CHAPTER 8

Analytic Geometry in Two and Three Dimensions

The oval-shaped lawn behind the White House in Washington, D.C. is called *the Ellipse*. It has views of the Washington Monument, the Jefferson Memorial, the Department of Commerce, and the Old Post Office Building. The Ellipse is 616 ft long, 528 ft wide, and is in the shape of a conic section. Its shape can be modeled using the methods of this chapter. See page 652.

Equations in the three variables x, y, and z generally graph as surfaces in three-dimensional space. Just as in the plane, second-degree equations are of particular interest. Recall that second-degree equations in two variables yield conic sections in the Cartesian plane. In space, second-degree equations in *three* variables yield **quadric surfaces**: The paraboloids, ellipsoids, and hyperboloids of revolution that have special reflective properties are all quadric surfaces, as are such exotic-sounding surfaces as hyperbolic paraboloids and elliptic hyperboloids.

Other surfaces of interest include graphs of **functions of two variables**, whose equations have the form $z = f(x, y)$. Some examples are $z = x \ln y$, $z = \sin(xy)$, and $z = \sqrt{1 - x^2 - y^2}$. The last equation graphs as a hemisphere (see Exercise 63). Equations of the form $z = f(x, y)$ can be graphed using some graphing calculators and most computer algebra software. Quadric surfaces and functions of two variables are studied in most university-level calculus course sequences.

Vectors in Space

In space, just as in the plane, the sets of equivalent directed line segments (or arrows) are *vectors*. They are used to represent forces, displacements, and velocities in three dimensions. In space, we use ordered triples to denote vectors:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

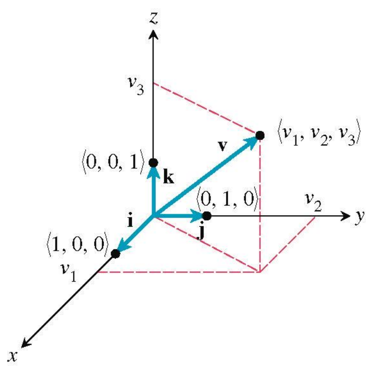

FIGURE 8.50 The vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

The **zero vector** is $\mathbf{0} = \langle 0, 0, 0 \rangle$, and the **standard unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. As shown in Figure 8.50, the vector $\mathbf{v}$ can be expressed in terms of these standard unit vectors:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

The vector $\mathbf{v}$ that is represented by the arrow from $P(a, b, c)$ to $Q(x, y, z)$ is

$$\mathbf{v} = \overrightarrow{PQ} = \langle x - a, y - b, z - c \rangle = (x - a)\mathbf{i} + (y - b)\mathbf{j} + (z - c)\mathbf{k}.$$

A vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be multiplied by a scalar (real number) c as follows:

$$c\mathbf{v} = c\langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle.$$

Many other properties of vectors generalize in a natural way when we move from two to three dimensions:

Vector Relationships in Space

For vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$,

- **Equality:** $\mathbf{v} = \mathbf{w}$ if and only if $v_1 = w_1$, $v_2 = w_2$, and $v_3 = w_3$
- **Addition:** $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
- **Subtraction:** $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$
- **Magnitude:** $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- **Dot product:** $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$
- **Unit vector:** $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$, $\mathbf{v} \neq \mathbf{0}$, is the unit vector in the direction of $\mathbf{v}$.

EXAMPLE 5 Computing with Vectors

(a) $3\langle -2, 1, 4\rangle = \langle 3\cdot -2, 3\cdot 1, 3\cdot 4\rangle = \langle -6, 3, 12\rangle$

(b) $\langle 0, 6, -7\rangle + \langle -5, 5, 8\rangle = \langle 0-5, 6+5, -7+8\rangle = \langle -5, 11, 1\rangle$

(c) $\langle 1, -3, 4\rangle - \langle -2, -4, 5\rangle = \langle 1+2, -3+4, 4-5\rangle = \langle 3, 1, -1\rangle$

(d) $|\langle 2, 0, -6\rangle| = \sqrt{2^2+0^2+6^2} = \sqrt{40} \approx 6.32$

(e) $\langle 5, 3, -1\rangle \cdot \langle -6, 2, 3\rangle = 5\cdot(-6) + 3\cdot 2 + (-1)\cdot 3$

$$= -30 + 6 - 3 = -27$$

Now try Exercises 23–26.

EXAMPLE 6 Using Vectors in Space

A jet airplane just after takeoff is pointed due east. Its air velocity vector makes an angle of 30° with flat ground with an airspeed of 250 mph. If the wind is out of the southeast at 32 mph, calculate a vector that represents the plane's velocity relative to the point of takeoff.

SOLUTION Let **i** point east, **j** point north, and **k** point up. The plane's air velocity is

$$\mathbf{a} = \langle 250\cos 30°, 0, 250\sin 30°\rangle \approx \langle 216.506, 0, 125\rangle,$$

and the wind velocity, which is pointing northwest, is

$$\mathbf{w} = \langle 32\cos 135°, 32\sin 135°, 0\rangle \approx \langle -22.627, 22.627, 0\rangle.$$

The velocity relative to the ground is $\mathbf{v} = \mathbf{a} + \mathbf{w}$, so

$$\begin{aligned}\mathbf{v} &\approx \langle 216.506, 0, 125\rangle + \langle -22.627, 22.627, 0\rangle\\ &\approx \langle 193.88, 22.63, 125\rangle\\ &= 193.88\mathbf{i} + 22.63\mathbf{j} + 125\mathbf{k}\end{aligned}$$

Now try Exercise 33.

In Exercise 64, you will be asked to interpret the meaning of the velocity vector obtained in Example 6.

Lines in Space

We have seen that first-degree equations in three variables graph as planes in space. So how do we get lines? There are several ways. First notice that to specify the x-axis, which is a line, we could use the pair of first-degree equations $y = 0$ and $z = 0$. As alternatives to using a pair of Cartesian equations, we can specify any line in space using

- one vector equation, or
- a set of three parametric equations.

seconds	0	1	2	3	4
meters	80	75.1	60.4	35.9	1.6

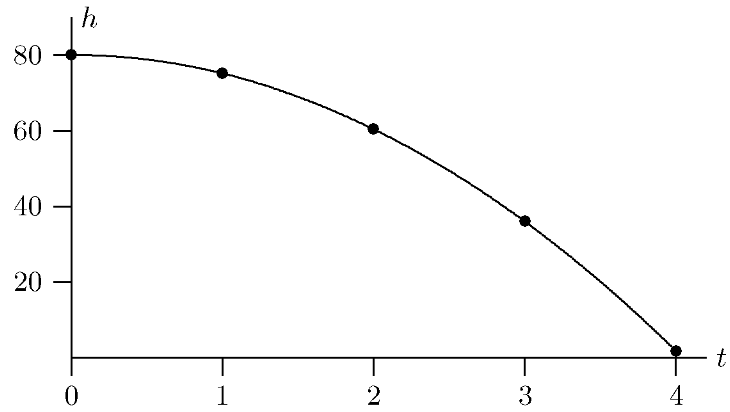

Figure 1.0.1 A data plot, height versus time.

is often written Δx, where the meaning of Δ (a capital delta in the Greek alphabet) is "change in". (Thus, Δx can be read as "change in x" although it usually is read as "delta x". The point is that Δx denotes a single number, and should not be interpreted as "delta times x".) For example, if $A = (2, 1)$ and $B = (3, 3)$, $\Delta x = 3 - 2 = 1$. Similarly, the "change in y" is written Δy. In our example, $\Delta y = 3 - 1 = 2$, the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B. The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) are:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that either or both of these might be negative.

1.1 Lines

If we have two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then we can draw one and only one line through both points. By the *slope* of this line we mean the ratio of Δy to Δx. The slope is often denoted m: $m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. For example, the line joining the points $(1, -2)$ and $(3, 5)$ has slope $(5 + 2)/(3 - 1) = 7/2$.

EXAMPLE 1.1.1 According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to \$26050. If taxable income was between \$26050 and \$134930, then, in addition, 28% was to be paid on the amount between \$26050 and \$67200, and 33% paid on the amount over \$67200 (if any). Interpret the tax bracket information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the y-axis against the taxable income on the x-axis.

The percentages, when converted to decimal values 0.15, 0.28, and 0.33, are the *slopes* of the straight lines which form the graph of the tax for the corresponding tax brackets. The tax graph is what's called a *polygonal line*, i.e., it's made up of several straight line segments of different slopes. The first line starts at the point (0,0) and heads upward with slope 0.15 (i.e., it goes upward 15 for every increase of 100 in the x-direction), until it reaches the point above $x = 26050$. Then the graph "bends upward," i.e., the slope changes to 0.28. As the horizontal coordinate goes from $x = 26050$ to $x = 67200$, the line goes upward 28 for each 100 in the x-direction. At $x = 67200$ the line turns upward again and continues with slope 0.33. See figure 1.1.1. □

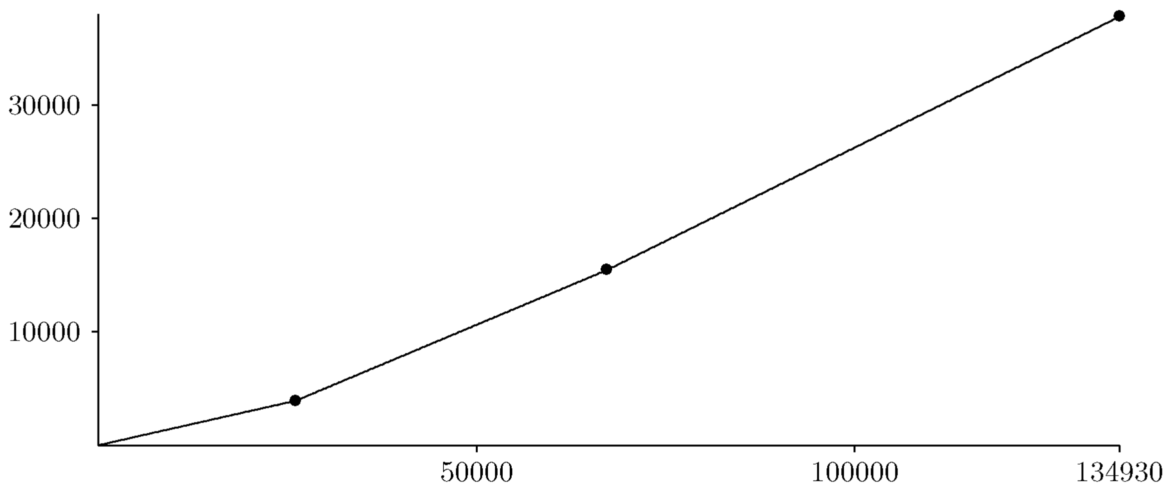

Figure 1.1.1 Tax vs. income.

The most familiar form of the equation of a straight line is: $y = mx + b$. Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m. If you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the **y-intercept**, because it is where the line crosses the y-axis. If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of either point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the "point-slope" form of the equation of a straight line: start with $(y - y_1)/(x - x_1) = m$ and then multiply to get $(y - y_1) = m(x - x_1)$, the point-slope form. Of course, this may be further manipulated to get $y = mx - mx_1 + y_1$, which is essentially the "$mx + b$" form.

It is possible to find the equation of a line between two points directly from the relation $(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$, which says "the slope measured between the point (x_1, y_1) and the point (x_2, y_2) is the same as the slope measured between the point (x_1, y_1)

and any other point (x, y) on the line." For example, if we want to find the equation of the line joining our earlier points $A(2,1)$ and $B(3,3)$, we can use this formula:

$$\frac{y-1}{x-2} = \frac{3-1}{3-2} = 2, \qquad \text{so that} \qquad y - 1 = 2(x-2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

Of course, this is really just the point-slope formula, except that we are not computing m in a separate step.

The slope m of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline, while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward; while if m is negative but near zero, then it points only a little downward. These four possibilities are illustrated in figure 1.1.2.

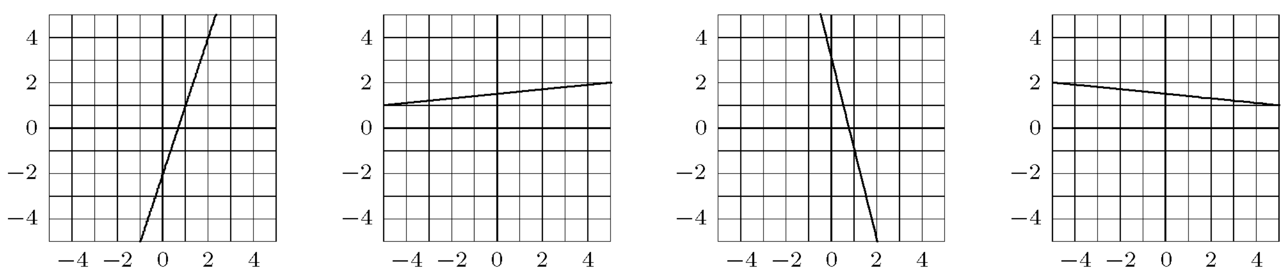

Figure 1.1.2 Lines with slopes 3, 0.1, −4, and −0.1.

If $m = 0$, then the line is horizontal: its equation is simply $y = b$.

There is one type of line that cannot be written in the form $y = mx + b$, namely, vertical lines. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

Sometimes it is useful to find the x-intercept of a line $y = mx + b$. This is the x-value when $y = 0$. Setting $mx + b$ equal to 0 and solving for x gives: $x = -b/m$. For example, the line $y = 2x - 3$ through the points $A(2,1)$ and $B(3,3)$ has x-intercept $3/2$.

EXAMPLE 1.1.2 Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., $t = 1$), you pass a sign saying it is 110 miles to Seattle, and after driving another half-hour you pass a sign saying it is 85 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from Seattle, graph and find the equation $y = mt + b$ for your distance from Seattle. Find the slope, y-intercept, and t-intercept, and describe the practical meaning of each.

The graph of y versus t is a straight line because you are traveling at constant speed. The line passes through the two points $(1, 110)$ and $(1.5, 85)$, so its slope is $m = (85 -$

110)/(1.5 − 1) = −50. The meaning of the slope is that you are traveling at 50 mph; m is negative because you are traveling *toward* Seattle, i.e., your distance y is *decreasing*. The word "velocity" is often used for $m = -50$, when we want to indicate direction, while the word "speed" refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

$$\frac{y-110}{t-1} = -50, \qquad \text{so that} \qquad y = -50(t-1) + 110 = -50t + 160.$$

The meaning of the y-intercept 160 is that when $t = 0$ (when you started the trip) you were 160 miles from Seattle. To find the t-intercept, set $0 = -50t+160$, so that $t = 160/50 = 3.2$. The meaning of the t-intercept is the duration of your trip, from the start until you arrive in Seattle. After traveling 3 hours and 12 minutes, your distance y from Seattle will be 0. □

Exercises 1.1.

1. Find the equation of the line through $(1,1)$ and $(-5,-3)$ in the form $y = mx + b$. ⇒

2. Find the equation of the line through $(-1,2)$ with slope -2 in the form $y = mx + b$. ⇒

3. Find the equation of the line through $(-1,1)$ and $(5,-3)$ in the form $y = mx + b$. ⇒

4. Change the equation $y - 2x = 2$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

5. Change the equation $x+y = 6$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept. ⇒

6. Change the equation $x = 2y - 1$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

7. Change the equation $3 = 2y$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

8. Change the equation $2x + 3y + 6 = 0$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

9. Determine whether the lines $3x + 6y = 7$ and $2x + 4y = 5$ are parallel. ⇒

10. Suppose a triangle in the x, y–plane has vertices $(-1,0)$, $(1,0)$ and $(0,2)$. Find the equations of the three lines that lie along the sides of the triangle in $y = mx + b$ form. ⇒

11. Suppose that you are driving to Seattle at constant speed. After you have been traveling for an hour you pass a sign saying it is 130 miles to Seattle, and after driving another 20 minutes you pass a sign saying it is 105 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from your starting point, graph and find the equation $y = mt + b$ for your distance from your starting point. How long does the trip to Seattle take? ⇒

12. Let x stand for temperature in degrees Celsius (centigrade), and let y stand for temperature in degrees Fahrenheit. A temperature of 0°C corresponds to 32°F, and a temperature of 100°C corresponds to 212°F. Find the equation of the line that relates temperature Fahrenheit y to temperature Celsius x in the form $y = mx + b$. Graph the line, and find the point at which this line intersects $y = x$. What is the practical meaning of this point? ⇒