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Mathematical Analysis I

Springer

The theorem asserts that if f is differentiable on I, the following logic equivalence holds:

$$f'(x) \geq 0, \quad \forall x \in I \quad \Longleftrightarrow \quad f \text{ is increasing on } I.$$

Furthermore,

$$f'(x) > 0, \quad \forall x \in I \quad \Longrightarrow \quad f \text{ is strictly increasing on } I.$$

The latter implication is not reversible: f *strictly* increasing on I does not imply $f'(x) > 0$ for all $x \in I$. We have elsewhere observed that $f(x) = x^3$ is everywhere strictly increasing, despite having vanishing derivative at the origin.

A similar statement to the above holds if we change the word 'increasing' with 'decreasing' and the symbols $\geq$, $>$ with $\leq$, $<$.

Corollary 6.27 *Let f be differentiable on I and x_0 an interior critical point. If $f'(x) \geq 0$ at the left of x_0 and $f'(x) \leq 0$ at its right, then x_0 is a maximum point for f. Similarly, $f'(x) \leq 0$ at the left, and ≥ 0 at the right of x_0 implies x_0 is a minimum point.*

Theorem 6.26 and Corollary 6.27 justify the search for extrema among the zeroes of f', and explain why the derivative's sign affects monotonicity intervals.

Example 6.28

The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x\mathrm{e}^{2x}$ differentiates to $f'(x) = (2x+1)\mathrm{e}^{2x}$, whence $x_0 = -\frac{1}{2}$ is the sole critical point. As $f'(x) > 0$ if and only if $x > -\frac{1}{2}$, $f(x_0)$ is an absolute minimum. The function is strictly decreasing on $(-\infty, -\frac{1}{2}]$ and strictly increasing on $[-\frac{1}{2}, +\infty)$. □

6.8 Higher-order derivatives

Let f be differentiable around x_0 and let its first derivative f' be also defined around x_0.

Definition 6.29 *If f' is a differentiable function at x_0, one says f* **is twice differentiable at** x_0. *The expression*

$$f''(x_0) = (f')'(x_0)$$

is called **second derivative of** f **at** x_0. *The* **second derivative of** f, *denoted f'', is the map associating to x the number $f''(x)$, provided the latter is defined.*

Other notations commonly used for the second derivative include

$$y''(x_0), \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0), \qquad \mathrm{D}^2 f(x_0).$$

The third derivative, where defined, is the derivative of the second derivative:

$$f'''(x_0) = (f'')'(x_0).$$

In general, for any $k \geq 1$, the **derivative of order k (kth derivative) of f at** x_0 is the first derivative, where defined, of the derivative of order $(k-1)$ of f at x_0:

$$\boxed{f^{(k)}(x_0) = (f^{(k-1)})'(x_0).}$$

Alternative symbols are:

$$y^{(k)}(x_0), \qquad \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x_0), \qquad \mathrm{D}^k f(x_0).$$

For conveniency one defines $f^{(0)}(x_0) = f(x_0)$ as well.

Examples 6.30

We compute the derivatives of all orders for three elementary functions.

i) Choose $n \in \mathbb{N}$ and consider $f(x) = x^n$. Then

$$\begin{aligned} f'(x) &= nx^{n-1} = \frac{n!}{(n-1)!}x^{n-1} \\ f''(x) &= n(n-1)x^{n-2} = \frac{n!}{(n-2)!}x^{n-2} \\ &\vdots \quad \vdots \\ f^{(n)}(x) &= n(n-1)\cdots 2\cdot 1\, x^{n-n} = n!\,. \end{aligned}$$

More concisely,

$$f^{(k)}(x) = \frac{n!}{(n-k)!}x^{n-k}$$

with $0 \leq k \leq n$. Furthermore, $f^{(n+1)}(x) = 0$ for any $x \in \mathbb{R}$ (the derivative of the constant function $f^{(n)}(x)$ is 0), and consequently all derivatives $f^{(k)}$ of order $k > n$ exist and vanish identically.

ii) The sine function $f(x) = \sin x$ satisfies $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f^{(4)}(x) = \sin x$. Successive derivatives of f clearly reproduce this cyclical pattern. The same phenomenon occurs for $y = \cos x$.

iii) Because $f(x) = \mathrm{e}^x$ differentiates to $f'(x) = \mathrm{e}^x$, it follows that $f^{(k)}(x) = \mathrm{e}^x$ for every $k \geq 0$, proving the remarkable fact that all higher-order derivatives of the exponential function are equal to e^x. □

A couple of definitions wrap up the section.

Definition 6.31 *A map f is* **of class** $\mathcal{C}^k$ *($k \geq 0$) on an interval I if f is differentiable k times everywhere on I and its kth derivative $f^{(k)}$ is continuous on I. The collection of all $\mathcal{C}^k$ maps on I is denoted by $\mathcal{C}^k(I)$.*
A map f is **of class** $\mathcal{C}^\infty$ *on I if it is arbitrarily differentiable everywhere on I. One indicates by $\mathcal{C}^\infty(I)$ the collection of such maps.*

In virtue of Proposition 6.3, if $f \in \mathcal{C}^k(I)$ all derivatives of order smaller or equal than k are continuous on I. Similarly, if $f \in \mathcal{C}^\infty(I)$, all its derivatives are continuous on I.

Moreover, the elementary functions are differentiable any number of times (so they are of class $\mathcal{C}^\infty$) at every interior point of their domains.

6.9 Convexity and inflection points

Let f be differentiable at the point x_0 of the domain. As customary, we indicate by $y = t(x) = f(x_0) + f'(x_0)(x - x_0)$ the equation of the tangent to the graph of f at x_0.

Definition 6.32 *The map f is* **convex at** x_0 *if there is a neighbourhood $I_r(x_0) \subseteq \operatorname{dom} f$ such that*

$$\forall x \in I_r(x_0), \qquad f(x) \geq t(x);$$

f is **strictly convex** *if $f(x) > t(x)$, $\forall x \neq x_0$.*

The definitions for **concave** and **strictly concave** functions are alike (just change $\geq, >$ to $\leq, <$).

What does this say geometrically? A map is convex at a point if around that point the graph lies 'above' the tangent line, concave if its graph is 'below' the tangent (Fig. 6.9).

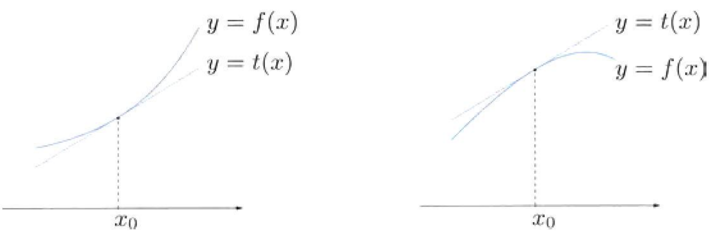

Figure 6.9. Strictly convex (left) and strictly concave (right) maps at x_0