7.6 Nonhomogeneous Systems

Now consider the nonhomogeneous linear system of n equations in m unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n. \end{aligned}$$

In matrix form,

$$\mathbf{AX} = \mathbf{B} \tag{7.1}$$

where $\mathbf{A}$ is the coefficient matrix,

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

The system is *nonhomogeneous* if at least one $b_j \neq 0$. Nonhomogeneous systems differ from linear systems in two significant ways.

1. A nonhomogeneous system may have no solution. For example, the system

$$\begin{aligned} 2x_1 - 3x_2 &= 6 \\ 4x_1 - 6x_2 &= 8 \end{aligned}$$

We call $\mathbf{AX} = \mathbf{B}$ *consistent* if there is a solution. If there is no solution, the system is *inconsistent*.

2. A linear combination of solutions of a nonhomogeneous system $\mathbf{AX} = \mathbf{B}$ need not be a solution. Therefore the solutions do not have the vector space structure seen in the homogeneous case.

Nevertheless, solutions of $\mathbf{AX} = \mathbf{B}$ do have a property that parallels that for solutions of linear second order differential equations. We will call $\mathbf{AX} = \mathbf{O}$ the *associated homogeneous system* of the nonhomogeneous system $\mathbf{AX} = \mathbf{B}$. Although a sum of solutions of the nonhomogeneous system need not be a solution, we claim that the *difference* of any two solutions of the nonhomogeneous system is a solution, not of the system, but of the associated *homogeneous* system. The reason for this is that, if $\mathbf{AU}_1 = \mathbf{B}$ and $\mathbf{AU}_2 = \mathbf{B}$, then

$$\mathbf{A}(\mathbf{U}_1 - \mathbf{U}_2) = \mathbf{AU}_1 - \mathbf{AU}_2 = \mathbf{B} - \mathbf{B} = \mathbf{O}.$$

This is the key to the fundamental theorem for writing the general solution of $\mathbf{AX} = \mathbf{B}$.

THEOREM 7.13

Let $\mathbf{H}$ be the general solution of the associated homogeneous system. Let $\mathbf{U}_p$ be any particular solution of $\mathbf{AX} = \mathbf{B}$. Then the expression $\mathbf{H} + \mathbf{U}_p$ contains every solution of the nonhomogeneous system $\mathbf{AX} = \mathbf{B}$. ♦

system $\mathbf{AX} = \mathbf{B}$.

Proof Suppose $\mathbf{H}_1, \cdots, \mathbf{H}_k$ form a basis for the solution space of $\mathbf{AX} = \mathbf{O}$, where $k = m -$ number of nonzero rows of $(\mathbf{A}_R)$. Then the general solution of the homogeneous system is

$$\mathbf{H} = \alpha_1 \mathbf{H}_1 + \cdots + \alpha_k \mathbf{H}_k.$$

If $\mathbf{U}$ is any solution of $\mathbf{AX} = \mathbf{B}$, then $\mathbf{U} - \mathbf{U}_p$ is a solution of the associated homogeneous system, and therefore has the form

$$\mathbf{U} - \mathbf{U}_p = c_1 \mathbf{H}_1 + \cdots + c_k \mathbf{H}_k$$

for some constants $c_1, \cdots, c_k$. But then

$$\mathbf{U} = c_1 \mathbf{H}_1 + \cdots + c_k \mathbf{H}_k + \mathbf{U}_p,$$

and this solution is contained in the general expression $\mathbf{H} + \mathbf{U}_p$. ♦

As an immediate consequence, Theorem 7.13 tells us when a nonhomogeneous system can have only one solution.

COROLLARY 7.6

A consistent nonhomogeneous system $\mathbf{AX} = \mathbf{B}$ has a unique solution if and only if the associated homogeneous system has only the trivial solution. ♦

The corollary follows from the fact that the nonhomogeneous system has a unique solution exactly when $\mathbf{H}$ is the zero vector in Theorem 7.13.

Theorem 7.13 suggests a strategy for finding all solutions of $\mathbf{AX} = \mathbf{B}$, when the system is consistent.

inverse.

1. $\begin{pmatrix} 2 & -1 \\ 1 & 6 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$

3. $\begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 5 \\ -7 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 6 & -1 & 3 \\ 0 & 1 & -4 \\ 2 & 2 & -3 \end{pmatrix}$

7. $\begin{pmatrix} 0 & -4 & 3 \\ 2 & -1 & 6 \\ 1 & -1 & 7 \end{pmatrix}$

8. $\begin{pmatrix} 11 & 0 & -5 \\ 0 & 1 & 0 \\ 4 & -7 & 9 \end{pmatrix}$

9. $\begin{pmatrix} 3 & 1 & -2 & 1 \\ 4 & 6 & -3 & 9 \\ -2 & 1 & 7 & 4 \\ 13 & 0 & 1 & 5 \end{pmatrix}$

10. $\begin{pmatrix} 7 & -3 & -4 & 1 \\ 8 & 2 & 0 & 0 \\ 1 & 5 & -1 & 7 \\ 3 & -2 & -5 & 9 \end{pmatrix}$

8.5 Cramer's Rule

Cramer's rule is a determinant formula for the unique solution of a nonhomogeneous system $\mathbf{AX} = \mathbf{B}$ when $\mathbf{A}$ is nonsingular. Of course, this is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, but the following method is sometimes convenient.

THEOREM 8.5 Cramer's Rule

Let **A** be a nonsingular $n \times n$ matrix of numbers, and **B** be an $n \times 1$ matrix of numbers. Then the unique solution of $\mathbf{AX} = \mathbf{B}$ is determined by

$$x_k = \frac{1}{|\mathbf{A}|}|\mathbf{A}(k; \mathbf{B})| \tag{8.7}$$

for $k = 1, 2, \cdots, n$, where $\mathbf{A}(k; \mathbf{B})$ is the matrix obtained from **A** by replacing column k of **A** with **B**. ♦

It is easy to see why this works. Let

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Multiply column k of **A** by x_k. This multiplies the determinant of **A** by x_k:

$$x_k|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k}x_k & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k}x_k & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk}x_k & \cdots & a_{nn} \end{vmatrix}.$$

For each $j \neq k$ add x_j times column j to column k in the last determinant. Since this operation does not change the value of a determinant, then

does not change the value of a determinant, then

$$x_k|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{11}x_1 + \cdots + a_{1n}x_n & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{21}x_1 + \cdots + a_{2n}x_n & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n1}x_1 + \cdots + a_{nn}x_n & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} = |\mathbf{A}(k; \mathbf{B})|$$

and this gives us equation (8.7).

EXAMPLE 8.7

Solve the system

$$x_1 - 3x_2 - 4x_3 = 1$$
$$-x_1 + x_2 - 3x_3 = 14$$
$$x_2 - 3x_3 = 5.$$

The matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 1 & -3 \\ 0 & 1 & -3 \end{pmatrix}.$$

We find that $|\mathbf{A}| = 13$, so this system has a unique solution. By Cramer's rule,

$$x_1 = \frac{1}{13}\begin{vmatrix} 1 & -3 & -4 \\ 14 & 1 & -3 \\ 5 & 1 & -3 \end{vmatrix} = -\frac{117}{13} = -9,$$

$$x_2 = \frac{1}{13}\begin{vmatrix} 1 & 1 & -4 \\ -1 & 14 & -3 \\ 0 & 5 & -3 \end{vmatrix} = -\frac{10}{13},$$

$$x_3 = \frac{1}{13}\begin{vmatrix} 1 & -3 & 1 \\ -1 & 1 & 14 \\ 0 & 1 & 5 \end{vmatrix} = -\frac{25}{13}. \blacklozenge$$

SECTION 8.5 PROBLEMS

In each of Problems 1 through 10, solve the system using Cramer's rule, or show that the rule does not apply because the matrix of coefficients is singular.

1. $15x_1 - 4x_2 = 5$
   $8x_1 + x_2 = -4$

2. $x_1 + 4x_2 = 3$ …

$x_1 - 3x_2 + x_3 - 4x_5 = 0$
$-2x_1 + x_3 - 2x_5 = 4$
$x_3 - x_4 - x_5 = 8$

7. $2x_1 - 4x_2 + x_3 - x_4 = 6$
   $x_2 - 3x_3 = 10$
   $x_1 - 4x_3 = 0$
   $x_2 - x_3 + 2x_4 = 4$

the matrix of coefficients is singular.

1. $15x_1 - 4x_2 = 5$
   $8x_1 + x_2 = -4$

2. $x_1 + 4x_2 = 3$
   $x_1 + x_2 = 0$

3. $8x_1 - 4x_2 + 3x_3 = 0$
   $x_1 + 5x_2 - x_3 = -5$
   $-2x_1 + 6x_2 + x_3 = -4$

4. $5x_1 - 6x_2 + x_3 = 4$
   $-x_1 + 3x_2 - 4x_3 = 5$
   $2x_1 + 3x_2 + x_3 = -8$

5. $x_1 + x_2 - 3x_3 = 0$
   $x_2 - 4x_3 = 0$
   $x_1 - x_2 - x_3 = 5$

6. $6x_1 + 4x_2 - x_3 + 3x_4 - x_5 = 7$
   $x_1 - 4x_2 + x_5 = -5$

7. $2x_1 - 4x_2 + x_3 - x_4 = 6$
   $x_2 - 3x_3 = 10$
   $x_1 - 4x_3 = 0$
   $x_2 - x_3 + 2x_4 = 4$

8. $2x_1 - 3x_2 + x_4 = 2$
   $x_2 - x_3 + x_4 = 2$
   $x_3 - 2x_4 = 5$
   $x_1 - 3x_2 + 4x_3 = 0$

9. $14x_1 - 3x_3 = 5$
   $2x_1 - 4x_3 + x_4 = 2$
   $x_1 - x_2 + x_3 - 3x_4 = 1$
   $x_3 - 4x_4 = -5$

10. $x_2 - 4x_4 = 18$
    $x_1 - x_2 + 3x_3 = -1$
    $x_1 + x_2 - 3x_3 + x_4 = 5$
    $x_2 + 3x_4 = 0$

8.6 The Matrix Tree Theorem

In 1847, G.R. Kirchhoff published a classic paper in which he derived many of the electrical circuit laws that bear his name, including the matrix tree theorem we will now discuss.

Figure 8.1 shows a typical electrical circuit. The underlying geometry of the circuit if shown in Figure 8.2. Such a diagram of points and interconnecting lines is called a *graph*, and was seen in the context of atoms moving through crystals in Section 7.1.3. A *labeled graph* has symbols attached to the points.