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Mathematical Analysis I

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We next discuss differentiability in terms of operations (algebraic operations, composition, inversion) on functions. We shall establish certain *differentiation rules* to compute derivatives of functions that are built from the elementary ones, without resorting to the definition each time. The proofs may be found at ⇝ `Derivatives`.

Theorem 6.4 (Algebraic operations) *Let $f(x), g(x)$ be differentiable maps at $x_0 \in \mathbb{R}$. Then the maps $f(x) \pm g(x)$, $f(x)g(x)$ and, if $g(x_0) \neq 0$, $\frac{f(x)}{g(x)}$ are differentiable at x_0. To be precise,*

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0), \tag{6.3}$$

$$(f\,g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0), \tag{6.4}$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}. \tag{6.5}$$

Corollary 6.5 ('Linearity' of the derivative) *If $f(x)$ and $g(x)$ are differentiable at $x_0 \in \mathbb{R}$, the map $\alpha f(x) + \beta g(x)$ is differentiable at x_0 for any $\alpha, \beta \in \mathbb{R}$ and*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0). \tag{6.6}$$

Proof. Consider (6.4) and recall that differentiating a constant gives zero; then $(\alpha f)'(x_0) = \alpha f'(x_0)$ and $(\beta g)'(x_0) = \beta g'(x_0)$ follow. The rest is a consequence of (6.3). □

Examples 6.6

i) To differentiate a polynomial, we use the fact that $D\,x^n = nx^{n-1}$ and apply the corollary repeatedly. So, $f(x) = 3x^5 - 2x^4 - x^3 + 3x^2 - 5x + 2$ differentiates to

$$f'(x) = 3 \cdot 5x^4 - 2 \cdot 4x^3 - 3x^2 + 3 \cdot 2x - 5 = 15x^4 - 8x^3 - 3x^2 + 6x - 5.$$

ii) For rational functions, we compute the numerator and denominator's derivatives and then employ rule (6.5), to the effect that

$$f(x) = \frac{x^2 - 3x + 1}{2x - 1}$$

has derivative

$$f'(x) = \frac{(2x-3)(2x-1) - (x^2 - 3x + 1)2}{(2x-1)^2} = \frac{2x^2 - 2x + 1}{4x^2 - 4x + 1}.$$

iii) Consider $f(x) = x^3 \sin x$. The product rule (6.4) together with $(\sin x)' = \cos x$ yield

$$f'(x) = 3x^2 \sin x + x^3 \cos x.$$

iv) The function

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

can be differentiated with (6.5)

$$f'(x) = \frac{\cos x \, \cos x - \sin x \, (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x.$$

Another possibility is to use $\cos^2 x + \sin^2 x = 1$ to obtain

$$f'(x) = \frac{1}{\cos^2 x}.$$

□

Theorem 6.7 (**"Chain rule"**) *Let $f(x)$ be differentiable at $x_0 \in \mathbb{R}$ and $g(y)$ a differentiable map at $y_0 = f(x_0)$. Then the composition $g \circ f(x) = g\big(f(x)\big)$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0) = g'\big(f(x_0)\big) f'(x_0). \tag{6.7}$$

Examples 6.8

i) The map $h(x) = \sqrt{1 - x^2}$ is the composite of $f(x) = 1 - x^2$, whose derivative is $f'(x) = -2x$, and $g(y) = \sqrt{y}$, for which $g'(y) = \dfrac{1}{2\sqrt{y}}$. Then (6.7) directly gives

$$h'(x) = \frac{1}{2\sqrt{1 - x^2}}(-2x) = -\frac{x}{\sqrt{1 - x^2}}.$$

ii) The function $h(x) = \mathrm{e}^{\cos 3x}$ is composed by $f(x) = \cos 3x$, $g(y) = \mathrm{e}^y$. But $f(x)$ is in turn the composite of $\varphi(x) = 3x$ and $\psi(y) = \cos y$; thus (6.7) tells $f'(x) = -3 \sin 3x$. On the other hand $g'(y) = \mathrm{e}^y$. Using (6.7) once again we conclude

$$h'(x) = -3\mathrm{e}^{\cos 3x} \sin 3x.$$

□

Theorem 6.9 (**Derivative of the inverse function**) *Suppose $f(x)$ is a continuous, invertible map on a neighbourhood of $x_0 \in \mathbb{R}$, and differentiable at x_0, with $f'(x_0) \neq 0$. Then the inverse map $f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}. \tag{6.8}$$

Examples 6.10

i) The function $y = f(x) = \tan x$ has derivative $f'(x) = 1 + \tan^2 x$ and inverse $x = f^{-1}(y) = \arctan y$. By (6.8)

$$(f^{-1})'(y) = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}.$$

Setting for simplicity $f^{-1} = g$ and denoting the independent variable with x, the derivative of $g(x) = \arctan x$ is the function $g'(x) = \frac{1}{1+x^2}$.

ii) We are by now acquainted with the function $y = f(x) = \sin x$: it is invertible on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, namely $x = f^{-1}(y) = \arcsin y$. Moreover, f differentiates to $f'(x) = \cos x$. Using $\cos^2 x + \sin^2 x = 1$, and taking into account that on that interval $\cos x \geq 0$, one can write the derivative of f in the equivalent form $f'(x) = \sqrt{1-\sin^2 x}$. Now (6.8) yields

$$(f^{-1})'(y) = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.$$

Put once again $f^{-1} = g$ and change names to the variables: the derivative of $g(x) = \arcsin x$ is $g'(x) = \frac{1}{\sqrt{1-x^2}}$.

In similar fashion $g(x) = \arccos x$ differentiates to $g'(x) = -\frac{1}{\sqrt{1-x^2}}$.

iii) Consider $y = f(x) = a^x$. It has derivative $f'(x) = (\log a)a^x$ and inverse $x = f^{-1}(y) = \log_a y$. The usual (6.8) gives

$$(f^{-1})'(y) = \frac{1}{(\log a)a^x} = \frac{1}{(\log a)y}.$$

Defining $f^{-1} = g$ and renaming x the independent variable gives $g'(x) = \frac{1}{(\log a)x}$ as derivative of $g(x) = \log_a x$ $(x > 0)$.

Take now $h(x) = \log_a(-x)$ (with $x < 0$), composition of $x \mapsto -x$ and $g(y)$: then $h'(x) = \frac{1}{(\log a)(-x)}(-1) = \frac{1}{(\log a)x}$. Putting all together shows that $g(x) = \log_a |x|$ $(x \neq 0)$ has derivative $g'(x) = \frac{1}{(\log a)x}$.

With the choice of base $a = e$ the derivative of $g(x) = \log |x|$ is $g'(x) = \frac{1}{x}$. □

Remark 6.11 Let $f(x)$ be differentiable and strictly positive on an interval I. Due to the previous result and the Chain rule, the derivative of the composite map $g(x) = \log f(x)$ is

$$g'(x) = \frac{f'(x)}{f(x)}.$$

The expression $\frac{f'}{f}$ is said **logarithmic derivative** of the map f. □

The section ends with a useful corollary to the Chain rule 6.7.

Property 6.12 *If f is an even (or odd) differentiable function on all its domain, the derivative f' is odd (resp. even).*

Proof. Since f is even, $f(-x) = f(x)$ for any $x \in \operatorname{dom} f$. Let us differentiate both sides. As $f(-x)$ is the composition of $x \mapsto -x$ and $y \mapsto f(y)$, its derivative reads $-f'(-x)$. Then $f'(-x) = -f'(x)$ for all $x \in \operatorname{dom} f$, so f' is odd. Similarly if f is odd. □

We reckon it could be useful to collect the derivatives of the main elementary functions in one table, for reference.

$$\mathrm{D}\, x^\alpha = \alpha x^{\alpha-1} \qquad (\forall \alpha \in \mathbb{R})$$

$$\mathrm{D}\, \sin x = \cos x$$

$$\mathrm{D}\, \cos x = -\sin x$$

$$\mathrm{D}\, \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\mathrm{D}\, \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arctan x = \frac{1}{1+x^2}$$

$$\mathrm{D}\, a^x = (\log a)\, a^x \qquad \text{in particular,} \quad \mathrm{D}\, \mathrm{e}^x = \mathrm{e}^x$$

$$\mathrm{D}\, \log_a |x| = \frac{1}{(\log a)\, x} \qquad \text{in particular,} \quad \mathrm{D}\, \log |x| = \frac{1}{x}$$

6.3 Where differentiability fails

It was noted earlier that the function $f(x) = |x|$ is continuous but not differentiable at the origin. At each other point of the real line f is differentiable, for it coincides with the line $y = x$ when $x > 0$, and with $y = -x$ for $x < 0$. Therefore $f'(x) = +1$