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Mathematical Analysis I

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6.5 Theorems of Rolle and of the Mean Value

The results we present in this section, called Theorem of Rolle and Mean Value Theorem, are fundamental for the study of differentiable maps on an interval.

Theorem 6.22 (Rolle) *Let f be a function defined on a closed bounded interval $[a, b]$, continuous on $[a, b]$ and differentiable on (a, b) (at least). If $f(a) = f(b)$, there exists an $x_0 \in (a, b)$ such that*

$$f'(x_0) = 0.$$

In other words, f admits at least one critical point in (a, b).

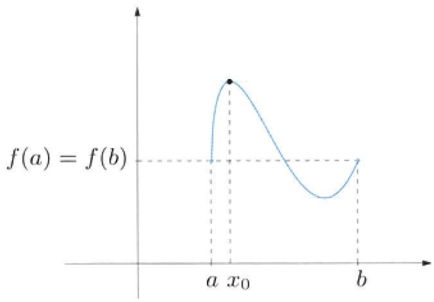

Figure 6.5. Rolle's Theorem

Proof. By the Theorem of Weierstrass the range $f([a, b])$ is the closed interval $[m, M]$ bounded by the minimum and maximum values m, M of the map:

$$m = \min_{x \in [a,b]} f(x) = f(x_m), \qquad M = \max_{x \in [a,b]} f(x) = f(x_M),$$

for suitable $x_m, x_M \in [a, b]$.
In case $m = M$, f is constant on $[a, b]$, so in particular $f'(x) = 0$ for any $x \in (a, b)$ and the theorem follows.
Suppose then $m < M$. Since $m \leq f(a) = f(b) \leq M$, one of the strict inequalities $f(a) = f(b) < M$, $m < f(a) = f(b)$ will hold.
If $f(a) = f(b) < M$, the absolute maximum point x_M cannot be a nor b; thus, $x_M \in (a, b)$ is an interior extremum point at which f is differentiable. By Fermat's Theorem 6.21 we have that $x_M = x_0$ is a critical point.
If $m < f(a) = f(b)$, one proves analogously that x_m is the critical point x_0 of the claim. □

The theorem proves the existence of one critical point in (a, b); Fig. 6.5 shows that there could actually be more.

Theorem 6.23 (**Mean Value Theorem** or **Lagrange Theorem**) *Let f be defined on the closed and bounded interval $[a,b]$, continuous on $[a,b]$ and differentiable (at least) on (a,b). Then there is a point $x_0 \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(x_0). \tag{6.9}$$

Every such point x_0 we shall call **Lagrange point for f in (a,b)**.

Proof. Introduce an auxiliary map

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$$

defined on $[a,b]$. It is continuous on $[a,b]$ and differentiable on (a,b), as difference of f and an affine map, which is differentiable on all of $\mathbb{R}$. Note

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

It is easily seen that

$$g(a) = f(a), \qquad g(b) = f(a),$$

so Rolle's Theorem applies to g, with the consequence that there is a point $x_0 \in (a,b)$ satisfying

$$g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0.$$

But this is exactly (6.9). □

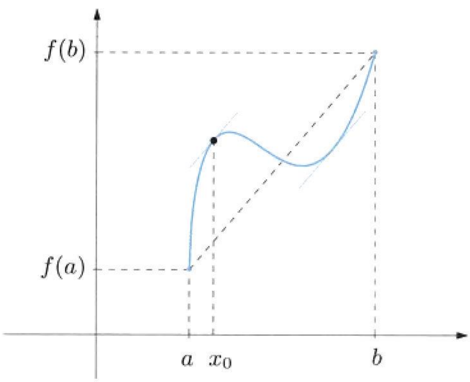

Figure 6.6. Lagrange point for f in (a,b)

The meaning of the Mean Value Theorem is clarified in Fig. 6.6. At each Lagrange point, the tangent to the graph of f is *parallel* to the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.

Example 6.24

Consider $f(x) = 1 + x + \sqrt{1-x^2}$, a continuous map on its domain $[-1, 1]$ as composite of elementary continuous functions. It is also differentiable on the open interval $(-1, 1)$ (not at the end-points), in fact

$$f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}.$$

Thus f fulfills the Mean Value Theorem's hypotheses, and must admit a Lagrange point in $(-1, 1)$. Now (6.9) becomes

$$1 = \frac{f(1) - f(-1)}{1-(-1)} = f'(x_0) = 1 - \frac{x_0}{\sqrt{1-x_0^2}},$$

satisfied by $x_0 = 0$. □

6.6 First and second finite increment formulas

We shall discuss a couple of useful relations to represent how a function varies when passing from one point to another of its domain.

Let us begin by assuming f is differentiable at x_0. By definition

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

that is to say

$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0.$$

Using the Landau symbols of Sect. 5.1, this becomes

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0), \quad x \to x_0.$$

An equivalent formulation is

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0), \quad x \to x_0, \tag{6.10}$$

or

$$\Delta f = f'(x_0)\Delta x + o(\Delta x), \quad \Delta x \to 0, \tag{6.11}$$

by putting $\Delta x = x - x_0$ and $\Delta f = f(x) - f(x_0)$.

Equations (6.10)-(6.11) are equivalent writings of what we call the **first formula of the finite increment** , the geometric interpretation of which can be found in