Aniq integralda o’zgaruvchini almashtirish va bo‘laklab integrallash. Xosmas integrallar

REJA

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1. Aniq integralda o'zgaruvchini almashtirish

Teorema. Ushbu

$$\int_a^b f(x)dx$$

integral berilgan bo'lsin, bu yerda $f(x)$ - $[a,b]$ kesmada uzluksiz funksiya.

Endi t o'zgaruvchi kiritamiz

$$x=\varphi(t)$$

Agar

1) $\varphi(\alpha)=a$, $\varphi(\beta)=b$

2) $\varphi(t)$ va $\varphi'(t)$ funksiyalar $[\alpha,\beta]$da uzluksiz bo'lsa,

3) $f[\varphi(t)]$ funksiya $[\alpha,\beta]$ kesmada aniqlangan va uzluksiz bo'lsa, u holda

$$\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt \qquad (1)$$

Isbot. Agar $F(x)$ funksiya $f(x)$ funksiya uchun boshlang’ich bo’lsa, u holda quyidagi tengliklarni yozishimiz mumkin:

$$\int f(x)dx = F(x) + C \qquad (2)$$

$$\int f[\varphi(t)]\varphi'(t)dt = F[\varphi(t)] + C \qquad (3)$$

Oxirgi tenglikning to’g’riligini tekshirish uchun ikkala tomondan t bo’yicha hosila olamiz. (2) tenglikdan olamiz:

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

(3) tenglikdan olamiz:

$$\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt = F[\varphi(t)]|_a^b = F[\varphi(\beta)] - F[\varphi(\alpha)] = F(b) - F(a)$$

Oxirgi ifodalarning o’ng tomonlari teng, demak, chap tomonlari ham teng.

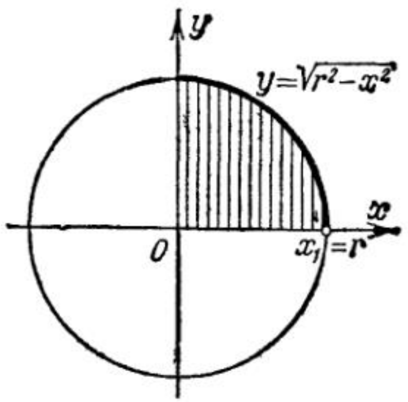
Izoh. Aniq integral (1) formula bo’yicha hisoblaganda biz eski o’zgaruvchiga qaytmayapmiz. Agar biz (1) tengsizlikdagi aniq integrallardan ikkinchisini hisoblasak, u holda biz biror sonni olamiz; birinchi integral ham shu songa teng bo’ladi.

Misol. Integralni hisoblang

$$\int_0^r \sqrt{r^2 - x^2}\,dx$$

Yechish. O'zgaruvchini almashtiramiz

$$x = r\sin t\,,\; dx = r\cos t dt$$



Yangi integrallash chegaralarini aniqlaymiz:

$$x = 0,\; t = 0.$$

$$x = r\,,\; t = \frac{\pi}{2}.$$

Demak,

$$\int_0^r \sqrt{r^2 - x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t}\, r\cos t dt = r^2 \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t}\cos t dt =$$

$$= r^2 \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = r^2 \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt = r^2 \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi r^2}{4}$$

Hisoblagan integral geometrik nuqtai nazardan $x^2 + y^2 = r^2$ aylana bilan chegaralangan chorak doiraning yuzini bildiradi

2. Bo’laklab integrallash

u va v - x ga bo’gliq differensiallanuvchi funksiya bo’lsin. U holda

$$(uv)' = u'v + uv'$$

Ayniyatning ikkala tomonini a dan b gacha integrallab, topamiz:

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx \qquad (1)$$

Endi $\int_a^b (uv)'dx = uv + C$ bo’lganligi uchun $\int_a^b (uv)'dx = uv\,|_a^b$ bo’ladi. Shuning uchun (1) tenglik

$$uv\,|_a^b = \int_a^b vdu + \int_a^b udv$$

yoki

$$\int_a^b udv = uv\,|_a^b - \int_a^b vdu$$

ko’rinishida yozilishi mumkin.

Misol. $I_n = \int\limits_0^{\frac{\pi}{2}} \sin^n x dx$ integralni hisoblang

$$I_n = \int\limits_0^{\frac{\pi}{2}} \sin^n x dx = \int\limits_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \, dx = -\int\limits_0^{\frac{\pi}{2}} \underbrace{\sin^{n-1} x}_{u} \underbrace{d \cos x}_{du} =$$

$$= -\sin^{n-1} x \cos x \Big|_0^{\frac{\pi}{2}} + (n-1)\int\limits_0^{\frac{\pi}{2}} \sin^{n-2} x \cos x \cos x \, dx =$$

$$= (n-1)\int\limits_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx = (n-1)\int\limits_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, dx =$$

$$= (n-1)\int\limits_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1)\int\limits_0^{\frac{\pi}{2}} \sin^n x \, dx$$

Biz tanlagan belgilashlarda oxirgi tenglikni quyidagicha yozish mumkin:

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

Bu yerdan topamiz:

$$I_n = \frac{n-1}{n} I_{n-2} \qquad (2)$$

Xuddi shu yo'l bilan topamiz:

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

Shuning uchun

$$I_n = \frac{n-1}{n} \frac{n-3}{n-2} I_{n-4}$$

Shu tarzda davom ettirib, so'ngra biz yo I_0 gacha yo I_1 gacha, n soni juft yoki toqligiga qarab, yetib boramiz.

Ikkita holni qaraymiz:

1) n - juft son, $n = 2m$

$$I_{2m} = \frac{2m-1}{2m} \frac{2m-3}{2m-2} \cdots \frac{3}{4} \frac{1}{2} I_0$$

2) n -toq son, $n = 2m+1$

$$I_{2m+1} = \frac{2m}{2m+1} \frac{2m-2}{2m-1} \cdots \frac{4}{5} \frac{2}{3} I_1$$

Ammo

$$I_0=\int_0^{\frac{\pi}{2}}\sin^0 x\,dx=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}$$

$$I_1=\int_0^{\frac{\pi}{2}}\sin^1 x\,dx=1$$

bo'lganligi uchun

$$I_{2m}=\int_0^{\frac{\pi}{2}}\sin^{2m} x\,dx=\frac{2m-1}{2m}\frac{2m-3}{2m-2}..\frac{5}{6}.\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$I_{2m+1}=\int_0^{\frac{\pi}{2}}\sin^{2m+1} x\,dx=\frac{2m}{2m+1}\frac{2m-2}{2m-1}...\frac{6}{7}\frac{4}{5}\frac{2}{3}$$

Bu formulalardan $\frac{\pi}{2}$ sonni cheksiz ko'paytma ko'rinishida tasvirlovchi Vallis formulasi kelib chiqadi.

Haqiatan ham, oxirgi ikkita tengliklardan hadma-had bo'lish yordamida topamiz:

$$\frac{\pi}{2}=\left(\frac{2\cdot 4\cdot 6...2m}{3\cdot 5...(2m-1)}\right)^2\frac{1}{2m+1}\frac{I_{2m}}{I_{2m+1}} \qquad (3)$$

Endi

$$\lim_{m\to\infty}\frac{I_{2m}}{I_{2m+1}}=1$$

Ekanligini isbotlaymiz.

$(0,\frac{\pi}{2})$ intervaldagi barcha x lar uchun $\sin^{2m-1}x>\sin^{2m}x>\sin^{2m+1}x$ tengsizliklar o'rinli. 0 dan $\frac{\pi}{2}$ gacha bo'lgan chegaralarda integrallab, topamiz:

$$I_{2m-1}\geq I_{2m}\geq I_{2m+1}$$

Bu yerdan

$$\frac{I_{2m-1}}{I_{2m+1}} \geq \frac{I_{2m}}{I_{2m+1}} \geq 1 \qquad (4)$$

(2) tenglikdan kelib chiqadi:

$$\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}$$

Demak,

$$\lim_{m\to\infty}\frac{I_{2m-1}}{I_{2m+1}} = \lim_{m\to\infty}\frac{2m+1}{2m} = 1$$

(4) tenglikdan, olamiz:

$$\lim\frac{I_{2m}}{I_{2m+1}} = 1$$

(3) formulada limitga o'tib, Vallis formulasini olamiz:

$$\frac{\pi}{2} = \lim_{m\to\infty}\left[\left(\frac{2\cdot 4\cdot 6 ... 2m}{3\cdot 5 ... (2m-1)}\right)^2 \frac{1}{2m+1}\right]$$

Bu formulani quyidagicha yozish mumkin:

$$\frac{\pi}{2} = \lim_{m\to\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5} ... \frac{2m-2}{2m-1}\frac{2m}{2m-1}\frac{2m}{2m+1}\right)$$

3. Xosmas integrallar

1. Chegaralari cheksiz bo’lgan integrallar. $f(x)$ funksiya x ning ($a \le x < +\infty$) qiymatlarida aniqlangan va uzluksiz bo’ladi. Ushbu

$$I(b) = \int_a^b f(x)dx$$

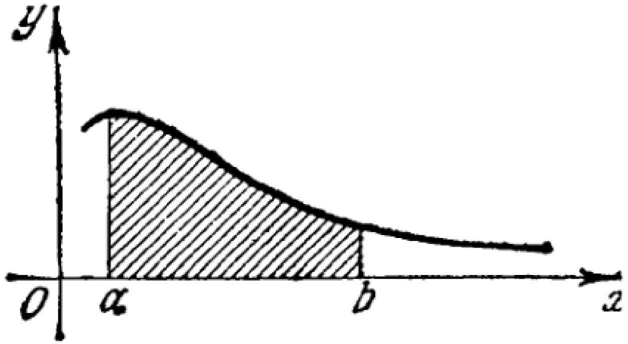
Integralni qaraymiz. Bu integral $b > a$ da ma’noga ega. b o’zgarganda integral o’zgaradi, u b ning uzluksiz funksiyasi bo’ladi. $b \to +\infty$ bo’lganda bu integralni o’rganamiz.

Ta’rif. Agar $$\lim_{b \to +\infty} \int_a^b f(x)dx$$

chekili limit mavjud bo’lsa, u holda bu limit $f(x)$ ning $[a, +\infty)$ intervaldagi xomas integrali deyiladi va

$$\int_a^{+\infty} f(x)dx$$

bilan belgilanadi.

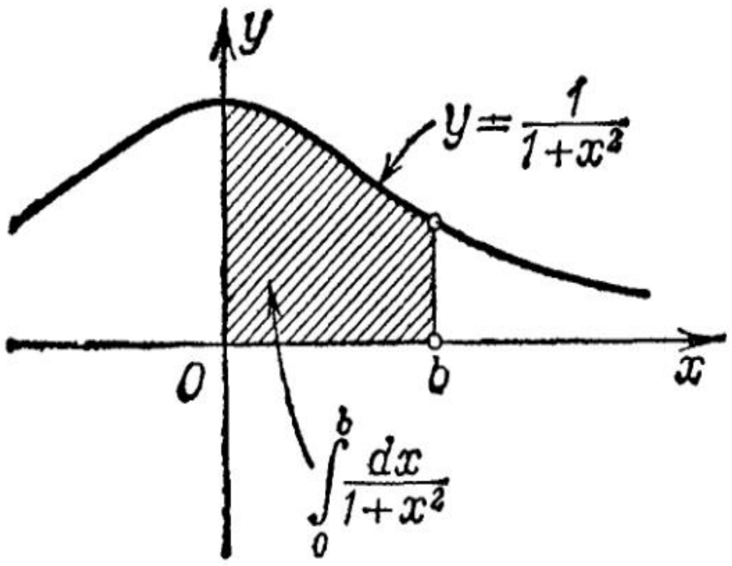


Shunday qilib, ta'rifga ko'ra

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx$$

Shu holda $\int_a^{+\infty} f(x)dx$ xosmas integral mavjud yoki yaqinlashadi deyiladi. Agar $b \to +\infty$ da $\int_a^b f(x)dx$ chekli limitga ega bo'lmasa, u holda $\int_a^{+\infty} f(x)dx$ mavjud emas yoki uzoqlashadi deyiladi.

$f(x) \geq 0$ bo'lganda xosmas integralning geometrik ma'nosiga tushinish qiyin emas. Agar $\int_a^b f(x)dx$ integral $y = f(x)$ egri chiziq, absissalar o'qi, $x = a, x = b$ to'g'ri chiziqlar bilan chegaralangan sohaning yuzasini beradi, u holda $\int_a^{+\infty} f(x)dx$ xosmas integral $y = f(x)$, $x = a$ va absissalar o'qi orasida joylashgan chegaralanmagan yuzani ifodalaydi.



Boshqa cheksiz intervallar uchun ham xosmas integrallar shu tarzda aniqlanadi:

$$\int_{-\infty}^{a} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$$

Oxirgi tenglikni quyifagicha tushinish mumkin: agar o'ng tomonda turgan xosmas integrallardan har biri mavjuda bo'lsa, u holda o'ng tomonda turgan integral ham mavjud (yaqinlashadi).

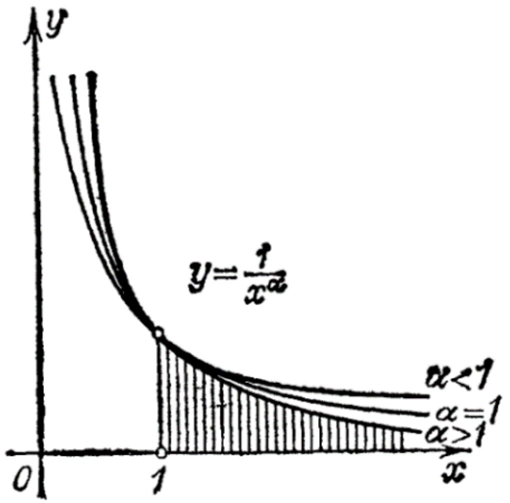
Misol. Integralni hisoblang $\int_0^{+\infty} \frac{dx}{1+x^2}$.

Xosmas integralning ta'rifidan foydalanib, topamiz:

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_0^{b} \frac{dx}{1+x^2} = \lim_{b\to+\infty} arctgx \Big|_0^b = \lim_{b\to+\infty} arctg\, b = \frac{\pi}{2}$$

Bu integral cheksiz egri chiziqli trapetsiyani yuzasini beradi.

Misol 2. α ning qanday qiymatlarida $\int_1^{+\infty}\frac{dx}{x^2}$ integral yaqinlashadi va uzoqlashadi.



Yechish. $\alpha \neq 1$bo'lganda

$$\int_1^b \frac{dx}{x^2} = \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^b = \frac{1}{1-\alpha}[b^{1-\alpha} - 1]$$

bo'lgani uchun

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{b\to+\infty} \frac{1}{1-\alpha}[b^{1-\alpha} - 1]$$

Shunday qilib, $\alpha > 1$da $\int_1^{+\infty} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}$, ya'ni integral yaqinlashadi;

$a < 1$da $\int_1^{+\infty} \frac{dx}{x^\alpha} = \infty$, ya'ni uzoqlashadi;

$\alpha = 1$ da $\int_1^b \frac{dx}{x} = \ln x \Big|_1^{+\infty} = \infty$, ya'ni integral uzoqlashadi.

Misol 3. Hisoblang $\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Yechish.

$$\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int\limits_{-\infty}^{0} \frac{dx}{1+x^2} + \int\limits_{0}^{+\infty} \frac{dx}{1+x^2}$$

Ikkinchi integral $\frac{\pi}{2}$ ga teng. Birinchi integralni hisoblaymiz:

$$\int\limits_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int\limits_{a}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} arctgx \Big|_a^0 = \lim_{a\to-\infty} (arctg0 - arctga) = \frac{\pi}{2}$$

Shunday qilib,

$$\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Ko’p hollarda berilgan integralning yaqinlashuvchi yoki uzoqlashuvchi ekanligini ko’rsatish, uning qiymatini baholash yetarli. Buning uchun quyidagi 2ta teorema foydali bo’lishi mumkin.

Teorema 1. Agar barcha $x(x \geq a)$ uchun

$$0 \leq f(x) \leq \varphi(x)$$

Tengsizlik bajarilsa va agar $\int_a^{+\infty} \varphi(x)dx$ yaqinlashsa, u holda $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi, bunda

$$\int_a^{+\infty} f(x)dx \leq \int_a^{+\infty} \varphi(x)dx$$

bo’ladi.

Misol 4. Integralni yaqinlashishiga tekshiring.

$$\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$$

Yechish. $1 \le x$ da

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$$

So’ngra

$$\int_1^{+\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{+\infty} = 1$$

Shunday qilib,

$$\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$$

yaqinladi va uning qiymati 1dan kichik.

Teorema 2. Agar barcha $x(x \geq a)$ uchun

$$0 \leq \varphi(x) \leq f(x)$$

tengsizlik bajarilsa va agar $\int_a^{+\infty} \varphi(x)dx$ uzoqlashadi, u holda $\int_a^{+\infty} f(x)dx$ integral ham uzoqlashadi.

Misol 5. Integralni yaqinlashini tekshiring

$$\int_1^{+\infty} \frac{x+1}{\sqrt{x^3}} dx$$

Ma'lumki,

$$\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}$$

Ammo

$$\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim_{b \to +\infty} 2\sqrt{x} \Big|_1^b = +\infty$$

Demak, berilgan integral uzoqlashadi.

Teorema 3. Agar $\int_a^{+\infty} |f(x)|\,dx$ integral yaqinlashsa, u holda $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi. Bu holda oxirgi integral absolyut yaqinlashuvchi integral deyiladi.

Misol 6. Integralni yaqinlashishini tekshirish

$$\int_1^{+\infty} \frac{\sin x}{x^3} dx$$

Yechish. Bu yerda integral ostidagi funksiya – ishora o'zgartiruvchili funksiya. Ma'lumki $\left|\frac{\sin x}{x^3}\right| \le \left|\frac{1}{x^3}\right|$, ammo $\int_1^{+\infty} \frac{dx}{x^3} = -\frac{1}{2x^2} \Big|_1^{+\infty} = \frac{1}{2}$. Demak, $\int_1^{+\infty} \left|\frac{\sin x}{x^3}\right| dx$ integral yaqinlashadi. Bundan berilgan integralning yaqinlashishi kelib chiqadi.

Uzilishli funksiyaning integrali

$f(x)$ funksiya $a \le x < c$ da aniqlangan va uzluksiz bo'lib, $x = c$ da funksiya yo aniqlanmagan, yo uzilishga ega bo'lsin. Bu holda $f(x)$ funksiya $[a,c]$ uzluksiz bo'lmaganligi uchun $\int_a^c f(x)dx$ integralni integral yig'indilarning limiti haqida gapirib bo'lmaydi va shuning uchun bu limit mavjud bo'lmasligi ham mumkin.

c nuqtada uzilishli bo'lgan $f(x)$ funksiyaning $\int_a^c f(x)dx$ integral quyidagicha topiladi

$$\int_a^c f(x)dx = \lim_{b \to c-0} \int_a^b f(x)dx$$

Agar o'ng tomonda turgan limit mavjud bo'lsa, u holda integral yaqinlashuvchi xosmas integral, aks holda uzoqlashuvchi xosmas deyiladi.

Agar $f(x)$ funksiya $[a,c]$ kesmaning chap chetida (ya'ni $x=a$ da) uzilishiga ega bo'lsa, u holda

$$\int_a^c f(x)dx = \lim_{b\to a+0}\int_b^c f(x)dx$$

Agar $f(x)$ funksiya $[a,c]$ kesmaning ichidagi biror $x=x_0$ nuqtada uzilishga ega bo'lsa, agar o'ng tomonda turgan ikkala xosmas integrallar mavjud

bo'lsa, u holda $\int_a^c f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^c f(x)dx$

deb olinadi.

Misol 7. Hisoblang $\int_0^1 \frac{dx}{\sqrt{1-x}}$

Yechish.

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{b\to 1-0} \int_0^b \frac{dx}{\sqrt{1-x}} = -\lim_{b\to 1-0} 2\sqrt{1-x}\,\Big|_0^b =$$

$$= -\lim_{b\to 1-0} 2[\sqrt{1-b} - 1] = 2$$