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Claudio Canuto
Anita Tabacco

Mathematical Analysis I

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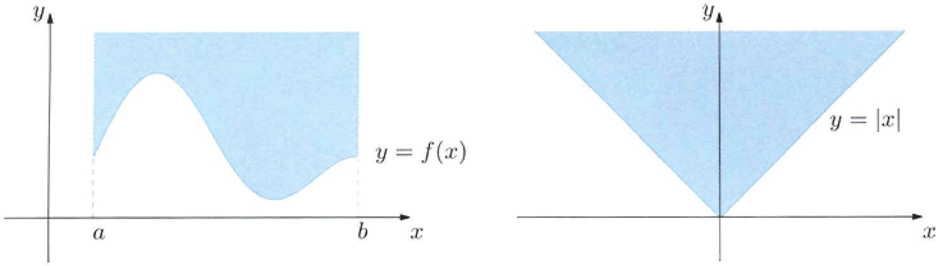

Figure 6.12. The set E_f for a generic f defined on I (left) and for $f(x) = |x|$ (right)

It is easy to convince oneself that the convexity of E_f can be checked by considering points P_1, P_2 belonging to the graph of f only. In other words, given x_1, x_2 in I, the segment S_{12} between $(x_1, f(x_1))$ and $(x_2, f(x_2)$ should lie above the graph.

Since one can easily check that any x between x_1 and x_2 can be represented as

$$x = (1-t)x_1 + tx_2 \qquad \text{with} \quad t = \frac{x - x_1}{x_2 - x_1} \in [0,1]\,,$$

the convexity of f reads

$$f\big((1-t)x_1 + tx_2\big) \le (1-t)f(x_1) + tf(x_2) \qquad \forall x_1, x_2 \in I\,, \forall t \in [0,1]\,.$$

If the inequality is strict for $x_1 \neq x_2$ and $t \in (0,1)$, the function is called **strictly convex on** I.

For *differentiable* functions on the interval I, Definitions 6.39, 6.32 can be proven to be equivalent. But a function may well be convex according to Definition 6.39 without being differentiable on I, like $f(x) = |x|$ on $I = \mathbb{R}$ (Fig. 6.12, right). Note, however, that convexity implies continuity at all interior points of I, although discontinuities may occur at the end-points.

6.10 Qualitative study of a function

We have hitherto supplied the reader with several analytical tools to study a map f on its domain and draw a relatively thorough – qualitatively speaking – graph. This section describes a step-by-step procedure for putting together all the information acquired.

Domain and symmetries

It should be possible to determine the domain of a generic function starting from the elementary functions that build it via algebraic operations and composition. The study is greatly simplified if one detects the map's possible symmetries and periodicity at the very beginning (see Sect. 2.6). For instance, an even or odd map

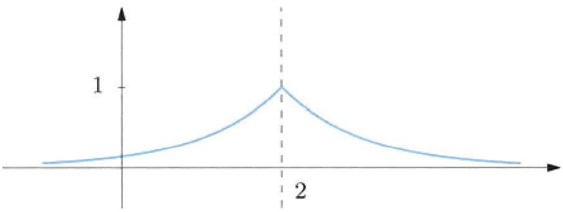

Figure 6.13. The function $f(x) = \mathrm{e}^{-|x-2|}$

can be studied only for positive values of the variable. We point out that a function might present different kinds of symmetries, like the symmetry with respect to a vertical line other than the y-axis: the graph of $f(x) = \mathrm{e}^{-|x-2|}$ is symmetric with respect to $x = 2$ (Fig. 6.13).

For the same reason the behaviour of a periodic function is captured by its restriction to an interval as wide as the period.

Behaviour at the end-points of the domain

Assuming the domain is a union of intervals, as often happens, one should find the one-sided limits at the end-points of each interval. Then the existence of asymptotes should be discussed, as in Sect. 5.3.

For instance, consider

$$f(x) = \frac{\log(2-x)}{\sqrt{x^2-2x}}.$$

Now, $\log(2-x)$ is defined for $2-x > 0$, or $x < 2$; in addition, $\sqrt{x^2-2x}$ has domain $x^2 - 2x \geq 0$, so $x \leq 0$ or $x \geq 2$, and being a denominator, $x \neq 0, 2$. Thus $\operatorname{dom} f = (-\infty, 0)$. Since $\lim_{x\to 0^-} f(x) = +\infty$, the line $x = 0$ is a vertical left asymptote, while $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\log(2-x)}{|x|} = 0$ yields the horizontal left asymptote $y = 0$.

Monotonicity and extrema

The first step consists in computing the derivative f' and its domain $\operatorname{dom} f'$. Even if the derivative's analytical expression might be defined on a larger interval, one should in any case have $\operatorname{dom} f' \subseteq \operatorname{dom} f$. For example $f(x) = \log x$ has $f'(x) = \frac{1}{x}$ and $\operatorname{dom} f = \operatorname{dom} f' = (0, +\infty)$, despite $g(x) = \frac{1}{x}$ makes sense for any $x \neq 0$. After that, the zeroes and sign of f' should be determined. They allow to find the intervals where f is monotone and discuss the nature of critical points (the zeroes of f'), in the light of Sect. 6.7.

A careless analysis might result in wrong conclusions. Suppose a map f is differentiable on the union $(a, b) \cup (b, c)$ of two bordering intervals where $f' > 0$. If f is not differentiable at the point b, deducing from that that f is increasing on $(a, b) \cup (b, c)$ **is wrong**. The function $f(x) = -\frac{1}{x}$ satisfies $f'(x) = \frac{1}{x^2} > 0$ on $(-\infty, 0) \cup (0, +\infty)$, but it is not globally increasing therein (e.g. $f(-1) > f(1)$); we can only say f is increasing on $(-\infty, 0)$ and on $(0, +\infty)$ *separately*.

Recall that extremum points need not only be critical points. The function $f(x) = \sqrt{\dfrac{x}{1+x^2}}$, defined on $x \geq 0$, has a critical point $x = 1$ giving an absolute maximum. At the other extremum $x = 0$, the function is not differentiable, although $f(0)$ is the absolute minimum.

Convexity and inflection points

Along the same lines one determines the intervals upon which the function is convex or concave, and its inflections. As in Sect. 6.9, we use the second derivative for this.

Sign of the function and its higher derivatives

When sketching the graph of f we might find useful (not compulsory) to establish the sign of f and its vanishing points (the x-coordinates of the intersections of the graph with the horizontal axis). The roots of $f(x) = 0$ are not always easy to find analytically. In such cases one may resort to the Theorem of existence of zeroes 4.23, and deduce the presence of a unique zero within a certain interval. Likewise can be done for the sign of the first or second derivatives.

The function $f(x) = x \log x - 1$ is defined for $x > 0$. One has $f(x) < 0$ when $x \leq 1$. On $x \geq 1$ the map is strictly increasing (in fact $f'(x) = \log x + 1 > 0$ for $x > 1/\mathrm{e}$); besides, $f(1) = -1 < 0$ and $f(\mathrm{e}) = \mathrm{e} - 1 > 0$. Therefore there is exactly one zero somewhere in $(1, \mathrm{e})$, f is negative to the left of said zero and positive to the right.

6.10.1 Hyperbolic functions

An exemplary application of what seen so far is the study of a family of functions, called **hyperbolic**, that show up in various concrete situations.

We introduce the maps $f(x) = \sinh x$ and $g(x) = \cosh x$ by

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}.$$

They are respectively called **hyperbolic sine** and **hyperbolic cosine.** The terminology stems from the fundamental relation

$$\cosh^2 x - \sinh^2 x = 1\,, \qquad \forall x \in \mathbb{R}\,,$$

whence the point P of coordinates $(X, Y) = (\cosh x, \sinh x)$ runs along the right branch of the rectangular hyperbola $X^2 - Y^2 = 1$ as x varies.

The first observation is that $\operatorname{dom} f = \operatorname{dom} g = \mathbb{R}$; moreover, $f(x) = -f(-x)$ and $g(x) = g(-x)$, hence the hyperbolic sine is an odd map, whereas the hyperbolic cosine is even. Concerning the limit behaviour,

$$\lim_{x\to\pm\infty} \sinh x = \pm\infty\,, \qquad \lim_{x\to\pm\infty} \cosh x = +\infty\,.$$

and ask ourselves whether there are real numbers a and b rendering f differentiable at the origin. The continuity at the origin (recall: differentiable implies continuous) forces the two values

$$\lim_{x\to 0^-} f(x) = -4, \qquad \lim_{x\to 0^+} f(x) = f(0) = -b+1$$

to agree, hence $b = 5$. With b fixed, we may impose the equality of the right and left limits of $f'(x)$ for $x \to 0$, to the effect that $f'(x)$ admits finite limit for $x \to 0$. Then we use Theorem 6.15, which prescribes that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2a\cos 2x = 2a, \qquad \text{and} \qquad \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (5+\mathrm{e}^x) = 6$$

are the same, so $a = 3$. □

Remark 6.17 In using Theorem 6.15 one should not forget to impose continuity at the point x_0. The mere existence of the limit for f' is not enough to guarantee f will be differentiable at x_0. For example, $f(x) = x + \operatorname{sign} x$ is differentiable at every $x \neq 0$: since $f'(x) = 1$, it necessarily follows $\lim_{x\to 0} f'(x) = 1$. The function is nonetheless not differentiable, because not continuous, at $x = 0$. □

6.4 Extrema and critical points

Definition 6.18 *One calls* $x_0 \in \operatorname{dom} f$ *a* **relative** (*or* **local**) **maximum point for** f *if there is a neighbourhood* $I_r(x_0)$ *of* x_0 *such that*

$$\forall x \in I_r(x_0) \cap \operatorname{dom} f, \qquad f(x) \le f(x_0).$$

Then $f(x_0)$ *is a* **relative** (*or* **local**) **maximum of** f.
One calls x_0 *an* **absolute maximum point** (*or* **global maximum point**) **for** f *if*

$$\forall x \in \operatorname{dom} f, \qquad f(x) \le f(x_0),$$

and $f(x_0)$ *becomes the* (**absolute**) **maximum of** f. *In either case, the maximum is said* **strict** *if* $f(x) < f(x_0)$ *when* $x \neq x_0$.

Exchanging the symbols $\le$ with $\ge$ one obtains the definitions of **relative** and **absolute minimum point**. A minimum or maximum point shall be referred to generically as an **extremum** (**point**) of f.

Examples 6.19

i) The parabola $f(x) = 1+2x-x^2 = 2-(x-1)^2$ has a strict absolute maximum point at $x_0 = 1$, and 2 is the function's absolute maximum. Notice the derivative $f'(x) = 2(1-x)$ is zero at that point. There are no minimum points (relative or absolute).

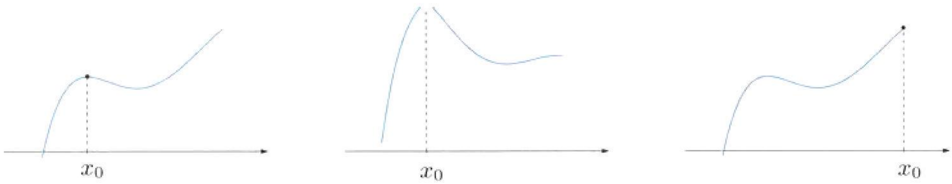

Figure 6.3. Types of maxima

ii) For $g(x) = \arcsin x$ (see Fig. 2.24), $x_0 = 1$ is a strict absolute maximum point, with maximum value $\frac{\pi}{2}$. The point $x_1 = -1$ is a strict absolute minimum, with value $-\frac{\pi}{2}$. At these extrema g is not differentiable. □

We are interested in finding the extremum points of a given function. Provided the latter is differentiable, it might be useful to look for the points where the first derivative vanishes.

Definition 6.20 *A* **critical point** (*or* **stationary point**) *of f is a point x_0 at which f is differentiable with derivative $f'(x_0) = 0$.*

The tangent at a critical point is horizontal.

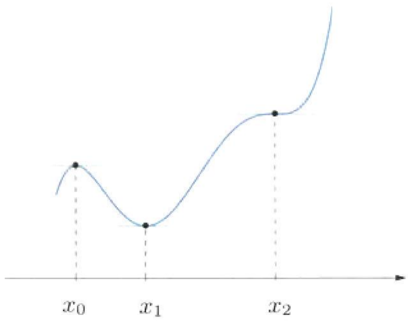

Figure 6.4. Types of critical points

Theorem 6.21 (Fermat) *Suppose f is defined in a full neighbourhood of a point x_0 and differentiable at x_0. If x_0 is an extremum point, then it is critical for f, i.e.,*

$$f'(x_0) = 0.$$

Proof. To fix ideas, assume x_0 is a relative maximum point and that $I_r(x_0)$ is a neighbourhood where $f(x) \leq f(x_0)$ for all $x \in I_r(x_0)$. On such neighbourhood then $\Delta f = f(x) - f(x_0) \leq 0$.

If $x > x_0$, hence $\Delta x = x - x_0 > 0$, the difference quotient $\dfrac{\Delta f}{\Delta x}$ is non-positive. Corollary 4.3 implies

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Vice versa, if $x < x_0$, i.e., $\Delta x < 0$, then $\dfrac{\Delta f}{\Delta x}$ is non-negative, so

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

By Property 6.14,

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0},$$

so $f'(x_0)$ is simultaneously ≤ 0 and ≥ 0, hence zero.
A similar argument holds for relative minima. □

Fermat's Theorem 6.21 ensures that the extremum points of a differentiable map which belong to the *interior* of the domain should be searched for among critical points.

A function can nevertheless have critical points that are not extrema, as in Fig. 6.4. The map $f(x) = x^3$ has the origin as a critical point ($f'(x) = 3x^2 = 0$ if and only if $x = 0$), but admits no extremum since it is strictly increasing on the whole $\mathbb{R}$.

At the same time though, a function may have non-critical extremum point (Fig. 6.3); this happens when a function is not differentiable at an extremum that lies inside the domain (e.g. $f(x) = |x|$, whose absolute minimum is attained at the origin), or when the extremum point is on the boundary (as in Example 6.19 ii)). The upshot is that in order to find *all* extrema of a function, browsing through the critical points might not be sufficient.

To summarise, extremum points are contained among the points of the domain at which either

i) the first derivative vanishes,

ii) or the function is not differentiable,

iii) or among the domain's boundary points (inside $\mathbb{R}$).