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Mathematical Analysis I

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Thirdly, the halving procedure used in the proof can be transformed into an algorithm of approximation, known in Numerical Analysis under the name *Bisection method.*

A first application of the Theorem of existence of zeroes comes next.

Example 4.24

The function $f(x) = x^4 + x^3 - 1$ on $[0,1]$ is a polynomial, hence continuous. As $f(0) = -1$ and $f(1) = 1$, f must vanish somewhere on $[0,1]$. The zero is unique because the map is strictly increasing (it is sum of the strictly increasing functions $y = x^4$ and $y = x^3$, and of the constant function $y = -1$). □

Our theorem can be generalised usefully as follows.

Corollary 4.25 *Let f be continuous on the interval I and suppose it admits non-zero limits (finite or infinite) that are different in sign for x tending to the end-points of I. Then f has a zero in I, which is unique if f is strictly monotone on I.*

Proof. The result is a consequence of Theorems 4.2 and 4.23 (Existence of zeroes). For more details ⇝ `Continuous functions.` □

Example 4.26

Consider the map $f(x) = x + \log x$, defined on $I = (0, +\infty)$. The functions $y = x$ and $y = \log x$ are continuous and strictly increasing on I, and so is f. Since $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$, f has exactly one zero on its domain. □

Corollary 4.27 *Consider f and g continuous maps on the closed bounded interval $[a, b]$. If $f(a) < g(a)$ and $f(b) > g(b)$, there exists at least one point x_0 in the open interval (a, b) with*

$$f(x_0) = g(x_0). \tag{4.14}$$

Proof. Consider the auxiliary function $h(x) = f(x) - g(x)$, which is continuous in $[a, b]$ as sum of continuous maps. By assumption, $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$. So, h satisfies the Theorem of existence of zeroes and admits in (a, b) a point x_0 such that $h(x_0) = 0$. But this is precisely (4.14).

Note that if h is strictly increasing on $[a, b]$, the solution of (4.14) has to be unique in the interval. □

$g(x) = x$ and $|g(x) - \ell| = |g(x)| = |x| < \varepsilon$. We say that g has limit 0 for x going to 0.

As for the function h, it cannot be continuous at the origin, since comparing the values $h(x)$, for x near 0, with the value at the origin simply makes no sense, for the latter is not even defined. Neverthless, the graph allows to 'conjecture' that these values might estimate $\ell = 1$ increasingly better, the closer we choose x to the origin. We are lead to say h has a limit for x going to 0, and this limit is 1. We shall substantiate this claim later on.

The examples just seen introduce us to the definition of continuity and of (finite) limit.

Definition 3.14 *Let x_0 be a point in the domain of a function f. This function is called* **continuous at** x_0 *if for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\forall x \in \operatorname{dom} f, \qquad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon. \tag{3.6}$$

In neighbourhood-talk: for any neighbourhood $I_\varepsilon(f(x_0))$ of $f(x_0)$ there exists a neighbourhood $I_\delta(x_0)$ of x_0 such that

$$\boxed{\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \quad \Rightarrow \quad f(x) \in I_\varepsilon(f(x_0)).} \tag{3.7}$$

Definition 3.15 *Let f be a function defined on a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. Then f* **has limit** $\ell \in \mathbb{R}$ (*or* **tends to** ℓ *or* **converges to** ℓ) **for** x **approaching** x_0, *written*

$$\lim_{x \to x_0} f(x) = \ell,$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon. \tag{3.8}$$

Alternatively: for any given neighbourhood $I_\varepsilon(\ell)$ of ℓ there is a neighbourhood $I_\delta(x_0)$ of x_0 such that

$$\boxed{\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \setminus \{x_0\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).}$$

The definition of limit is represented in Fig. 3.6.

Let us compare the notions just seen. To have continuity one looks at the values $f(x)$ from the point of view of $f(x_0)$, whereas for limits these $f(x)$ are compared to ℓ, which *could* be different from $f(x_0)$, provided f is defined in x_0. To test the limit, moreover, the comparison with $x = x_0$ is excluded: requiring $0 < |x - x_0|$

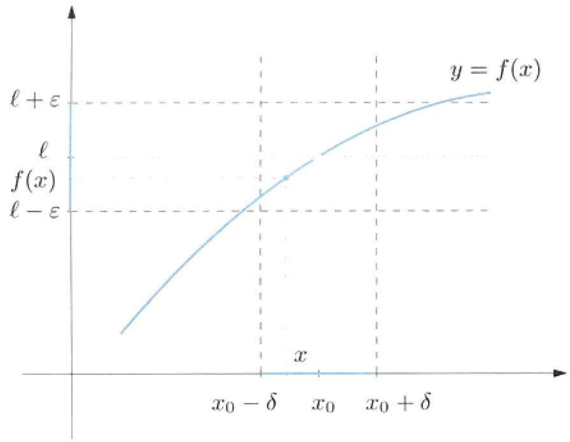

Figure 3.6. Definition of finite limit of a function

means exactly $x \neq x_0$; on the contrary, the implication (3.6) is obviously true for $x = x_0$.

Let f be defined in a neighbourhood of x_0. If f is continuous at x_0, then (3.8) is certainly true with $\ell = f(x_0)$; vice versa if f has limit $\ell = f(x_0)$ for x going to x_0, then (3.6) holds. Thus the continuity of f at x_0 is tantamount to

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{3.9}$$

In both definitions, after fixing an arbitrary $\varepsilon > 0$, one is asked to find *at least* one positive number δ ('there is a δ') for which (3.6) or (3.8) holds. If either implication holds for a certain δ, it will also hold for every $\delta' < \delta$. The definition does not require to find the biggest possible δ satisfying the implication. With this firmly in mind, testing continuity or verifying a limit can become much simpler.

Returning to the functions f, g, h of the beginning, we can now say that f is continuous at $x_0 = 0$,

$$\lim_{x \to 0} f(x) = 1 = f(0),$$

whereas g, despite having limit 0 for $x \to 0$, is not continuous:

$$\lim_{x \to 0} g(x) = 0 \neq g(0).$$

We shall prove in Example 4.6 i) that h admits a limit for x going to 0, and actually

$$\lim_{x \to 0} h(x) = 1.$$

The functions g and h suggest the following definition.

Definition 3.16 *Let f be defined on a neighbourhood of x_0, excluding the point x_0. If f admits limit $\ell \in \mathbb{R}$ for x approaching x_0, and if a) f is defined in x_0 but $f(x_0) \neq \ell$, or b) f is not defined in x_0, then we say x_0 is a* **(point of) removable discontinuity for** f.

The choice of terminology is justified by the fact that one can *modify* the function at x_0 by *defining* it in x_0, so that to obtain a continuous map at x_0. More precisely, the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0, \\ \ell & \text{if } x = x_0, \end{cases}$$

is such that

$$\lim_{x \to x_0} \tilde{f}(x) = \lim_{x \to x_0} f(x) = \ell = \tilde{f}(x_0),$$

hence it is continuous at x_0.

For the above functions we have $\tilde{g}(x) = x$ in a neighbourhood of the origin, while

$$\tilde{h}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

In the latter case, we have defined the **continuous prolongation** of $y = \dfrac{\sin x}{x}$, by assigning the value that renders it continuous at the origin. From now on when referring to the function $y = \dfrac{\sin x}{x}$, we will always understand it as continuously prolonged in the origin.

Examples 3.17

We show that the main elementary functions are continuous.

i) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ and $x_0 \in \mathbb{R}$ be given. For any $\varepsilon > 0$, $|f(x) - f(x_0)| < \varepsilon$ if and only if $|a|\,|x - x_0| < \varepsilon$. When $a = 0$, the condition holds for any $x \in \mathbb{R}$; if $a \neq 0$ instead, it is equivalent to $|x - x_0| < \dfrac{\varepsilon}{|a|}$, and we can put $\delta = \dfrac{\varepsilon}{|a|}$ in (3.6). The map f is thus continuous at every $x_0 \in \mathbb{R}$.

ii) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous at $x_0 = 2$. We shall prove this fact in two different ways. Given $\varepsilon > 0$, $|f(x) - f(2)| < \varepsilon$, or $|x^2 - 4| < \varepsilon$, means

$$4 - \varepsilon < x^2 < 4 + \varepsilon. \tag{3.10}$$

We can suppose $\varepsilon \leq 4$ (for if $|f(x) - f(2)| < \varepsilon$ for a certain ε, the same will be true for all $\varepsilon' > \varepsilon$); as we are looking for x in a neighbourhood of 2, we can furthermore assume $x > 0$. Under such assumptions (3.10) yields

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon},$$

hence

$$-(2-\sqrt{4-\varepsilon}) < x-2 < \sqrt{4+\varepsilon}-2. \qquad (3.11)$$

This suggests to take $\delta = \min(2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2)$ $(= \sqrt{4+\varepsilon}-2$, easy to verify). If $|x-2| < \delta$, then (3.11) holds, which was equivalent to $|x^2-4| < \varepsilon$. With a few algebraic computations, this furnishes the *greatest* δ for which the inequality $|x^2-4| < \varepsilon$ is true.
We have already said that the largest value of δ is not required by the definitions, so we can also proceed alternatively. Since

$$|x^2-4| = |(x-2)(x+2)| = |x-2|\,|x+2|\,,$$

by restricting x to a neighbourhood of 2 of radius < 1, we will have $-1 < x-2 < 1$, hence $1 < x < 3$. The latter will then give $3 < x+2 = |x+2| < 5$. Thus

$$|x^2-4| < 5|x-2|. \qquad (3.12)$$

To obtain $|x^2-4| < \varepsilon$ it will suffice to demand $|x-2| < \frac{\varepsilon}{5}$; since (3.12) holds when $|x-2| < 1$, we can set $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$ and the condition (3.6) will be satisfied. The neighbourhood of radius < 1 was arbitrary: we could have chosen any other sufficiently small neighbourhood and obtain another δ, still respecting the continuity requirement.
Note at last that a similar reasoning tells f is continuous at every $x_0 \in \mathbb{R}$.

iii) We verify that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$ is continuous at every $x_0 \in \mathbb{R}$. We establish first a simple but fundamental inequality.

Lemma 3.18 *For any $x \in \mathbb{R}$,*

$$|\sin x| \le |x|, \qquad (3.13)$$

with equality holding if and only if $x = 0$.

Proof. Let us start assuming $0 < x \le \frac{\pi}{2}$ and look at the right-angled triangle PHA of Fig. 3.7. The vertical side PH is shorter than the hypotenuse PA, whose length is in turn less than the length of the arc PA (the shortest distance between two points is given by the straight line joining them):

$$\overline{PH} < \overline{PA} < \stackrel{\frown}{PA}.$$

By definition $\overline{PH} = \sin x > 0$, and $\stackrel{\frown}{PA} = x > 0$ (angles being in radians). Thus (3.13) is true. The case $-\frac{\pi}{2} \le x < 0$ is treated with the same argument observing $|\sin x| = \sin|x|$ for $0 < |x| \le \frac{\pi}{2}$. At last, when $|x| > \frac{\pi}{2}$ one has $|\sin x| \le 1 < \frac{\pi}{2} < |x|$, ending the proof. □