ADVANCED ENGINEERING MATHEMATICS

CHAPTER 7

MATRICES ELEMENTARY ROW OPERATIONS REDUCED ROW ECHELON FORM ROW AND COLUMN SPACES HOMOGENEOUS SYSTEMS NONHOMOGENEOUS SYSTEMS MATRIX

Matrices and Linear Systems

7.1 Matrices

An n by m (or $n \times m$) *matrix* is a rectangular array of objects arranged in n rows and m columns.

We will denote matrices in boldface. For example,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & \pi \\ 1 & \sqrt{2} & -5 \end{pmatrix}$$

is a 2×3 matrix (two rows, three columns) and

$$\mathbf{B} = \begin{pmatrix} e^t & 1 & -1 & \cos(t) \\ 0 & 4t & -7 & 1-t \end{pmatrix}$$

is a 2×4 matrix.

The object located in the row i and column j place of a matrix is called its i, j *element*. Often we write $\mathbf{A} = [a_{ij}]$, meaning that the i, j element of $\mathbf{A}$ is a_{ij}. In the above matrices $\mathbf{A}$ and $\mathbf{B}$, $a_{11} = 2$, $a_{22} = \sqrt{2}$, $a_{23} = -5$, $b_{14} = \cos(t)$ and $b_{21} = 0$.

If the elements of an $n \times m$ matrix are real numbers, then each row can be thought of as a vector in R^m and each column as a vector in R^n. In the first example, $\mathbf{A}$ has two rows that are vectors in R^3 and columns forming three vectors in R^2. This vector point of view is often useful in dealing with matrices.

Two matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are *equal* if they have the same number of rows, the same number of columns, and for each i and j, $a_{ij} = b_{ij}$. Equal matrices have the same dimensions, and objects located in the same positions in the matrices must be equal.

There are three operations we will define for matrices: addition, multiplication by a real or complex number, and multiplication. These are defined as follows.

Addition of Matrices

If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are both $n \times m$ matrices, then their sum is defined to be the $n \times m$ matrix $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

We add two matrices of the same dimensions by adding objects in the same locations in the matrices. For example,

$$\begin{pmatrix} 1 & 2 & -3 \\ 4 & 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 6 & 3 \\ 8 & 12 & 14 \end{pmatrix} = \begin{pmatrix} 0 & 8 & 0 \\ 12 & 12 & 16 \end{pmatrix}.$$

We can think of this as adding respective row vectors, or respective column vectors, of the matrix.

Multiplication by a Scalar

Multiply a matrix by a scalar quantity (say a number or function) by multiplying each matrix element by the scalar. If $\mathbf{A} = [a_{ij}]$, then $c\mathbf{A} = [ca_{ij}]$. For example,

$$\sqrt{2}\begin{pmatrix} -3 \\ 4 \\ 2t \\ \sin(2t) \end{pmatrix} = \begin{pmatrix} -3\sqrt{2} \\ 4\sqrt{2} \\ 2t\sqrt{2} \\ \sqrt{2}\sin(2t) \end{pmatrix}.$$

This is the same as multiplying each row vector, or each column vector, by c. As another example,

$$\cos(t)\begin{pmatrix} 2 & e^t \\ \sin(t) & 4 \end{pmatrix} = \begin{pmatrix} 2\cos(t) & e^t\cos(t) \\ \cos(t)\sin(t) & 4\cos(t) \end{pmatrix}.$$

Multiplication of Matrices

Let $\mathbf{A} = [a_{ij}]$ be $n \times k$ and $\mathbf{B} = [b_{ij}]$ be $k \times m$. Then the product $\mathbf{AB}$ is the $n \times m$ matrix whose i, j element is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj},$$

or

$$\sum_{s=1}^{k} a_{is}b_{sj}.$$

This is the dot product of row i of $\mathbf{A}$ with column j of $\mathbf{B}$ (both are vectors in R^k):

$$\begin{aligned} i, j \text{ element of } \mathbf{AB} &= (\text{ row } i \text{ of } \mathbf{A}) \cdot (\text{ column } j \text{ of } \mathbf{B}) \\ &= (a_{i1}, a_{i2}, \cdots, a_{ik}) \cdot (b_{1j}, b_{2j}, \cdots, b_{kj}) \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}. \end{aligned}$$

This clarifies why the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$ for the product $\mathbf{AB}$ to be defined. We can only take the dot product of two vectors of the same dimension.

EXAMPLE 7.1

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}.$$

Here $\mathbf{A}$ is 2×2 and $\mathbf{B}$ is 2×3, so we can compute $\mathbf{AB}$, which is 2×3 (number of rows of $\mathbf{A}$, number of columns of $\mathbf{B}$). In terms of dot products of rows with columns,

$$\mathbf{AB} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} <1,3>\cdot<1,2> & <1,3>\cdot<1,1> & <1,3>\cdot<3,4> \\ <2,5>\cdot<1,2> & <2,5>\cdot<1,1> & <2,5>\cdot<3,4> \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & 15 \\ 12 & 7 & 26 \end{pmatrix}.$$

In this example, $\mathbf{BA}$ is not defined because the number of columns of $\mathbf{B}$ does not equal the number of rows of $\mathbf{A}$. ♦

EXAMPLE 7.2

Let

$$\mathbf{A}=\begin{pmatrix}1&1&2&1\\4&1&6&2\end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix}-1&8\\2&1\\1&1\\12&6\end{pmatrix}.$$

Because **A** is 2×4 and **B** is 4×2, then **AB** is defined and is 2×2:

$$\mathbf{AB}=\begin{pmatrix}<1,1,2,1>\cdot<-1,2,1,12> & <1,1,2,1>\cdot<8,1,1,6>\\ <4,1,6,2>\cdot<-1,2,1,12> & <4,1,6,2>\cdot<8,1,1,6>\end{pmatrix}$$

$$=\begin{pmatrix}15&17\\28&51\end{pmatrix}.$$

In this example, **BA** is also defined and is a 4×4 matrix:

$$\mathbf{BA}=\begin{pmatrix}-1&8\\2&1\\1&1\\12&6\end{pmatrix}\begin{pmatrix}1&1&2&1\\4&1&6&2\end{pmatrix}=\begin{pmatrix}31&7&46&15\\6&3&10&4\\5&2&8&3\\36&18&60&24\end{pmatrix}.$$

Even when both **AB** and **BA** are defined, these matrices may not be equal, and may not even have the same dimensions. Matrix multiplication is noncommutative. ♦

We will list some properties of these matrix operations.

THEOREM 7.1

Let **A**, **B** and **C** be matrices. Then, whenever the indicated operations are defined:

1. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ (matrix addition is commutative).
2. $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$.
3. $(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{AC}$.
4. $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$.
5. $c\mathbf{AB}=(c\mathbf{A})\mathbf{B}=\mathbf{A}(c\mathbf{B})$ for any scalar c. ♦

Proof Proofs of these conclusions are straightforward. To illustrate, we will prove operation (3):

$$\begin{aligned} i,j \text{ element of } \mathbf{A}(\mathbf{B}+\mathbf{C}) &= (\text{row } i \text{ of } \mathbf{A})\cdot(\text{column } j \text{ of } \mathbf{B}+\mathbf{C})\\ &=(\text{row } i \text{ of } \mathbf{A})\cdot(\text{column } j \text{ of } \mathbf{B}+\text{ column } j \text{ of } \mathbf{C})\\ &=(\text{row } i \text{ of } \mathbf{A})\cdot(\text{column } j \text{ of } \mathbf{B})+((\text{row } i \text{ of } \mathbf{A})\cdot(\text{column } j \text{ of } \mathbf{C})\\ &=(i,j \text{ element of } \mathbf{AB})+(i,j \text{ element of } \mathbf{AC})\\ &=i,j \text{ element of } \mathbf{AB}+\mathbf{AC}.\ ♦\end{aligned}$$

We have already noted that in some ways matrix multiplication does not behave like multiplication of real numbers. The following examples illustrate other differences.

EXAMPLE 7.3

Even when **AB** and **BA** are defined and have the same dimensions, it is possible that $\mathbf{AB}\neq\mathbf{BA}$:

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Even when **AB** and **BA** are defined and have the same dimensions, it is possible that $\mathbf{AB} \neq \mathbf{BA}$:

$$\begin{pmatrix} 1 & 0 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} -2 & 6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 8 & 0 \end{pmatrix}$$

but

$$\begin{pmatrix} -2 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} -14 & 24 \\ -5 & 12 \end{pmatrix}. \ \blacklozenge$$

EXAMPLE 7.4

There is in general no cancelation in products: if $\mathbf{AB} = \mathbf{AC}$, it does not follow that $\mathbf{A} = \mathbf{C}$. To illustrate,

$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 3 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ 5 & 11 \end{pmatrix} = \begin{pmatrix} 7 & 18 \\ 21 & 54 \end{pmatrix},$$

even though

$$\begin{pmatrix} 4 & 2 \\ 3 & 16 \end{pmatrix} \neq \begin{pmatrix} 2 & 7 \\ 5 & 11 \end{pmatrix}. \ \blacklozenge$$

EXAMPLE 7.5

The product of two nonzero matrices may be a zero matrix:

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 6 & 4 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \ \blacklozenge$$

As an example, let

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ 1 & 7 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -3 & 6 & 7 \\ -5 & 1 & 2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 2 & -4 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 14 \\ -38 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -4 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix},$$

and

$$\begin{pmatrix} 2 & -4 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 21 \end{pmatrix}.$$

These are the columns of **AB**:

$$\begin{pmatrix} 2 & -4 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} -3 & 6 & 7 \\ -5 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 8 & 6 \\ -38 & 13 & 21 \end{pmatrix}.$$

We also will sometimes find it useful to think of a product **AX**, when **X** is a $k \times 1$ column matrix, as a linear combination of the columns $\mathbf{A}_1, \cdots, \mathbf{A}_k$ of **A**. In particular, if

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix},$$

7.1.2 Terminology and Special Matrices

We will define some terms and special matrices that are encountered frequently.

> The $n \times m$ *zero matrix* $\mathbf{O}_{nm}$ is the $n \times m$ matrix having every element equal to zero.

For example

$$\mathbf{O}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\mathbf{A}$ is $n \times m$ then

$$\mathbf{A} + \mathbf{O}_{nm} = \mathbf{O}_{nm} + \mathbf{A} = \mathbf{A}.$$

The negative of a matrix $\mathbf{A}$ is just the scalar product $(-1)\mathbf{A}$ formed by multiplying each matrix element by -1. We denote this matrix $-\mathbf{A}$. If $\mathbf{B}$ has the same dimensions as $\mathbf{A}$, then we denote $\mathbf{B} + (-\mathbf{A})$ as $\mathbf{B} - \mathbf{A}$, as we do with numbers.

> A *square matrix* is one having the same number of rows and columns. If $\mathbf{A} = [a_{ij}]$ is $n \times n$, the *main diagonal* of $\mathbf{A}$ consists of the matrix elements $a_{11}, a_{22}, \cdots, a_{nn}$. These are the matrix elements along the diagonal from the upper left corner to the lower right corner.

The $n \times n$ *identity matrix* is the $n \times n$ matrix $\mathbf{I}_n$ having each i, j element equal to zero if $i \neq j$ and each i, i element equal to 1. For example,

Then

$$\mathbf{I}_3\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} = \mathbf{A}$$

and

$$\mathbf{A}\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} = \mathbf{A}. \ \blacklozenge$$

If $\mathbf{A} = [a_{ij}]$ is an $n \times m$ matrix, the *transpose* of $\mathbf{A}$ is the $m \times n$ matrix defined by

$$\mathbf{A}^t = [a_{ji}].$$

We form the transpose by interchanging the rows and columns of $\mathbf{A}$.

EXAMPLE 7.7

Let

$$\mathbf{A} = \begin{pmatrix} -1 & 6 & 3 & -4 \\ 0 & \pi & 12 & -5 \end{pmatrix},$$

a 2×4 matrix. Then $\mathbf{A}^t$ is the 4×2 matrix

$$\mathbf{A}^t = \begin{pmatrix} -1 & 0 \\ 6 & \pi \\ 3 & 12 \\ -4 & -5 \end{pmatrix}. \ \blacklozenge$$

***THEOREM 7.3** Properties of the Transpose*

Proof of Conclusion (3) First observe that the conclusion is consistent with the definition of the matrix product. If $\mathbf{A} = [a_{ij}]$ is $n \times m$ and $\mathbf{B} = [b_{ij}]$ is $m \times k$, then $\mathbf{AB}$ is $n \times k$, so $(\mathbf{AB})^t$ is $k \times n$. However, $\mathbf{A}^t$ is $m \times n$ and $\mathbf{B}^t$ is $k \times m$, so $\mathbf{A}^t\mathbf{B}^t$ is defined only if $n = k$, while $\mathbf{B}^t\mathbf{A}^t$ is always defined and is $k \times n$.

Now, from the definition of matrix product

$$
\begin{aligned}
i, j \text{ element of } \mathbf{B}^t\mathbf{A}^t &= \sum_{s=1}^{k} (\mathbf{B}^t)_{is}(\mathbf{A}^t)_{sj} \\
&= \sum_{s=1}^{k} b_{si}a_{js} = \sum_{s=1}^{k} a_{js}b_{si} \\
&= j, i \text{ element of } \mathbf{AB} = i, j \text{ element of } (\mathbf{AB})^t.
\end{aligned}
$$

This argument can also be given conveniently in terms of dot products:

$$
\begin{aligned}
(\mathbf{B}^t\mathbf{A}^t)_{ij} &= (\text{ row } i \text{ of } \mathbf{B}^t) \cdot (\text{ column } j \text{ of } \mathbf{A}^t) \\
&= (\text{ column } i \text{ of } \mathbf{B}) \cdot (\text{ row } j \text{ of } \mathbf{A}) \\
&= (\text{ row } j \text{ of } \mathbf{A}) \cdot (\text{ column } i \text{ of } \mathbf{B}) \\
&= (\mathbf{AB})_{ji} = ((\mathbf{AB})^t)_{ij}. \ \blacklozenge
\end{aligned}
$$

In some contexts, it is useful to observe that the dot product of two n - vectors can be written as a matrix product. Write the n-vectors

$$\mathbf{X} = < x_1, x_2, \cdots, x_n > \text{ and } \mathbf{Y} = < y_1, y_2, \cdots, y_n > .$$

as $n \times 1$ column matrices

SECTION 7.1 *PROBLEMS*

In each of Problems 1 through 6, perform the requested computation.

1. $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -4 & 6 \\ -1 & 1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 & 0 & 0 \\ -2 & -1 & 6 \\ 8 & 15 & 4 \end{pmatrix}; 2\mathbf{A} - 3\mathbf{B}$

2. $\mathbf{A} = \begin{pmatrix} -2 & 2 \\ 0 & 1 \\ 14 & 2 \\ 6 & 8 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 & 4 \\ 2 & 1 \\ 14 & 16 \\ 1 & 25 \end{pmatrix}, -5\mathbf{A} + 3\mathbf{B}$

3. $\mathbf{A} = \begin{pmatrix} x & 1-x \\ 2 & e^x \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -6 \\ x & \cos(x) \end{pmatrix}, \mathbf{A}^2 + 2\mathbf{AB}$

4. $\mathbf{A} = (14), \mathbf{B} = (-12), -3\mathbf{A} - 5\mathbf{B}$

5. $\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 7 & -9 \\ 8 & 2 & -5 & 0 & 0 \end{pmatrix},$
   $\mathbf{B} = \begin{pmatrix} -5 & 1 & 8 & 21 & 7 \\ 12 & -6 & -2 & -1 & 9 \end{pmatrix}, 4\mathbf{A} + 8\mathbf{B}$

6. $\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 8 \\ -5 & 1 \end{pmatrix}, \mathbf{A}^3 - \mathbf{B}^2$

In each of Problems 7 through 16, determine which of $\mathbf{AB}$

8. $\mathbf{A} = \begin{pmatrix} -2 & -4 \\ 3 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 & 8 \\ 1 & -4 \end{pmatrix}$

9. $\mathbf{A} = \begin{pmatrix} -1 & 6 & 2 & 14 & -22 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 2 \\ 6 \\ 0 \\ -4 \end{pmatrix}$

10. $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 6 & 2 \\ 18 & -22 \\ 1 & 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -16 & 0 & 0 & 28 \\ 0 & 1 & 1 & 26 \end{pmatrix}$

11. $\mathbf{A} = \begin{pmatrix} -21 & 4 & 8 & -3 \\ 12 & 1 & 0 & 14 \\ 1 & 16 & 0 & -8 \\ 13 & 4 & 8 & 0 \end{pmatrix},$
    $\mathbf{B} = \begin{pmatrix} -9 & 16 & 3 & 2 \\ 5 & 9 & 14 & 0 \end{pmatrix}$

12. $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ 3 & 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -3 & 7 & 2 \\ 5 & 9 & 1 & 0 \end{pmatrix}$