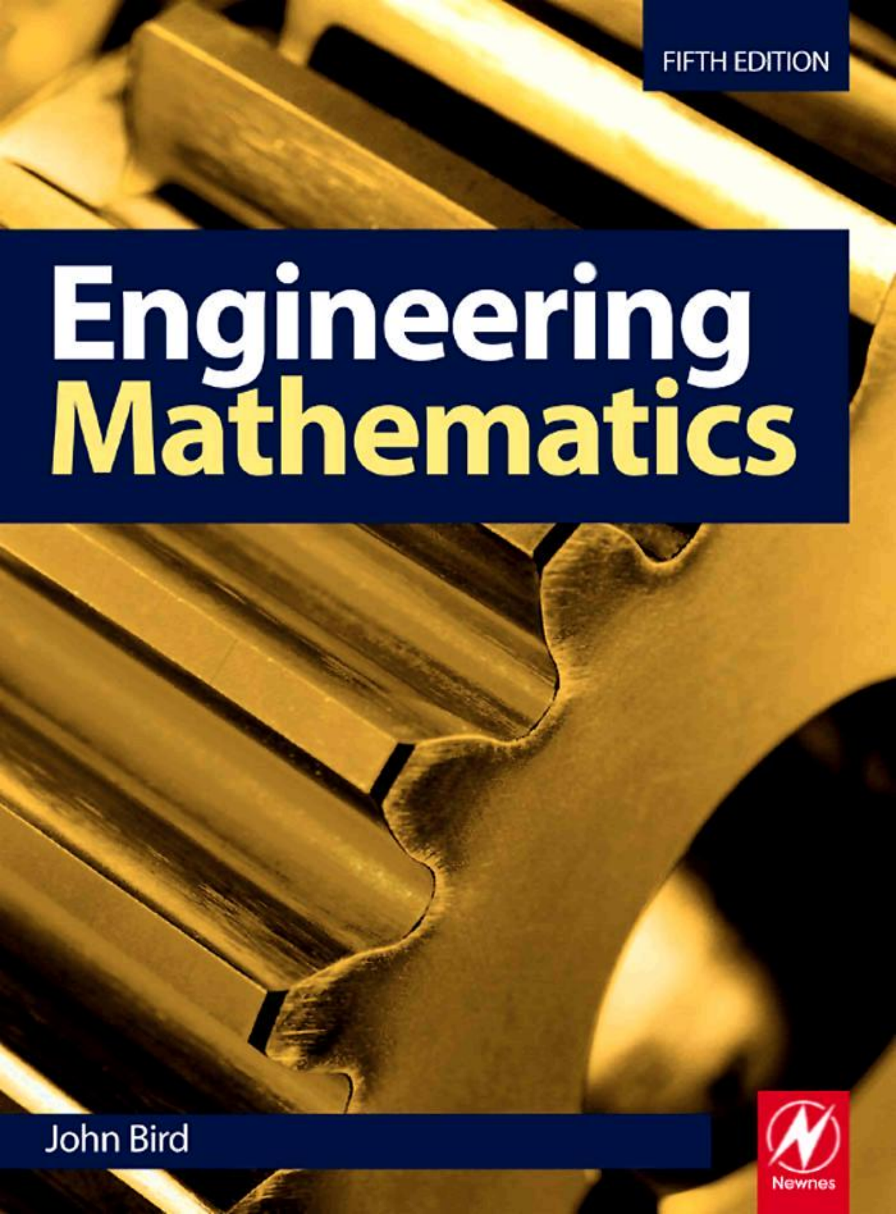
FIFTH EDITION

Engineering Mathematics



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Newnes

Chapter 35

Complex numbers

35.1 Cartesian complex numbers

(i) If the quadratic equation $x^2+2x+5=0$ is solved using the quadratic formula then:

$$\text{x} = \frac{-2 \pm \sqrt{(2)^2-(4)(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm \sqrt{(16)(-1)}}{2}$$

$$= \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2} = \frac{-2 \pm 4\sqrt{-1}}{2}$$

$$= -1 \pm 2\sqrt{-1}$$

It is not possible to evaluate $\sqrt{-1}$ in real terms. However, if an operator j is defined as $j=\sqrt{-1}$ then the solution may be expressed as $x=-1\pm j2$.

(ii) $-1+j2$ and $-1-j2$ are known as **complex numbers**. Both solutions are of the form $a+jb$, 'a' being termed the **real part** and jb the **imaginary part**. A complex number of the form $a+jb$ is called a **Cartesian complex number**.

(iii) In pure mathematics the symbol i is used to indicate $\sqrt{-1}$ (i being the first letter of the word imaginary). However i is the symbol of electric current in engineering, and to avoid possible confusion the next letter in the alphabet, j, is used to represent $\sqrt{-1}$

Problem 1. Solve the quadratic equation:

$$x^2+4=0$$

Since $x^2+4=0$ then $x^2=-4$ and $x=\sqrt{-4}$

i.e., $x = \sqrt{(-1)(4)} = \sqrt{-1}\sqrt{4} = j(\pm 2)$

$= \pm \boldsymbol{j2}$, (since $j=\sqrt{-1}$)

(Note that $\pm j2$ may also be written as $\pm \boldsymbol{2j}$).

Problem 2. Solve the quadratic equation:

$$2x^2+3x+5=0$$

Using the quadratic formula,

$$x = \frac{-3 \pm \sqrt{(3)^2-4(2)(5)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{-31}}{4} = \frac{-3 \pm \sqrt{-1}\sqrt{31}}{4}$$

$$= \frac{-3 \pm j\sqrt{31}}{4}$$

Hence $\boldsymbol{x = -\frac{3}{4} + j\frac{\sqrt{31}}{4}}$ or $\boldsymbol{-0.750 \pm j1.392}$, correct to 3 decimal places.

(Note, a graph of $y=2x^2+3x+5$ does not cross the x-axis and hence $2x^2+3x+5=0$ has no real roots).

Problem 3. Evaluate

(a) j^3 (b) j^4 (c) j^{23} (d) $\dfrac{-4}{j^9}$

(a) $j^3 = j^2 \times j = (-1) \times j = \mathbf{-j}$, since $j^2 = -1$

(b) $j^4 = j^2 \times j^2 = (-1) \times (-1) = \mathbf{1}$

(c) $j^{23} = j \times j^{22} = j \times (j^2)^{11} = j \times (-1)^{11}$

$= j \times (-1) = \mathbf{-j}$

(d) $j^9 = j \times j^8 = j \times (j^2)^4 = j \times (-1)^4$

$= j \times 1 = \mathbf{j}$

Hence $\dfrac{-4}{j^9} = \dfrac{-4}{j} = \dfrac{-4}{j} \times \dfrac{-j}{-j} = \dfrac{4j}{-j^2}$

$= \dfrac{4j}{-(-1)} = \mathbf{4j}$ or $\mathbf{j4}$

Now try the following exercise

Exercise 127 Further problems on the introduction to Cartesian complex numbers

In Problems 1 to 3, solve the quadratic equations.

1. $x^2 + 25 = 0$ $[\pm j5]$
2. $2x^2 + 3x + 4 = 0$
$\left[-\dfrac{3}{4} \pm j\dfrac{\sqrt{23}}{4} \text{ or } -0.750 \pm j1.199\right]$
3. $4t^2 - 5t + 7 = 0$
$\left[\dfrac{5}{8} \pm j\dfrac{\sqrt{87}}{8} \text{ or } 0.625 \pm j1.166\right]$
4. Evaluate (a) j^8 (b) $-\dfrac{1}{j^7}$ (c) $\dfrac{4}{2j^{13}}$
[(a) 1 (b) −j (c) $-j2$]

35.2 The Argand diagram

A complex number may be represented pictorially on rectangular or Cartesian axes. The horizontal (or x) axis is used to represent the real axis and the vertical (or y) axis is used to represent the imaginary axis. Such a diagram is called an **Argand diagram**. In Fig. 35.1, the point A represents the complex number $(3 + j2)$ and is obtained by plotting the co-ordinates $(3, j2)$ as in graphical work. Figure 35.1 also shows the Argand points B, C and D representing the complex numbers $(-2 + j4)$, $(-3 - j5)$ and $(1 - j3)$ respectively.

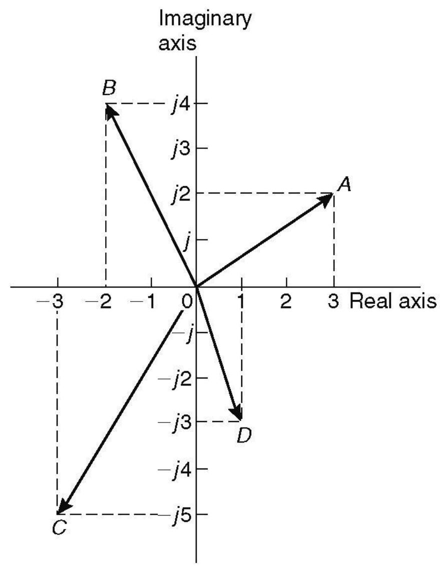

Figure 35.1

35.3 Addition and subtraction of complex numbers

Two complex numbers are added/subtracted by adding/subtracting separately the two real parts and the two imaginary parts.

For example, if $Z_1 = a + jb$ and $Z_2 = c + jd$,

then $Z_1 + Z_2 = (a + jb) + (c + jd)$

$= (a + c) + j(b + d)$

and $Z_1 - Z_2 = (a + jb) - (c + jd)$

$= (a - c) + j(b - d)$

Thus, for example,

$(2 + j3) + (3 - j4) = 2 + j3 + 3 - j4$

$= \mathbf{5 - j1}$

and $(2 + j3) - (3 - j4) = 2 + j3 - 3 + j4$

$= \mathbf{-1 + j7}$

The addition and subtraction of complex numbers may be achieved graphically as shown in the Argand diagram of Fig. 35.2. $(2 + j3)$ is represented by vector $\boldsymbol{OP}$ and $(3 - j4)$ by vector $\boldsymbol{OQ}$. In Fig. 35.2(a), by vector addition, (i.e. the diagonal of the parallelogram), $\boldsymbol{OP} + \boldsymbol{OQ} = \boldsymbol{OR}$. R is the point $(5, -j1)$.

Hence $(2 + j3) + (3 - j4) = \mathbf{5 - j1}$

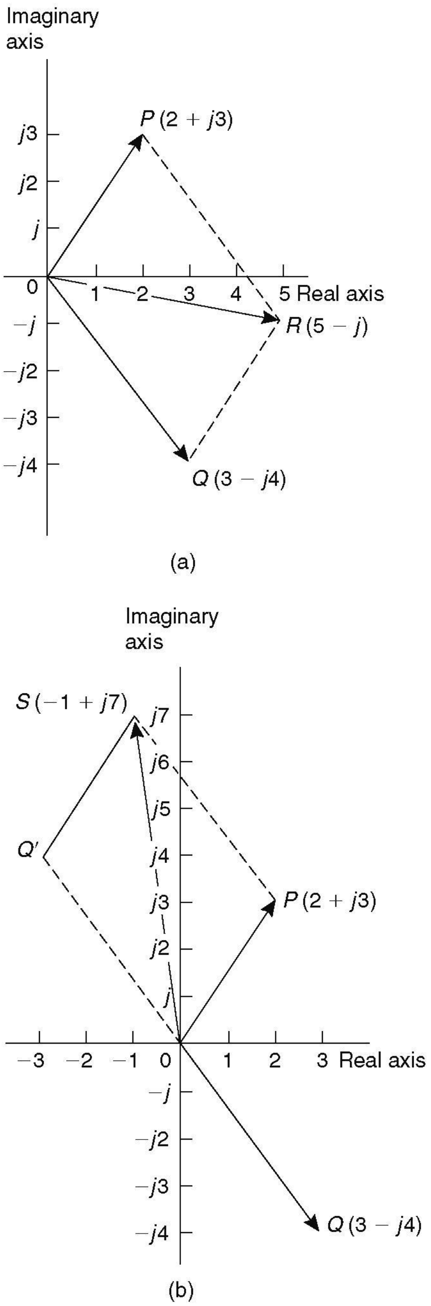

Figure 35.2

In Fig. 35.2(b), vector $\boldsymbol{OQ}$ is reversed (shown as OQ') since it is being subtracted. (Note $\boldsymbol{OQ} = 3 - j4$ and $\boldsymbol{OQ'} = -(3 - j4) = -3 + j4$).

$\boldsymbol{OP - OQ = OP + OQ' = OS}$ is found to be the Argand point $(-1, j7)$.

Hence $(2 + j3) - (3 - j4) = \mathbf{-1 + j7}$

Problem 4. Given $Z_1 = 2 + j4$ and $Z_2 = 3 - j$ determine (a) $Z_1 + Z_2$, (b) $Z_1 - Z_2$, (c) $Z_2 - Z_1$ and show the results on an Argand diagram

(a) $Z_1 + Z_2 = (2 + j4) + (3 - j)$

$= (2 + 3) + j(4 - 1) = \mathbf{5 + j3}$

(b) $Z_1 - Z_2 = (2 + j4) - (3 - j)$

$= (2 - 3) + j(4 - (-1)) = \mathbf{-1 + j5}$

(c) $Z_2 - Z_1 = (3 - j) - (2 + j4)$

$= (3 - 2) + j(-1 - 4) = \mathbf{1 - j5}$

Each result is shown in the Argand diagram of Fig. 35.3.

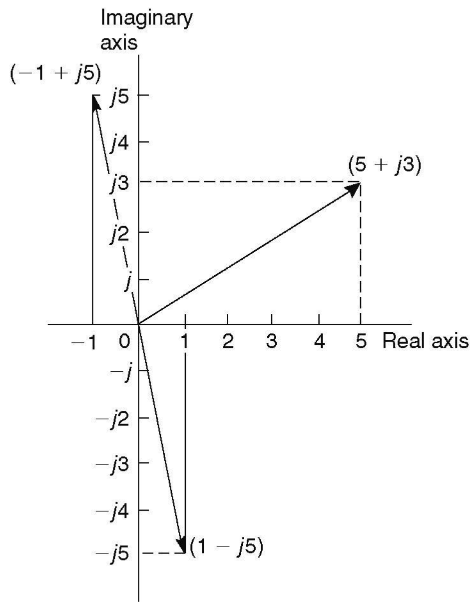

Figure 35.3

35.4 Multiplication and division of complex numbers

(i) **Multiplication of complex numbers** is achieved by assuming all quantities involved are real and then using $j^2 = -1$ to simplify.

Hence $(a + jb)(c + jd)$

$= ac + a(jd) + (jb)c + (jb)(jd)$

$= ac + jad + jbc + j^2bd$

$= (ac - bd) + j(ad + bc),$

since $j^2 = -1$

Thus $(3+j2)(4-j5)$

$$= 12 - j15 + j8 - j^2 10$$
$$= (12 - (-10)) + j(-15 + 8)$$
$$= \mathbf{22 - j7}$$

(ii) The **complex conjugate** of a complex number is obtained by changing the sign of the imaginary part. Hence the complex conjugate of $a+jb$ is $a-jb$. The product of a complex number and its complex conjugate is always a real number.

For example,

$$(3+j4)(3-j4) = 9 - j12 + j12 - j^2 16$$
$$= 9 + 16 = 25$$

$[(a+jb)(a-jb)$ may be evaluated 'on sight' as $a^2+b^2]$

(iii) **Division of complex numbers** is achieved by multiplying both numerator and denominator by the complex conjugate of the denominator.

For example,

$$\frac{2-j5}{3+j4} = \frac{2-j5}{3+j4} \times \frac{(3-j4)}{(3-j4)}$$
$$= \frac{6 - j8 - j15 + j^2 20}{3^2 + 4^2}$$
$$= \frac{-14 - j23}{25} = \frac{\mathbf{-14}}{\mathbf{25}} \mathbf{- j}\frac{\mathbf{23}}{\mathbf{25}}$$

or $\mathbf{-0.56 - j0.92}$

Problem 5. If $Z_1 = 1 - j3$, $Z_2 = -2 + j5$ and $Z_3 = -3 - j4$, determine in $a+jb$ form:

(a) $Z_1 Z_2$ (b) $\dfrac{Z_1}{Z_3}$

(c) $\dfrac{Z_1 Z_2}{Z_1 + Z_2}$ (d) $Z_1 Z_2 Z_3$

(a) $Z_1 Z_2 = (1 - j3)(-2 + j5)$

$$= -2 + j5 + j6 - j^2 15$$
$$= (-2 + 15) + j(5 + 6), \text{ since } j^2 = -1,$$
$$= \mathbf{13 + j11}$$

(b) $$\frac{Z_1}{Z_3} = \frac{1-j3}{-3-j4} = \frac{1-j3}{-3-j4} \times \frac{-3+j4}{-3+j4}$$
$$= \frac{-3 + j4 + j9 - j^2 12}{3^2 + 4^2}$$
$$= \frac{9 + j13}{25} = \frac{\mathbf{9}}{\mathbf{25}} \mathbf{+ j}\frac{\mathbf{13}}{\mathbf{25}}$$

or $\mathbf{0.36 + j0.52}$

(c) $$\frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(1-j3)(-2+j5)}{(1-j3)+(-2+j5)}$$
$$= \frac{13 + j11}{-1 + j2}, \text{ from part (a)},$$
$$= \frac{13 + j11}{-1 + j2} \times \frac{-1 - j2}{-1 - j2}$$
$$= \frac{-13 - j26 - j11 - j^2 22}{1^2 + 2^2}$$
$$= \frac{9 - j37}{5} = \frac{\mathbf{9}}{\mathbf{5}} \mathbf{- j}\frac{\mathbf{37}}{\mathbf{5}} \text{ or } \mathbf{1.8 - j7.4}$$

(d) $Z_1 Z_2 Z_3 = (13 + j11)(-3 - j4)$, since

$Z_1 Z_2 = 13 + j11$, from part (a)

$$= -39 - j52 - j33 - j^2 44$$
$$= (-39 + 44) - j(52 + 33) = \mathbf{5 - j85}$$

Problem 6. Evaluate:

(a) $\dfrac{2}{(1+j)^4}$ (b) $j\left(\dfrac{1+j3}{1-j2}\right)^2$

(a) $(1+j)^2 = (1+j)(1+j) = 1 + j + j + j^2$

$$= 1 + j + j - 1 = j2$$

$$(1+j)^4 = [(1+j)^2]^2 = (j2)^2 = j^2 4 = -4$$

Hence $$\frac{2}{(1+j)^4} = \frac{2}{-4} = \mathbf{-\frac{1}{2}}$$

(b) $$\frac{1+j3}{1-j2} = \frac{1+j3}{1-j2} \times \frac{1+j2}{1+j2}$$
$$= \frac{1 + j2 + j3 + j^2 6}{1^2 + 2^2} = \frac{-5 + j5}{5}$$
$$= -1 + j1 = -1 + j$$