OLIY MATEMATIKA

1.ANIQMASLIKLARNI OCHISH (Lopital qoidalari).
2.MUSTAQIL YECHISH UCHUN MISOL VA MASALALAR.

ANIQMASLIKLARNI OCHISH
(Lopital qoidalari)

1. Ikki funksiya orttirmalarining nisbati haqida teorema (Koshi teoeremasi)

Koshi teoremasi. $f(x)$ va $g(x)$ funksiyalar $[a,b]$ segmentda aniqlangan va uzluksiz bo`lsin. Agar bu funksiyalar $(a,b)$ intervalda chekli $f'(x)$ va $g'(x)$ hosilalarga ega bo`lib, $\forall x\in(a,b)$ uchun $g'(x)\neq 0$ bo`lsa, u holda shunday c $(a<c<b)$ nuqta topiladiki,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

bo`ladi.

1-misol. Ushbu

$$f(x)=e^x \qquad \text{va} \qquad g(x)=\frac{x^2}{1+x^2}$$

funksiyalar $[-3,3]$ segmentda Koshi teoremasining shartlarini qanoatlantiradimi?

Berilgan funksiyalar $[-3,3]$ segmentda uzluksiz, $(-3,3)$ da

$$f'(x)=e^x \qquad \text{va} \qquad g'(x)=\frac{2x}{(1+x^2)^2}$$

hosilalarga ega. Biroq, $g'(0)=0$. Demak, $f(x)$ va $g(x)$ funksiyalar Koshi teoremasining shartlarini qanoatlantirmaydi.

Ma`lumki, funksiyalarning limitini topish muhim masalalardan biri bo`lib, ayni paytda ularni hisoblashda ancha qiyinchiliklar yuzaga keladi. Funksiyalarning hosilalaridan foydalanib ularning limitlarini topishni osonlashtiradigan qoidalar mavjud bo`lib, ular ***Lopital qoidalari*** deyiladi. Biz quyida shu qoidalar bayonini keltiramiz.

2. $\frac{0}{0}$ ko`rinishidagi aniqmasliklar.

1-teorema. $f(x)$ va $g(x)$ funksiyalar uchun quyidagi shartlar o`rinli bo`lsin:

1) $f(x)$ va $g(x)$ funksiyalar a nuqtaning biror atrofida aniqlangan va chekli hosilaga ega.
2) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$;
3) a nuqtaning biror atrofida $g'(x) \neq 0$
4) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ -chekli yoki cheksiz.

U holda

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

tenglik o`rinli bo`ladi.

Isboti. $f(x)$ hamda $g(x)$ funksiyalarning $x = a$ nuqtadagi qiymatlarini nolga teng deb olamiz

$$f(a) = 0, \qquad g(a) = 0.$$

Natijada

$$\lim_{x \to a} f(x) = 0 = f(a), \qquad \lim_{x \to a} g(x) = 0 = g(a).$$

bo`lib, $f(x)$ va $g(x)$ funksiyalar $x = a$ nuqtada uzluksiz bo`lib qoladi. Endi ixtiyoriy $x \in (a, b)$ nuqta olib, $[a, x]$ segmentda $f(x)$ va $g(x)$ funksiyalarni qaraymiz. Bu segmentda $f(x)$ va $g(x)$ funksiyalar Koshi teoremasining shartlarini qanoatlantiradi. Demak a va x orasida shunday c $(a < c < x)$ nuqta topiladiki

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

tenglik o`rinli bo`ladi.

Agar $f(a)=0$, $g(a) = 0$ bo`lishini e`tiborga olsak, unda keying tenglik

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

ko`rinishga keladi.

Ravshanki, $x \to a$, da $c \to a$. Demak,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c)}{g'(c)} = k.$$

Bu esa teoremani isbotlaydi.

Izoh. Agar bu teoremaning shartlari a nuqtaning chap (yoki o`ng) yarim atrofida bajarilsa, u holda teorema $\frac{f(x)}{g(x)}$ ning $a$ nuqtadagi chap (yoki o`ng) limitiga nisbatan o`rinli bo`ladi.

1-misol. Ushbu limitni hisoblang.

$$\lim_{x\to 0}\frac{e^{\alpha x}-\cos\alpha x}{e^{\beta x}-\cos\beta x}$$

Bu holda $f(x)=e^{\alpha x}-\cos\alpha x$, $g(x)=e^{\beta x}-\cos\beta x$ bo`lib, ular uchun teorema shartlari bajariladi:

a) $\lim\limits_{x\to 0} f(x)=0=\lim\limits_{x\to 0} g(x)$,

b) $f'(x)=\alpha(e^{\alpha x}+\sin\alpha x)$, $\quad g'(x)=\beta(e^{\beta x}+\sin\beta x)$,

v) $\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}=\frac{\alpha}{\beta}$, $\quad \lim\limits_{x\to 0}\frac{(e^{\alpha x}+\sin\alpha x)}{(e^{\beta x}+\sin\beta x)}=\frac{\alpha}{\beta}$,

u holda teoremaga ko`ra:

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\alpha}{\beta}.$$

2-teorema. $f(x)$ va $g(x)$ funksiyalar uchun quyidagi shartlar o`rinli bo`lsin:

a. $f(x)$ va $g(x)$ funksiyalar $(a,+\infty)$ da aniqlangan va chekli hosilaga ega;

b. $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty} g(x)=0$;

c. $g'(x) \neq 0$, $\forall x \in (a, +\infty)$;

d. $\lim\limits_{x \to +\infty} \dfrac{f'(x)}{g'(x)}$ -chekli yoki cheksiz.

U holda

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

tenglik o`rinli bo`ladi.

2-misol. Ushbu

$$\lim_{x \to +\infty} \frac{\pi - 2\,arctgx}{\ln\left(1 + \dfrac{1}{x}\right)} \qquad \text{limitni hisoblang.}$$

Bu yerda

$$f(x) = \pi - 2arctgx, \qquad g(x) = \ln\left(x + \frac{1}{x}\right)$$

funksiyalar teoremaning 1)-3) shartlarini qanoatlantirishini tekshirish qiyin emas. Ravshanki,

$$\lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to +\infty} \frac{-2\dfrac{1}{1+x^2}}{\dfrac{x}{1+x} \cdot \left(-\dfrac{1}{x^2}\right)} = \lim_{x \to +\infty} \frac{\dfrac{2}{1+x^2}}{-\dfrac{1}{(1+x)x}} = \lim_{x \to +\infty} \frac{2}{1+x^2}(1+x)x = 2.$$

Demak, 4) shart ham bajariladi. Shuning uchun teoremaga ko`ra:

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{\pi-2arctgx}{\ln\left(1+\frac{1}{x}\right)}=2.$$

3. $\frac{\infty}{\infty}$ - ko`rinishidagi aniqmasliklar.

3-teorema. $f(x)$ va $g(x)$ funksiyalari uchun quyidagi shartlar o`rinli bo`lsin:

a. bu funksiyalar a nuqtaning biror atrofida aniqlangan va chekli hosilaga ega.

b. $\lim_{x\to a} f(x)=\infty$, $\lim_{x\to a} g(x)=\infty$;

c. a nuqtaning shu atrofida $g'(x)\neq 0$;

d. $\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$ -chekli yoki cheksiz.

U holda
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$
tenglik o`rinli bo`ladi.

3-misol. Ushbu

$$\lim_{x\to\frac{\pi}{2}}\frac{\ln(x-\frac{\pi}{2})}{tgx}$$ limit hisoblansin.

Bu yerda

$$f(x)=\ln(x-\frac{\pi}{2}), \qquad g(x)=tgx$$

bo`lib, ular teoremaning 1)-3) shartlarini qanoatlantiradi va

$$\lim_{x\to\frac{\pi}{2}}\frac{f'(x)}{g'(x)}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{x-\frac{\pi}{2}}}{\frac{1}{\cos^2 x}}=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{x-\frac{\pi}{2}}.$$

Keyingi limit $\frac{0}{0}$ ko`rinishidagi aniqmaslik bo`lib,

$$f_1(x)=\cos^2 x, \qquad g_1(x)=x-\frac{\pi}{2}$$

funksiyalar 1-teoremaning barcha shartlarini qanoatlantiradi. Shu teoremaga asosan:

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{x-\frac{\pi}{2}}=\lim_{x\to\frac{\pi}{2}}\frac{2\cos x(-\sin x)}{1}=0.$$

Demak, 3-teoremaga ko`ra,

$$\lim_{x\to\frac{\pi}{2}}\frac{\ln(x-\frac{\pi}{2})}{tgx}=0.$$

4. Boshqa ko`rinishdagi aniqmasliklar.

Ma`lumki,

$$\lim_{x\to a} f(x)=0, \qquad \lim_{x\to a} g(x)=\infty$$ bo` lsa, $f(x)\cdot g(x)$

ko`paytma $0\cdot\infty$ ko`rinishdagi aniqmaslik bo`lib, uni

$$f(x)\cdot g(x)=\frac{f(x)}{\frac{1}{g(x)}}=\frac{g(x)}{\frac{1}{f(x)}}$$

ko`rinishda ifodalash orqali $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko`rinishdagi aniqmaslikka keltirish mumkin.

Shuningdek, $\lim_{x\to a} f(x)=+\infty$, $\lim_{x\to a} g(x)=+\infty$ bo`lsa, $f(x)-g(x)$ ayirma $\infty-\infty$ ko`rinishdagi aniqmaslik bo`lib, uni ham

$$f(x)-g(x)=\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)}\cdot\frac{1}{g(x)}}$$

ko`rinishda ifodalab, $\frac{0}{0}$ ko`rinishdagi aniqmaslikka keltirish mumkin.

Shunday qilib, funksiya hosilalari yordamida $0\cdot\infty$ va $\infty-\infty$ ko`rinishdagi aniqmasliklarni ochishda, ularni $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko`rinishdagi aniqmasliklarga keltirilib, so`ngra yuqoridagi teoremalar qo`llaniladi.

Ma`lumki, $x \to a$ da $f(x)$ funksiya 1, 0 va ∞ ga, $g(x)$ funksiya esa mos ravishda ∞, 0 va 0 ga intilganda

$[f(x)]^{g(x)}$ darajali-ko`rsatkichli ifodada $1^{\infty}$, $0^{0}$, $\infty^{0}$ ko`rinishdagi aniqmasliklar kelishi mumkin. Bu ko`rinishdagi aniqmasliklarni ochish uchun avvalo

$$y = [f(x)]^{g(x)}$$

ifoda logarifmlanadi:

$$\ln y = g(x) \cdot \ln f(x)$$

$x \to a$ da $g(x)\ln f(x)$ ifoda $0 \cdot \infty$ ko`rinishdagi aniqmaslikni ifodalashi ravshan.

Izoh. Agar $f(x)$ va $g(x)$ funksiyaning $f'(x)$ va $g'(x)$ hosilalari ham $f(x)$ va $g(x)$ lardek yuqorida keltirilgan teoremalarning barcha shartlarini qanoatlantirsa, u holda

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = \lim \frac{f''(x)}{g''(x)}$$

tengliklar o`rinli bo`ladi, ya`ni bu holda Lopital qoidasini takror qo`llash mumkin bo`ladi.

4-misol. Ushbu

$$\lim_{x\to\frac{\pi}{4}} (tgx)^{tg\,2x}$$

limitni hisoblang.

Bu limit 1^{∞} ko`rinishdagi aniqmaslik bo`lib, yuqorida aytilganlarga asosan:

$$y = (tgx)^{tg\,2x}$$

ifodani logarifmlash natijasida

$$\lim_{x\to\frac{\pi}{4}} \ln y = \lim_{x\to\frac{\pi}{4}} \frac{\ln tgx}{(tg\,2x)^{-1}}$$

$\frac{0}{0}$ ko`rinishdagi aniqmaslikka keladi.

Endi Lopital qoidasini qo`llasak:

$$\lim_{x\to\frac{\pi}{4}} \frac{(\ln tgx)'}{\left[(tg\,2x)^{-1}\right]'} = \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{tgx}\cdot\frac{1}{\cos^2 x}}{\frac{\frac{2}{\cos^2 x}}{(tg\,2x)^2}} = -\lim_{x\to\frac{\pi}{4}} \sin 2x = -1 .$$

Demak,

$$\lim_{x\to\frac{\pi}{4}} (tgx)^{tg\,2x} = \frac{1}{e}$$

MUSTAQIL YECHISH UCHUN MISOL VA MASALALAR

1. $\lim\limits_{x\to 0}\frac{xctgx-1}{x^2}$.

2. $\lim\limits_{x\to 0}\frac{3tg4x-12tgx}{3\sin 4x-12\sin x}$.

3. $\lim\limits_{x\to \frac{\pi}{4}}\frac{\sqrt[3]{tgx}-1}{2\sin^2 x-1}$.

4. $\lim\limits_{x\to 0}\frac{x(e^x+1)-2(e^x-1)}{x^3}$.

5. $\lim\limits_{x\to 0}\frac{1-\cos x^2}{x^2\sin x^2}$.

6. $\lim\limits_{x\to 0}\frac{a^x-a^{\sin x}}{x^3}$. $(a>0)$.

7. $\lim\limits_{x\to 0}\frac{\arcsin 2x-2\arcsin x}{x^2}$.

8. $\lim\limits_{x\to 0}\frac{\ln x}{x^{\varepsilon}}$ $(\varepsilon>0)$

9. $\lim\limits_{x\to 1}\frac{x^x-1}{\ln x}$.

10. $\lim\limits_{x\to 1}\frac{x^{\alpha}-1}{x^{\beta}-1}$, $(\beta\neq 0)$.

11. $\lim\limits_{x\to \frac{\pi}{2}}\frac{\cos(2m+1)x}{\cos(2n+1)x}$. $m\in N$, $n\in N$

12. $\lim\limits_{x\to 1}\frac{\ln x-x+1}{x-x^x}$.

13. $\lim\limits_{x\to +0}\frac{\ln x}{\ln\sin x}$.

14. $\lim\limits_{x\to +\infty}\frac{x^{\alpha}\ln^{\beta}x}{e^{\alpha x}}$.

15. $\lim\limits_{x\to 1}\left(\frac{\alpha}{1-x^{\alpha}}-\frac{\beta}{1-x^{\beta}}\right)$.