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8.1 Definition of the Determinant

Determinants are scalars (numbers or sometimes functions) formed from square matrices according to a rule we will develop. The Wronskian of two functions, seen in Chapter 2, is a determinant, and we will shortly see determinants in other important contexts. This chapter develops some properties of determinants that we will need to evaluate and make use of them.

Let n be an integer with $n \geq 2$. A *permutation* of the integers $1, 2, \cdots, n$ is a rearrangement of these integers. For example, if p is the permutation that rearranges

$$1, 2, 3, 4, 5, 6 \rightarrow 3, 1, 4, 5, 2, 6,$$

then $p(1) = 3$, $p(2) = 1$, $p(3) = 4$, $p(4) = 5$, $p(5) = 2$ and $p(6) = 6$.

A permutation is characterized as even or odd according to a rule we will illustrate. Consider the permutation

$$p : 1, 2, 3, 4, 5 \rightarrow 2, 5, 1, 4, 3$$

of the integers $1, 2, 3, 4, 5$. For each k in the permuted list on the right, count the number of integers to the right of k that are smaller than k. There is one number to the right of 2 smaller than 2, three numbers to the right of 5 smaller than 5, no numbers to the right of 1 smaller than 1, one number to the right of 4 smaller than 4, and no numbers to the right of 3 smaller than 3. Since $1 + 3 + 0 + 1 + 0 = 5$ is odd, p is an *odd permutation*. When this sum is even, p is an *even permutation*.

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The *determinant* of an $n \times n$ matrix $\mathbf{A}$ is defined to be

$$\det \mathbf{A} = \sum_{p} \sigma(p) a_{1p(1)} a_{2p(2)} \cdots a_{np(n)} \qquad (8.1)$$

with this sum extending over all permutations p of $1, 2, \cdots, n$. Note that $\det \mathbf{A}$ is a sum of terms, each of which is plus or minus a product containing one element from each row and each column of $\mathbf{A}$.

We often denote $\det \mathbf{A}$ as $|\mathbf{A}|$. This is not to be confused with the absolute value, as a determinant can be negative.

EXAMPLE 8.1

We will use the definition to evaluate the general 2×2 and 3×3 determinants. For the 2×2 case, we have a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}.$$

There are only two permutations on the numbers 1, 2, namely

$$p_1 : 1, 2 \rightarrow 1, 2 \text{ and } p_2 : 1, 2 \rightarrow 2, 1.$$

It is easy to check that p_1 is even and p_2 is odd. Therefore

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THEOREM 8.1 *Some Fundamental Properties of Determinants*

Let $\mathbf{A}$ be an $n \times n$ matrix. Then

1. $|\mathbf{A}^t| = |\mathbf{A}|$.
2. If $\mathbf{A}$ has a zero row or column then $|\mathbf{A}| = 0$.
3. If $\mathbf{B}$ is formed from $\mathbf{A}$ by interchanging two rows or columns (a type I operation, extended to include columns) then
$$|\mathbf{B}| = -|\mathbf{A}|.$$
4. If two rows of $\mathbf{A}$ are the same, or if two columns of $\mathbf{A}$ are the same, then $|\mathbf{A}| = 0$.
5. If $\mathbf{B}$ is formed from $\mathbf{A}$ by multiplying a row or column by a nonzero number α (a type II operation), then
$$|\mathbf{B}| = \alpha|\mathbf{A}|.$$
6. If one row (or column) of $\mathbf{A}$ is a constant multiple of another row (or column), then $|\mathbf{A}| = 0$.
7. Suppose each element of row k of $\mathbf{A}$ is written as a sum
$$a_{kj} = b_{kj} + c_{kj}.$$
Define a matrix $\mathbf{B}$ from $\mathbf{A}$ by replacing each a_{kj} of $\mathbf{A}$ by b_{kj}. Define a matrix $\mathbf{C}$ from $\mathbf{A}$ by replacing each a_{kj} by c_{kj}. Then
$$|\mathbf{A}| = |\mathbf{B}| + |\mathbf{C}|.$$

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In determinant notation,

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1}+c_{k1} & \cdots & b_{kj}+c_{kj} & \cdots & b_{kn}+c_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{k1} & \cdots & c_{kj} & \cdots & c_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{vmatrix}. \quad (8.2)$$

8. If $\mathbf{D}$ is formed from $\mathbf{A}$ by adding α times one row (or column) to another row (or column) (a type III operation), then
$$|\mathbf{D}| = |\mathbf{A}|.$$
9. $\mathbf{A}$ is nonsingular if and only if $|\mathbf{A}| \neq 0$.
10. If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$, then
$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|. \ ♦$$

The determinant of a product is the product of the determinants.

We will give informal arguments for these conclusions.

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Proof Conclusion (1) follows from the observation that each term in the sum of equation (8.1) is a product of matrix elements, one element from each row and one from each column. We therefore obtain the same terms from both **A** and $\mathbf{A}^t$.

The reason for conclusion (2) is that a zero row or column puts a zero factor in each term of the defining sum in equation (8.1).

Conclusion (3) states that interchanging two rows, or two columns, changes the sign of the determinant. We will illustrate this for the 3×3 case. Let $\mathbf{A} = [a_{ij}]$ be 3×3 matrix and let $\mathbf{B} = [b_{ij}]$ be formed by interchanging rows one and three of **A**. Then

$$b_{11} = a_{31},\ b_{12} = a_{32},\ b_{13} = a_{33},$$
$$b_{21} = a_{21},\ b_{22} = a_{22},\ b_{23} = a_{23},$$

and

$$b_{31} = a_{11},\ b_{32} = a_{12},\ b_{33} = a_{13}.$$

From Example 8.1,

$$\begin{aligned} |\mathbf{B}| &= b_{11}b_{22}b_{33} - b_{11}b_{23}b_{32} + b_{12}b_{23}b_{31} \\ &= -b_{12}b_{21}b_{33} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} \\ &= a_{31}a_{22}a_{13} - a_{31}a_{23}a_{12} + a_{32}a_{23}a_{11} \\ &= -a_{32}a_{21}a_{13} + a_{33}a_{21}a_{12} - a_{33}a_{22}a_{11} \\ &= -|\mathbf{A}|. \end{aligned}$$

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Conclusion (4) follows immediately from (3). Form **B** from **A** by interchanging the two identical rows or columns. Since $\mathbf{A} = \mathbf{B}$, $|\mathbf{A}| = |\mathbf{B}|$. But by (3), $|\mathbf{A}| = -|\mathbf{B}| = |\mathbf{A}|$. Then $|\mathbf{A}| = 0$.

Conclusion (5) is true because multiplying a row or column of **A** by α puts a factor of α in every term of the sum (8.1) defining the determinant.

Conclusion (6) follows from (2) if $\alpha = 0$, so suppose that $\alpha \neq 0$. Now the conclusion follows from (4) and (5). Suppose that row k of **A** is α times row i. Form **B** from **A** by multiplying row k by $1/\alpha$. Then **B** has two identical rows, hence zero determinant by (4). But by (5), $|\mathbf{B}| = (1/\alpha)|\mathbf{A}| = 0$, so $|\mathbf{A}| = 0$.

Conclusion (7) follows by replacing each a_{kj} in the defining sum (8.1) with $b_{kj} + c_{kj}$. Note here that k is fixed, so only one factor in each term of (8.1) is replaced. In particular, generally the determinant of a sum is <u>not</u> the sum of the determinants. Conclusion (7) also holds if each element of a specified column is written as a sum of two terms.

Conclusion (8) follows from (4) and (7). To see this we will deal with rows to be specific. Suppose α times row i is added to row k of **A** to form **D**. On the right side of equation (8.2), replace each b_{kj} with αa_{ij}, and each c_{kj} with a_{kj}, resulting in the following:

$$\mathbf{D} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{i1} + a_{k1} & \alpha a_{i2} + a_{k2} & \cdots & \alpha a_{in} + a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix}$$

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$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Then $|\mathbf{A}|$ is the sum of the determinants of the matrices on the right. But the second determinant on the right is just $|\mathbf{A}|$ and the first is 0 by (4) because row k is a multiple of row i.

For conclusion (9), note that, by (3), (5) and (8), every time we produce $\mathbf{B}$ from $\mathbf{A}$ by an elementary row operation, $|\mathbf{B}|$ is equal to a nonzero multiple of $\mathbf{A}$. Since we reduce a matrix by a sequence of elementary row operations, then $|\mathbf{A}|$ is always a nonzero multiple of $|\mathbf{A}_R|$. This means that $|\mathbf{A}|$ is nonzero if and only if $|\mathbf{A}_R|$ is nonzero. But this is the case exactly when $\mathbf{A}$ is nonsingular, since in this case $\mathbf{A}_R = \mathbf{I}_n$. If $\mathbf{A}_R \neq \mathbf{I}_n$, then $\mathbf{A}_R$ has at least one zero row and has determinant zero.

Vanishing or non-vanishing of the determinant is an important test for existence of an inverse, and we will use it when we discuss eigenvalues in the next chapter.

Finally, we will sketch a proof of conclusion (10). If $\mathbf{A}$ is nonsingular, then there is a product of elementary matrices that reduces $\mathbf{A}$ to $\mathbf{I}_n$:

$$\mathbf{E}_r\mathbf{E}_{r-1}\cdots\mathbf{E}_1\mathbf{A} = \mathbf{I}_n.$$

Then

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8.2 Evaluation of Determinants I

The more zero elements a matrix has, the easier it is to evaluate its determinant. The reason for this is that every zero element causes some terms in the sum of equation (8.1) to vanish. For example, in Example 8.1, if $a_{12} = a_{13} = 0$,

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and

$$|\mathbf{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32})$$

with four of the six terms of $|\mathbf{A}|$ being 0 cancelling because of the zeroes in the first row of $\mathbf{A}$.

A generalization of this observation will form the basis of a useful method for evaluating determinants.

LEMMA 8.1

Let $\mathbf{A}$ be $n \times n$, and suppose row k or column r has all zero elements, except perhaps for a_{kr}. Then

$$|\mathbf{A}| = (-1)^{k+r} a_{kr} |\mathbf{A}_{kr}|, \tag{8.3}$$

where $\mathbf{A}_{kr}$ is the $n-1 \times n-1$ matrix formed by deleting row k and column r of $\mathbf{A}$. ♦

This reduces the problem of evaluating an $n \times n$ determinant to one of evaluating a smaller,

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But this is really just a permutation of the $n-1$ numbers $2, 3, \cdots, n$, since 1 is fixed and only $2, 3, \cdots, n$ are acted upon. In the definition of equation (8.1), we may therefore sum over only the permutations q of $2, 3, \cdots, n$, and factor a_{11} from all of the terms of the sum, to obtain

$$|\mathbf{A}| = \sum_q a_{11} \sum_q a_{2q(2)} a_{3q(3)} \cdots a_{nq(n)} = |\mathbf{A}_{11}|.$$

This is a_{11} times the determinant of the $n-1 \times n-1$ matrix formed by deleting row one and column one of $\mathbf{A}$.

In the general case that a_{kr} is an element of a row or column whose other elements are all zero, we can interchange $k-1$ rows and then $r-1$ columns to obtain a new matrix with a_{kr} in the 1, 1 position of a row or column having its other elements equal to zero. Since each interchange incurs a factor of -1 in the determinant, then by the preceding result,

$$|\mathbf{A}| = (-1)^{k-1+r-1} a_{kr} |\mathbf{A}_{kr}| = (-1)^{k+r} a_{kr} |\mathbf{A}_{kr}|.$$

We are rarely lucky enough to encounter a matrix $\mathbf{A}$ having a row or column with all but possibly one element equal to zero. However, we can use elementary row and column operations to obtain such a matrix $\mathbf{B}$ from $\mathbf{A}$. Furthermore from properties (3), (5), and (8) of determinants, we can track the effect of each row and column operation on the value of the determinant. This and the lemma enable us to reduce the evaluation of an $n \times n$ determinant to a constant times an $n-1 \times n-1$ determinant. We can then repeat this strategy, eventually obtaining a constant times a determinant small enough to evaluate conveniently.

EXAMPLE 8.2

SECTION 8.2 PROBLEMS

In each of Problems 1 through 10, use the method of this section to evaluate the determinant. In each problem there are many different sequences of operations that can be used to make the evaluation.

1. $\begin{vmatrix} -2 & 4 & 1 \\ 1 & 6 & 3 \\ 7 & 0 & 4 \end{vmatrix}$

2. $\begin{vmatrix} 2 & -3 & 7 \\ 14 & 1 & 1 \\ -13 & -1 & 5 \end{vmatrix}$

3. $\begin{vmatrix} -4 & 5 & 6 \\ -2 & 3 & 5 \\ 2 & -2 & 6 \end{vmatrix}$

4. $\begin{vmatrix} 2 & -5 & 8 \\ 4 & 3 & 8 \\ 13 & 0 & -4 \end{vmatrix}$

5. $\begin{vmatrix} 17 & -2 & 5 \\ 1 & 12 & 0 \\ 14 & 7 & -7 \end{vmatrix}$

6. $\begin{vmatrix} -3 & 3 & 9 & 6 \\ 1 & -2 & 15 & 6 \end{vmatrix}$

7. $\begin{vmatrix} 0 & 1 & 1 & -4 \\ 6 & -3 & 2 & 2 \\ 1 & -5 & 1 & -2 \\ 4 & 8 & 2 & 2 \end{vmatrix}$

8. $\begin{vmatrix} 2 & 7 & -1 & 0 \\ 3 & 1 & 1 & 8 \\ -2 & 0 & 3 & 1 \\ 4 & 8 & -1 & 0 \end{vmatrix}$

9. $\begin{vmatrix} 10 & 1 & -6 & 2 \\ 0 & 3 & 3 & 9 \\ 0 & 1 & 1 & 7 \\ -2 & 6 & 8 & 8 \end{vmatrix}$

10. $\begin{vmatrix} -7 & 16 & 2 & 4 \\ 1 & 0 & 0 & 5 \\ 0 & 3 & -4 & 4 \\ 6 & 1 & 1 & -5 \end{vmatrix}$

11. Fill in the details of the following argument that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

    First, if $\mathbf{AB}$ is singular, show that at least one of $\mathbf{A}$ or $\mathbf{B}$ is singular, hence that the determinant of the product and the product of the determinants are both zero.

    Thus, suppose that $\mathbf{AB}$ is nonsingular. Show that $\mathbf{A}$ and $\mathbf{B}$ can be written as products of elementary matrices, and then show that the determinant of a prod-

THEOREM 8.2 *Cofactor Expansion by a Row*

For any k with $1 \le i \le n$.

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}. \quad \blacklozenge \tag{8.4}$$

Equation (8.4) states that the determinant of **A** is the sum, along any row k, of the matrix elements of that row, each multiplied by its cofactor. This holds for any row of the matrix, although of course this sum is easier to evaluate if we choose a row with as many zero elements as possible. Equation (8.4) is called *expansion by cofactors along row k*. If we write out a few terms for fixed k we get

$$|\mathbf{A}| = (-1)^{k+1} a_{k1} M_{k1} + (-1)^{k+2} a_{k2} M_{k2} + \cdots + (-1)^{k+n} a_{kn} M_{kn}.$$

EXAMPLE 8.4

Let

$$\mathbf{A} = \begin{pmatrix} -6 & 3 & 7 \\ 12 & -5 & -6 \\ 2 & 4 & -6 \end{pmatrix}$$

If we expand by cofactors along row one, we get

$$\begin{aligned} |\mathbf{A}| &= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} M_{1j} \\ &= (-1)^{1+1}(-6)\begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} + (-1)^{1+2}(3)\begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} \\ &\quad + (-1)^{1+3}(7)\begin{vmatrix} 12 & -5 \\ 2 & 4 \end{vmatrix} \\ &= (-6)(30+36) - 3(-72+18) + 7(-48+10) = 172. \end{aligned}$$

If we expand by row three, we get

$$\begin{aligned} |\mathbf{A}| &= \sum_{j=1}^{3} (-1)^{3+j} a_{3j} M_{3j} \\ &= (-1)^{3+1}(2)\begin{vmatrix} 3 & 7 \\ -5 & -9 \end{vmatrix} + (-1)^{3+2}(4)\begin{vmatrix} -6 & 7 \\ 12 & -9 \end{vmatrix} \\ &\quad + (-1)^{3+3}(-6)\begin{vmatrix} -6 & 3 \\ 12 & -5 \end{vmatrix} \\ &= (2)(-27+35) - 4(54-84) - 6(30-36) = 172. \quad \blacklozenge \end{aligned}$$

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We can also do a cofactor expansion along a column. Now fix j and sum the elements of column j times their cofactors.

THEOREM 8.3 *Cofactor Expansion by a Column*

For any j with $1 \le j \le n$,

$$|\mathbf{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}. \quad (8.5)$$

EXAMPLE 8.5

We will expand the determinant of the matrix of Example 8.3, using column 1:

$$\begin{aligned}|\mathbf{A}| &= \sum_{i=1}^{3} (-1)^{i+1} a_{i1} M_{i1} \\ &= (-1)^{1+1}(-6)\begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} + (-1)^{2+1}(12)\begin{vmatrix} 3 & 7 \\ 4 & -6 \end{vmatrix} \\ &\quad + (-1)^{3+1}(2)\begin{vmatrix} 3 & 7 \\ -5 & -9 \end{vmatrix} \\ &= (-6)(30+36) - 12(-18-28) + 2(-27+35) = 172.\end{aligned}$$

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If we expand by column two, we get

$$\begin{aligned}|\mathbf{A}| &= \sum_{i=1}^{3} (-1)^{i+2} a_{i2} M_{i2} \\ &= (-1)^{1+2}(3)\begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} + (-1)^{2+2}(-5)\begin{vmatrix} -6 & 7 \\ 2 & -6 \end{vmatrix} \\ &\quad + (-1)^{3+2}(4)\begin{vmatrix} -6 & 7 \\ 12 & -9 \end{vmatrix} \\ &= (-3)(-72+18) - 5(36-14) - 4(54-84) = 172.\end{aligned}$$

Sometimes we use row and column operations to produce a row or column with some zero elements, then write a cofactor expansion by that row or column. Each zero element eliminates one term from the cofactor expansion.

SECTION 8.3 PROBLEMS

In each of Problems 1 through 10, evaluate the determinant using a cofactor expansion by a row and again by a column. Elementary row and/or column operations may be performed first to simplify the cofactor expansion.

1. $\begin{vmatrix} -4 & 2 & -8 \\ 1 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix}$

$\begin{vmatrix} 1 & 1 & 6 \end{vmatrix}$

9. $\begin{vmatrix} -5 & 4 & 1 & 7 \\ -9 & 3 & 2 & -5 \\ -2 & 0 & -1 & 1 \\ 1 & 14 & 0 & 3 \end{vmatrix}$

10. $\begin{vmatrix} -8 & 5 & 1 & 7 & 2 \\ 0 & 1 & 3 & 5 & -6 \\ 2 & 2 & 1 & 5 & 3 \\ 0 & 4 & 3 & 7 & 2 \\ 1 & 1 & -7 & -6 & 5 \end{vmatrix}$

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1 −3 0

2. $\begin{vmatrix} 1 & 1 & 6 \\ 2 & -2 & 1 \\ 3 & -1 & 4 \end{vmatrix}$

3. $\begin{vmatrix} 7 & -3 & 1 \\ 1 & -2 & 4 \\ -3 & 1 & 0 \end{vmatrix}$

4. $\begin{vmatrix} 5 & -4 & 3 \\ -1 & 1 & 6 \\ -2 & -2 & 4 \end{vmatrix}$

5. $\begin{vmatrix} -5 & 0 & 1 & 6 \\ 2 & -1 & 3 & 7 \\ 4 & 4 & -5 & -8 \\ 1 & -1 & 6 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 4 & 3 & -5 & 6 \\ 1 & -5 & 15 & 2 \\ 0 & -5 & 1 & 7 \\ 8 & 9 & 0 & 15 \end{vmatrix}$

7. $\begin{vmatrix} -3 & 1 & 14 \\ 0 & 1 & 16 \\ 2 & -3 & 4 \end{vmatrix}$

8. $\begin{vmatrix} 14 & 13 & -2 & 5 \\ 7 & 1 & 1 & 7 \\ 0 & 2 & 12 & 3 \\ 1 & -6 & 5 & 23 \end{vmatrix}$

0 4 3 7 2
1 1 −7 −6 5

11. Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(c-a)(b-c).$$

This is called *Vandermonde's determinant*.

12. Show that

$$\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix}$$

$$= (a+b+c+d)(b-a+d-c)\begin{vmatrix} 0 & 1 & -1 & 1 \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix}.$$

13. Prove that the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) in the plane are collinear (lie on a line) if and only if

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0.$$

Hint: This determinant is zero exactly when one row or column is a linear combination of the others.

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8.4 A Determinant Formula for A^{-1}

When $|\mathbf{A}| \neq 0$, $\mathbf{A}$ has an inverse. Furthermore, there is a formula for the elements of this inverse in terms of determinants formed from elements of $\mathbf{A}$.

THEOREM 8.4 *Elements of a Matrix Inverse*

Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix and define an $n \times n$ matrix $\mathbf{B} = [b_{ij}]$ by

$$b_{ij} = \frac{1}{|\mathbf{A}|}(-1)^{i+j} M_{ji}.$$

Then $\mathbf{B} = \mathbf{A}^{-1}$. ♦

Note that the i, j element of $\mathbf{B}$ is defined in terms of $(-1)^{i+j} M_{ji}$, the cofactor of a_{ji} (not a_{ij}).

We can see why this construction yields $\mathbf{A}^{-1}$ by explicitly multiplying the two matrices. By the definition of matrix multiplication, the i, j element of $\mathbf{AB}$ is

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \frac{1}{|\mathbf{A}|} \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk}. \quad (8.6)$$

Now consider two cases. If $i = j$ the sum in equation (8.6) is exactly the cofactor expansion of $|\mathbf{A}|$ by row i. The main diagonal elements of $\mathbf{AB}$ are therefore 1.

If $i \neq j$, the sum in equation (8.6) is the cofactor expansion by row j of the determinant of the matrix formed from $\mathbf{A}$ by replacing row j by row i. But this matrix has two identical rows, so its determinant is zero and the off-diagonal elements of $\mathbf{AB}$ are all zero. This means that

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Let

$$\mathbf{A} = \begin{pmatrix} -2 & 4 & 1 \\ 6 & 3 & -3 \\ 2 & 9 & -5 \end{pmatrix}.$$

It is routine to compute $|\mathbf{A}| = 120$ so $\mathbf{A}$ is nonsingular. We will determine $\mathbf{A}^{-1}$ by computing the elements of the matrix $\mathbf{B}$ of Theorem 8.4:

$$b_{11} = \frac{1}{120} M_{11} = \frac{1}{120} \begin{vmatrix} 3 & -3 \\ 9 & -5 \end{vmatrix} = \frac{12}{120} = \frac{1}{10},$$

$$b_{12} = \frac{1}{120}(-1) M_{21} = -\frac{1}{120} \begin{vmatrix} 4 & 1 \\ 9 & -5 \end{vmatrix} = \frac{29}{120},$$

$$b_{13} = \frac{1}{120} M_{31} = \frac{1}{120} \begin{vmatrix} 4 & 1 \\ 3 & -3 \end{vmatrix} = -\frac{1}{8},$$

$$b_{21} = -\frac{1}{120} M_{12} = -\frac{1}{120} \begin{vmatrix} 6 & -3 \\ 2 & -5 \end{vmatrix} = \frac{1}{5},$$

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$$b_{22} = \frac{1}{120} M_{22} = \frac{1}{120} \begin{vmatrix} -2 & 1 \\ 2 & -5 \end{vmatrix} = \frac{1}{15},$$

$$b_{23} = -\frac{1}{120} M_{32} = -\frac{1}{120} \begin{vmatrix} -2 & 1 \\ 6 & -3 \end{vmatrix} = 0,$$

$$b_{31} = \frac{1}{120} M_{13} = \frac{1}{120} \begin{vmatrix} 6 & 3 \\ 2 & 9 \end{vmatrix} = \frac{2}{5},$$

$$b_{32} = -\frac{1}{120} M_{23} = -\frac{1}{120} \begin{vmatrix} -2 & 4 \\ 2 & 9 \end{vmatrix} = \frac{13}{60},$$

$$b_{33} = \frac{1}{120} M_{33} = \frac{1}{120} \begin{vmatrix} -2 & 4 \\ 6 & 3 \end{vmatrix} = -\frac{1}{4}.$$

Then

$$\mathbf{B} = \mathbf{A}^{-1} = \begin{pmatrix} 1/10 & 29/120 & -1/8 \\ 1/5 & 1/15 & 0 \\ 2/5 & 13/60 & -1/4 \end{pmatrix}. \ ♦$$

SECTION 8.4 PROBLEMS

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$(\,2/5 \quad 13/60 \quad -1/4\,)$

SECTION 8.4 *PROBLEMS*

In each of Problems 1 through 10, test the matrix for singularity by evaluating its determinant. If the matrix is nonsingular, use Theorem 8.4 to compute the inverse.

1. $\begin{pmatrix} 2 & -1 \\ 1 & 6 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$

3. $\begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 5 \\ -7 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 6 & -1 & 3 \\ 0 & 1 & -4 \\ 2 & 2 & -3 \end{pmatrix}$

6. $\begin{pmatrix} -14 & 1 & -3 \\ 2 & -1 & 3 \\ 1 & 1 & 7 \end{pmatrix}$

7. $\begin{pmatrix} 0 & -4 & 3 \\ 2 & -1 & 6 \\ 1 & -1 & 7 \end{pmatrix}$

8. $\begin{pmatrix} 11 & 0 & -5 \\ 0 & 1 & 0 \\ 4 & -7 & 9 \end{pmatrix}$

9. $\begin{pmatrix} 3 & 1 & -2 & 1 \\ 4 & 6 & -3 & 9 \\ -2 & 1 & 7 & 4 \\ 13 & 0 & 1 & 5 \end{pmatrix}$

10. $\begin{pmatrix} 7 & -3 & -4 & 1 \\ 8 & 2 & 0 & 0 \\ 1 & 5 & -1 & 7 \\ 3 & -2 & -5 & 9 \end{pmatrix}$

8.5 Cramer's Rule

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