx 149.598 - 2.498 = 147.100$ Gm.

Thus Earth's orbit is nearly a perfect circle, but the distance between the center of the Sun at one focus and the center of Earth's orbit is $c \approx 2.498$ Gm, more than 2 orders of magnitude greater than $a - b$. The eccentricity as a percentage is 1.67%; this measures how far *off-center* the Sun is.

Now try Exercise 53.

EXPLORATION 2 **Constructing Ellipses to Understand Eccentricity**

Each group will need a pencil, a centimeter ruler, scissors, some string, several sheets of unlined paper, two pushpins, and a foam board or other appropriate backing material.

1. Make a closed *loop* of string that is 20 cm in circumference.
2. Place a sheet of unlined paper on the backing material, and carefully place the two pushpins 2 cm apart near the center of the paper. Construct an ellipse using the loop of string and a pencil as shown in Figure 8.13. Measure and record the resulting values of a, b, and c for the ellipse, and compute the ratios $e = c/a$ and b/a for the ellipse.
3. On separate sheets of paper repeat step 2 three more times, placing the pushpins 4, 6, and 8 cm apart. Record the values of a, b, c and the ratios e and b/a for each ellipse.
4. Write your observations about the ratio b/a as the eccentricity ratio e increases. Which of these two ratios measures the shape of the ellipse? Which measures how off-center the foci are?
5. Plot the ordered pairs $(e, b/a)$, determine a formula for the ratio b/a as a function of the eccentricity e, and overlay this function's graph on the scatter plot.

EXPLORATION EXTENSIONS

Determine how far apart the pushpins would need to be for the ellipse to have a b/a ratio of 1/2. Draw this ellipse.

NOTE ON EXPLORATION

Have students work in groups on Exploration 2. To save time and trouble, make a 20-cm closed loop of string for each group in advance. Results will vary from group to group.

Reflective Property of an Ellipse

Because of their shape, ellipses are used to make reflectors of sound, light, and other waves. If we rotate an ellipse in three-dimensional space about its focal axis, the ellipse sweeps out an **ellipsoid of revolution**. If we place a signal source at one focus of a reflective ellipsoid, the signal reflects off the elliptical surface to the other focus, as illustrated in Figure 8.20. This property is used to make mirrors for optical equipment and to study aircraft noise in wind tunnels.

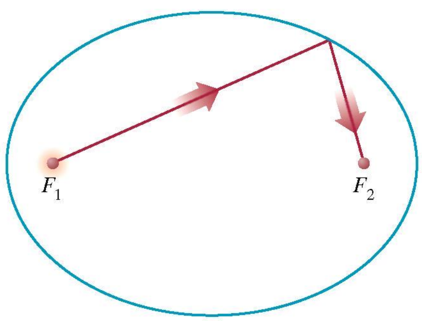


FIGURE 8.20 The reflective property of an ellipse.

WHISPERING GALLERIES

In architecture, ceilings in the shape of an ellipsoid are used to create *whispering galleries*. A person whispering at one focus can be heard across the room by a person at the other focus. An ellipsoid is part of the design of the Texas state capitol; a hand clap made in the center of the main vestibule (at one focus of the ellipsoid) bounces off the inner elliptical dome, passes through the other focus, bounces off the dome a second time, and returns to the person as a distinct echo.

Ellipsoids are used in health care to avoid surgery in the treatment of kidney stones. An elliptical *lithotripter* emits underwater ultrahigh-frequency (UHF) shock waves from one focus, with the patient's kidney carefully positioned at the other focus (Figure 8.21).

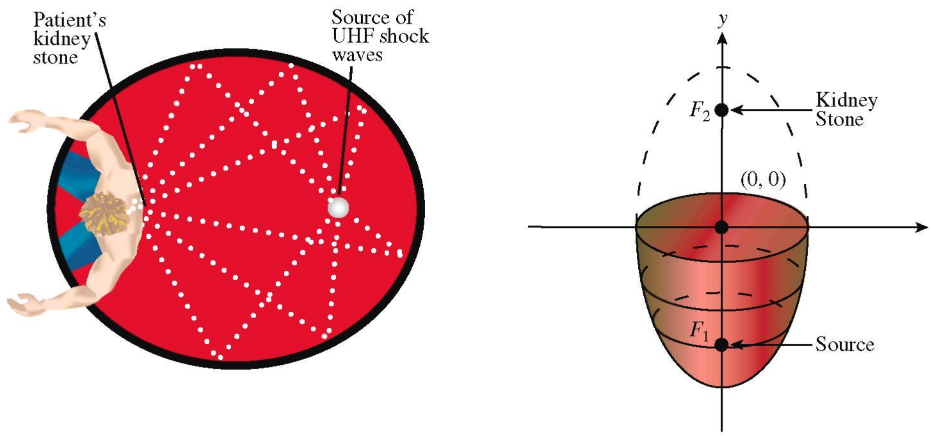


FIGURE 8.21 How a lithotripter breaks up kidney stones.

EXAMPLE 6 Focusing a Lithotripter

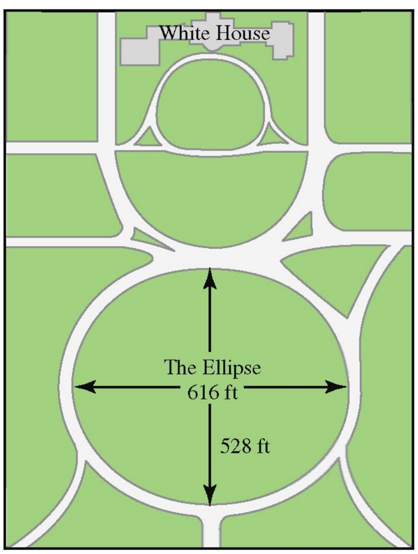
The ellipse used to generate the ellipsoid of a lithotripter has a major axis of 12 ft and a minor axis of 5 ft. How far from the center are the foci?

SOLUTION From the given information, we know $a = 12/2 = 6$ and $b = 5/2 = 2.5$. So

$$c = \sqrt{a^2 - b^2} \approx \sqrt{6^2 - 2.5^2} \approx 5.4544.$$

So the foci are about 5 ft 5.5 inches from the center of the lithotripter.

Now try Exercise 59.



CHAPTER OPENER PROBLEM (from page 631)

PROBLEM: If the Ellipse at the White House is 616 ft long and 528 ft wide, what is its equation?

SOLUTION: For simplicity's sake, we model the Ellipse as centered at (0, 0) with the x-axis as its focal axis. Because the Ellipse is 616 ft long, $a = 616/2 = 308$, and because the Ellipse is 528 ft wide, $b = 528/2 = 264$. Using $x^2/a^2 + y^2/b^2 = 1$, we obtain

$$\frac{x^2}{308^2} + \frac{y^2}{264^2} = 1,$$

$$\frac{x^2}{94{,}864} + \frac{y^2}{69{,}696} = 1.$$

Other models are possible.

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QUICK REVIEW 8.2 *(For help, go to Sections P.2 and P.5.)*

In Exercises 1 and 2, find the distance between the given points.

1. $(-3, -2)$ and $(2, 4)$ $\sqrt{61}$

2. $(-3, -4)$ and (a, b) $\sqrt{(a+3)^2 + (b+4)^2}$

In Exercises 3 and 4, solve for y in terms of x.

3. $\frac{y^2}{9} + \frac{x^2}{4} = 1$

4. $\frac{x^2}{36} + \frac{y^2}{25} = 1$

In Exercises 5–8, solve for x algebraically.

5. $\sqrt{3x+12} + \sqrt{3x-8} = 10$ $x = 8$

6. $\sqrt{6x+12} - \sqrt{4x+9} = 1$ $x = 4$

7. $\sqrt{6x^2+12} + \sqrt{6x^2+1} = 11$ $x = 2, x = -2$

8. $\sqrt{2x^2+8} + \sqrt{3x^2+4} = 8$ $x = 2, x = -2$

In Exercises 9 and 10, find exact solutions by completing the square.

9. $2x^2 - 6x - 3 = 0$

10. $2x^2 + 4x - 5 = 0$

SECTION 8.2 EXERCISES

In Exercises 1–6, find the vertices and foci of the ellipse.

1. $\frac{x^2}{16} + \frac{y^2}{7} = 1$

2. $\frac{y^2}{25} + \frac{x^2}{21} = 1$

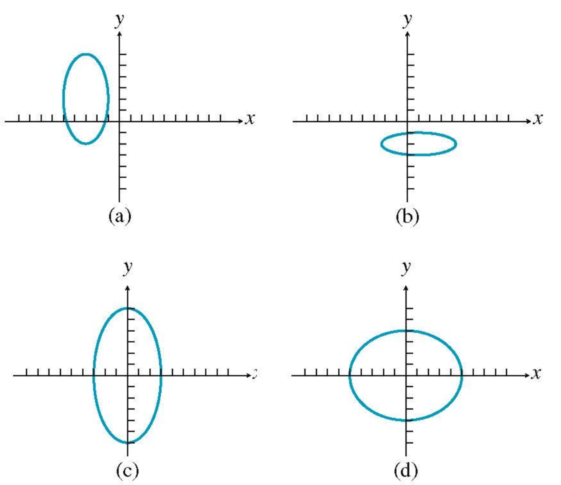
3. $\frac{y^2}{36} + \frac{x^2}{27} = 1$

4. $\frac{x^2}{11} + \frac{y^2}{7} = 1$

5. $3x^2 + 4y^2 = 12$

6. $9x^2 + 4y^2 = 36$

In Exercises 7–10, match the graph with its equation, given that the ticks on all axes are 1 unit apart.



7. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (d)

8. $\frac{y^2}{36} + \frac{x^2}{9} = 1$ (c)

9. $\frac{(y-2)^2}{16} + \frac{(x+3)^2}{4} = 1$

10. $\frac{(x-1)^2}{11} + (y+2)^2 = 1$

In Exercises 11–16, sketch the graph of the ellipse by hand.

11. $\frac{x^2}{64} + \frac{y^2}{36} = 1$

12. $\frac{x^2}{81} + \frac{y^2}{25} = 1$

13. $\frac{y^2}{9} + \frac{x^2}{4} = 1$

14. $\frac{y^2}{49} + \frac{x^2}{25} = 1$

15. $\frac{(x+3)^2}{16} + \frac{(y-1)^2}{4} = 1$

16. $\frac{(x-1)^2}{2} + \frac{(y+3)^2}{4} = 1$

In Exercises 17–20, graph the ellipse using a function grapher.

17. $\frac{x^2}{36} + \frac{y^2}{16} = 1$

18. $\frac{y^2}{64} + \frac{x^2}{16} = 1$

19. $\frac{(x+2)^2}{5} + 2(y-1)^2 = 1$

20. $\frac{(x-4)^2}{16} + 16(y+4)^2 = 8$

In Exercises 21–36, find an equation in standard form for the ellipse that satisfies the given conditions.

21. Major axis length 6 on y-axis, minor axis length 4

22. Major axis length 14 on x-axis, minor axis length 10

23. Foci $(\pm 2, 0)$, major axis length 10 $x^2/25 + y^2/21 = 1$

24. Foci $(0, \pm 3)$, major axis length 10 $y^2/25 + x^2/16 = 1$

25. Endpoints of axes are $(\pm 4, 0)$ and $(0, \pm 5)$

26. Endpoints of axes are $(\pm 7, 0)$ and $(0, \pm 4)$

27. Major axis endpoints $(0, \pm 6)$, minor axis length 8

28. Major axis endpoints $(\pm 5, 0)$, minor axis length 4

29. Minor axis endpoints $(0, \pm 4)$, major axis length 10

30. Minor axis endpoints $(\pm 12, 0)$, major axis length 26

31. Major axis endpoints $(1, -4)$ and $(1, 8)$, minor axis length 8

32. Major axis endpoints are $(-2, -3)$ and $(-2, 7)$, minor axis length 4 $(y-2)^2/25 + (x+2)^2/4 = 1$

33. The foci are $(1, -4)$ and $(5, -4)$; the major axis endpoints are $(0, -4)$ and $(6, -4)$. $(x-3)^2/9 + (y+4)^2/5 = 1$

34. The foci are $(-2, 1)$ and $(-2, 5)$; the major axis endpoints are $(-2, -1)$ and $(-2, 7)$. $(y-3)^2/16 + (x+2)^2/12 = 1$

35. The major axis endpoints are $(3, -7)$ and $(3, 3)$; the minor axis length is 6. $(y+2)^2/25 + (x-3)^2/9 = 1$

36. The major axis endpoints are $(-5, 2)$ and $(3, 2)$; the minor axis length is 6. $(x+1)^2/16 + (y-2)^2/9 = 1$

In Exercises 37–40, find the center, vertices, and foci of the ellipse.

37. $\dfrac{(x+1)^2}{25} + \dfrac{(y-2)^2}{16} = 1$ **38.** $\dfrac{(x-3)^2}{11} + \dfrac{(y-5)^2}{7} = 1$

39. $\dfrac{(y+3)^2}{81} + \dfrac{(x-7)^2}{64} = 1$ **40.** $\dfrac{(y-1)^2}{25} + \dfrac{(x+2)^2}{16} = 1$

In Exercises 41–44, graph the ellipse using a parametric grapher.

41. $\dfrac{y^2}{25} + \dfrac{x^2}{4} = 1$ **42.** $\dfrac{x^2}{30} + \dfrac{y^2}{20} = 1$

43. $\dfrac{(x+3)^2}{12} + \dfrac{(y-6)^2}{5} = 1$ **44.** $\dfrac{(y+1)^2}{15} + \dfrac{(x-2)^2}{6} = 1$

In Exercises 45–48, prove that the graph of the equation is an ellipse, and find its vertices, foci, and eccentricity.

45. $9x^2 + 4y^2 - 18x + 8y - 23 = 0$

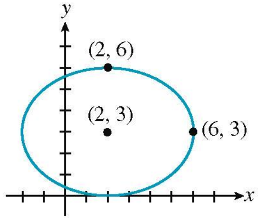
46. $3x^2 + 5y^2 - 12x + 30y + 42 = 0$

47. $9x^2 + 16y^2 + 54x - 32y - 47 = 0$

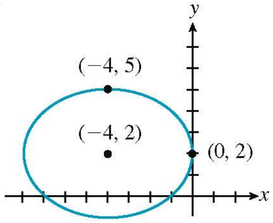
48. $4x^2 + y^2 - 32x + 16y + 124 = 0$

In Exercises 49 and 50, write an equation for the ellipse.

49.



50.



51. Writing to Learn Prove that an equation for the ellipse with center $(0, 0)$, foci $(0, \pm c)$, and semimajor axis $a > c \geq 0$ is $y^2/a^2 + x^2/b^2 = 1$, where $b^2 = a^2 - c^2$. [*Hint*: Refer to derivation at the beginning of the section.]

52. Writing to Learn Dancing Planets Using the data in Table 8.1, prove that the planet with the most eccentric orbit sometimes is closer to the Sun than the planet with the least eccentric orbit.

Table 8.1 Semimajor Axes and Eccentricities of the Planets

Planet	Semimajor Axis (Gm)	Eccentricity
Mercury	57.9	0.2056
Venus	108.2	0.0068
Earth	149.6	0.0167
Mars	227.9	0.0934
Jupiter	778.3	0.0485
Saturn	1427	0.0560
Uranus	2869	0.0461
Neptune	4497	0.0050
Pluto	5900	0.2484

Source: Shupe, et al., National Geographic Atlas of the World (rev. 6th ed.). Washington, DC: National Geographic Society, 1992, plate 116, and other sources.

53. The Moon's Orbit The Moon's apogee (farthest distance from Earth) is 252,710 miles, and perigee (closest distance to Earth) is 221,463 miles. Assuming the Moon's orbit of Earth is elliptical with Earth at one focus, calculate and interpret a, b, c, and e.

54. Hot Mercury Given that the diameter of the Sun is about 1.392 Gm, how close does Mercury get to the Sun's surface? ≈ 45.3 Gm

55. Saturn Find the perihelion and aphelion distances of Saturn. ≈ 1347 Gm, ≈ 1507 Gm

56. Venus and Mars Write equations for the orbits of Venus and Mars in the form $x^2/a^2 + y^2/b^2 = 1$.

57. Sungrazers One comet group, known as the sungrazers, passes within a Sun's diameter (1.392 Gm) of the solar surface. What can you conclude about $a - c$ for orbits of the sungrazers? $a - c < 1.5(1.392) = 2.088$

58. Halley's Comet The orbit of Halley's comet is 36.18 AU long and 9.12 AU wide. What is its eccentricity? ≈ 0.97

59. Lithotripter For an ellipse that generates the ellipsoid of a lithotripter, the major axis has endpoints $(-8, 0)$ and $(8, 0)$. One endpoint of the minor axis is $(0, 3.5)$. Find the coordinates of the foci. $(\pm\sqrt{51.75}, 0) \approx (\pm 7.19, 0)$

60. Lithotripter (Refer to Figure 8.21.) A lithotripter's shape is formed by rotating the portion of an ellipse below its minor axis about its major axis. If the length of the major axis is 26 in. and the length of the minor axis is 10 in., where should the shock-wave source and the patient be placed for maximum effect?

Group Activities In Exercises 61 and 62, solve the system of equations algebraically and support your answer graphically.

61. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$x^2 + y^2 = 4$

$(-2, 0), (2, 0)$

62. $\dfrac{x^2}{9} + y^2 = 1$

$x - 3y = -3$

$(-3, 0), (0, 1)$

63. Group Activity Consider the system of equations

$$x^2 + 4y^2 = 4$$
$$y = 2x^2 - 3$$

(a) Solve the system graphically.

(b) If you have access to a grapher that also does symbolic algebra, use it to find the exact solutions to the system.

64. Writing to Learn Look up the adjective *eccentric* in a dictionary and read its various definitions. Notice that the word is derived from *ex-centric*, meaning "out-of-center" or "off-center." Explain how this is related to the word's everyday meanings as well as its mathematical meaning for ellipses.

Standardized Test Questions

65. True or False The distance from a focus of an ellipse to the closer vertex is $a(1+e)$, where a is the semimajor axis and e is the eccentricity. Justify your answer.

66. True or False The distance from a focus of an ellipse to either endpoint of the minor axis is half the length of the major axis. Justify your answer.

In Exercises 67–70, you may use a graphing calculator to solve the problem.

67. Multiple Choice One focus of $x^2 + 4y^2 = 4$ is C

(A) $(4, 0)$. **(B)** $(2, 0)$.

(C) $(\sqrt{3}, 0)$. **(D)** $(\sqrt{2}, 0)$.

(E) $(1, 0)$.

68. Multiple Choice The focal axis of $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{16} = 1$ is C

(A) $y = 1$. **(B)** $y = 2$.

(C) $y = 3$. **(D)** $y = 4$.

(E) $y = 5$.

69. Multiple Choice The center of $9x^2 + 4y^2 - 72x - 24y + 144 = 0$ is B

(A) $(4, 2)$. **(B)** $(4, 3)$.

(C) $(4, 4)$. **(D)** $(4, 5)$.

(E) $(4, 6)$.

70. Multiple Choice The perimeter of a triangle with one vertex on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the other two vertices at the foci of the ellipse would be C

(A) $a + b$. **(B)** $2a + 2b$.

(C) $2a + 2c$. **(D)** $2b + 2c$.

(E) $a + b + c$.

Explorations

71. Area and Perimeter The area of an ellipse is $A = \pi ab$, but the perimeter cannot be expressed so simply:

$$P \approx \pi(a+b)\left(3 - \frac{\sqrt{(3a+b)(a+3b)}}{a+b}\right)$$

(a) Prove that, when $a = b = r$, these become the familiar formulas for the area and perimeter (circumference) of a circle.

(b) Find a pair of ellipses such that the one with greater area has smaller perimeter. Answers will vary.

72. Writing to Learn Kepler's Laws We have encountered Kepler's First and Third Laws (p. 193). Using a library or the Internet,

(a) Read about Kepler's life, and write in your own words how he came to discover his three laws of planetary motion. Answers will vary.

(b) What is Kepler's Second Law? Explain it with both pictures and words. Explanations and drawings will vary.

73. Pendulum Velocity vs. Position As a pendulum swings toward and away from a motion detector, its distance (in meters) from the detector is given by the position function $x(t) = 3 + \cos(2t - 5)$, where t represents time (in seconds). The velocity (in m/sec) of the pendulum is given by $y(t) = -2\sin(2t - 5)$,

(a) Using parametric mode on your grapher, plot the (x, y) relation for velocity versus position for $0 \le t \le 2\pi$

(b) Write the equation of the resulting conic in standard form, in terms of x and y, and eliminating the parameter t.

74. Pendulum Velocity vs. Position A pendulum that swings toward and away from a motion detector has a distance (in feet) from the detector of $x(t) = 5 + 3\sin(\pi t + \pi/2)$ and a velocity (in ft/sec) of $y(t) = 3\pi\cos(\pi t + \pi/2)$, where t represents time (in seconds).

(a) Prove that the plot of velocity versus position (distance) is an ellipse.

(b) Writing to Learn Describe the motion of the pendulum.

Extending the Ideas

75. Prove that a nondegenerate graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is an ellipse if $AC > 0$.

76. Writing to Learn The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 0$$

is considered to be a degenerate ellipse. Describe the graph. How is it like a full-fledged ellipse, and how is it different?