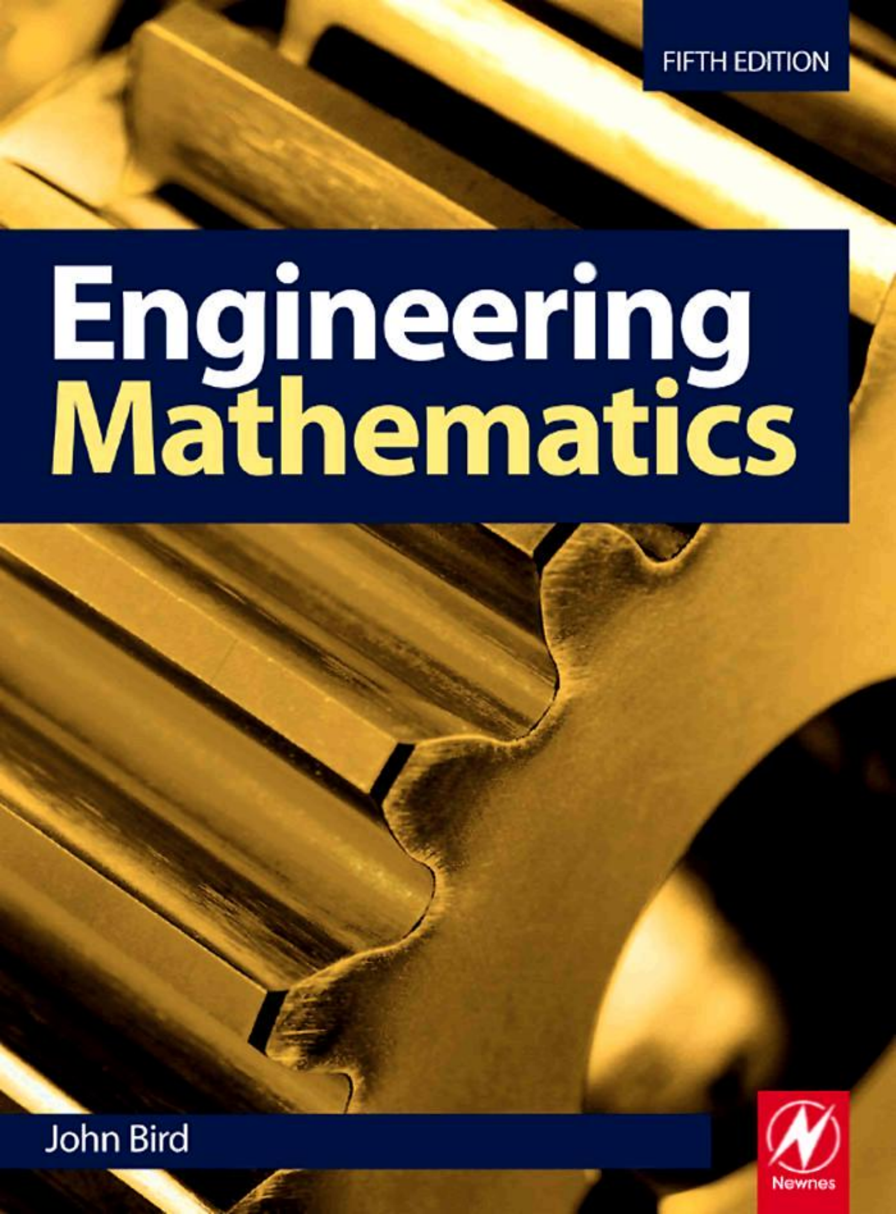
FIFTH EDITION

Engineering Mathematics



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Newnes

Chapter 55

Areas under and between curves

55.1 Area under a curve

The area shown shaded in Fig. 55.1 may be determined using approximate methods (such as the trapezoidal rule, the mid-ordinate rule or Simpson's rule) or, more precisely, by using integration.

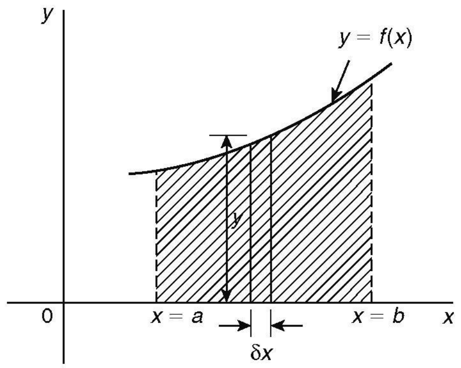

Figure 55.1

(i) Let A be the area shown shaded in Fig. 55.1 and let this area be divided into a number of strips each of width δx. One such strip is shown and let the area of this strip be δA.

Then: $\delta A \approx y\delta x$ (1)

The accuracy of statement (1) increases when the width of each strip is reduced, i.e. area A is divided into a greater number of strips.

(ii) Area A is equal to the sum of all the strips from $x = a$ to $x = b$,

i.e. $$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x \qquad (2)$$

(iii) From statement (1), $\dfrac{\delta A}{\delta x} \approx y$ (3)

In the limit, as δx approaches zero, $\dfrac{\delta A}{\delta x}$ becomes the differential coefficient $\dfrac{dA}{dx}$

Hence $\lim_{\delta x \to 0}\left(\dfrac{\delta A}{\delta x}\right) = \dfrac{dA}{dx} = y$, from statement (3).

By integration,

$$\int \frac{dA}{dx}\,dx = \int y\,dx \quad \text{i.e.} \quad A = \int y\,dx$$

The ordinates $x = a$ and $x = b$ limit the area and such ordinate values are shown as limits. Hence

$$A = \int_a^b y\,dx \qquad (4)$$

(iv) Equating statements (2) and (4) gives:

$$\textbf{Area } A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx$$

$$= \int_a^b f(x)\,dx$$

(v) If the area between a curve $x = f(y)$, the y-axis and ordinates $y = p$ and $y = q$ is required, then

$$\text{area} = \int_p^q x\,dy$$

Thus, determining the area under a curve by integration merely involves evaluating a definite integral.

There are several instances in engineering and science where the area beneath a curve needs to be accurately determined. For example, the areas between limits of a:

velocity/time graph gives distance travelled,
force/distance graph gives work done,
voltage/current graph gives power, and so on.

Should a curve drop below the x-axis, then $y\ (=f(x))$ becomes negative and $f(x)\,dx$ is negative. When determining such areas by integration, a negative sign is placed before the integral. For the curve shown in Fig. 55.2, the total shaded area is given by (area E + area F + area G).

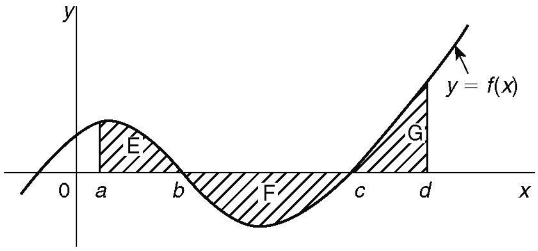

Figure 55.2

By integration, **total shaded area**

$$= \int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

(Note that this is **not** the same as $\int_a^d f(x)\,dx$.)
It is usually necessary to sketch a curve in order to check whether it crosses the x-axis.

55.2 Worked problems on the area under a curve

Problem 1. Determine the area enclosed by $y=2x+3$, the x-axis and ordinates $x=1$ and $x=4$

$y=2x+3$ is a straight line graph as shown in Fig. 55.3, where the required area is shown shaded.
By integration,

$$\text{shaded area} = \int_1^4 y\,dx$$
$$= \int_1^4 (2x+3)\,dx$$
$$= \left[\frac{2x^2}{2}+3x\right]_1^4$$
$$= [(16+12)-(1+3)]$$
$$= \textbf{24 square units}$$

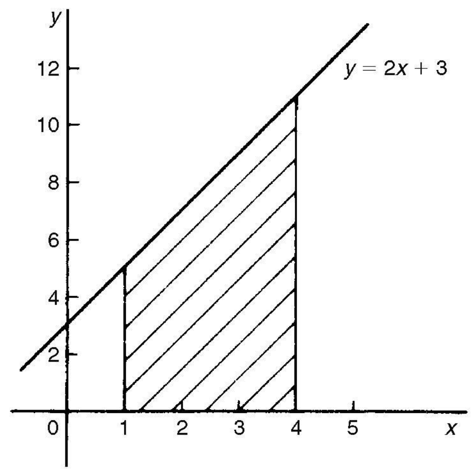

Figure 55.3

[This answer may be checked since the shaded area is a trapezium.

Area of trapezium

$$= \frac{1}{2}\begin{pmatrix}\text{sum of parallel}\\ \text{sides}\end{pmatrix}\begin{pmatrix}\text{perpendicular distance}\\ \text{between parallel sides}\end{pmatrix}$$
$$= \frac{1}{2}(5+11)(3)$$
$$= \textbf{24 square units}]$$

Problem 2. The velocity v of a body t seconds after a certain instant is: $(2t^2+5)$ m/s. Find by integration how far it moves in the interval from $t=0$ to $t=4$ s

Since $2t^2+5$ is a quadratic expression, the curve $v=2t^2+5$ is a parabola cutting the v-axis at $v=5$, as shown in Fig. 55.4.
The distance travelled is given by the area under the v/t curve (shown shaded in Fig. 55.4).
By integration,

$$\text{shaded area} = \int_0^4 v\,dt$$
$$= \int_0^4 (2t^2+5)\,dt$$
$$= \left[\frac{2t^3}{3}+5t\right]_0^4$$
$$= \left(\frac{2(4^3)}{3}+5(4)\right)-(0)$$

i.e. **distance travelled = 62.67 m**

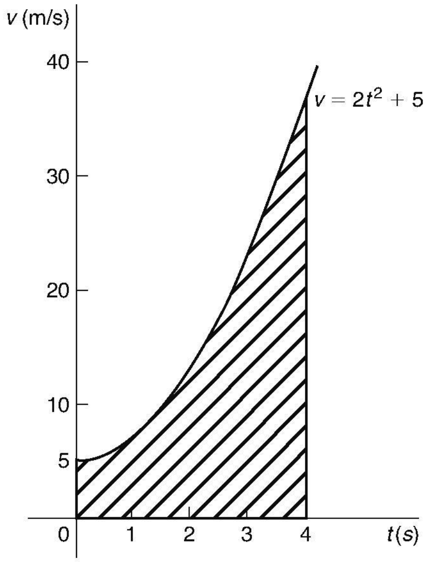

Figure 55.4

Problem 3. Sketch the graph $y=x^3+2x^2-5x-6$ between $x=-3$ and $x=2$ and determine the area enclosed by the curve and the x-axis

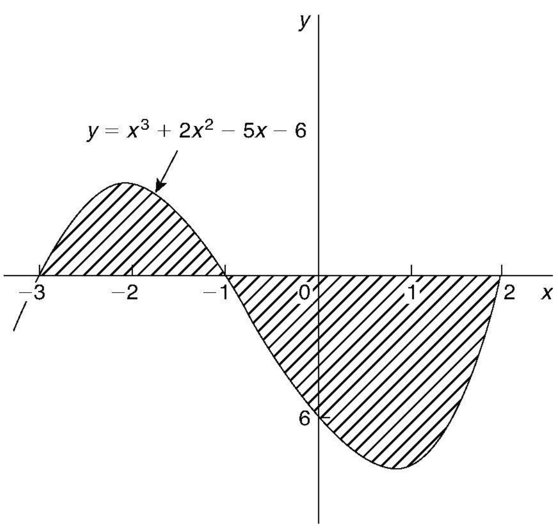

Figure 55.5

A table of values is produced and the graph sketched as shown in Fig. 55.5 where the area enclosed by the curve and the x-axis is shown shaded.

x	−3	−2	−1	0	1	2
x^3	−27	−8	−1	0	1	8
$2x^2$	18	8	2	0	2	8
$-5x$	15	10	5	0	−5	−10
−6	−6	−6	−6	−6	−6	−6
y	0	4	0	−6	−8	0

Shaded area $=\int_{-3}^{-1} y\,dx-\int_{-1}^{2} y\,dx$, the minus sign before the second integral being necessary since the enclosed area is below the x-axis.

Hence shaded area

$$=\int_{-3}^{-1}(x^3+2x^2-5x-6)\,dx-\int_{-1}^{2}(x^3+2x^2-5x-6)\,dx$$

$$=\left[\frac{x^4}{4}+\frac{2x^3}{3}-\frac{5x^2}{2}-6x\right]_{-3}^{-1}-\left[\frac{x^4}{4}+\frac{2x^3}{3}-\frac{5x^2}{2}-6x\right]_{-1}^{2}$$

$$=\left[\left\{\frac{1}{4}-\frac{2}{3}-\frac{5}{2}+6\right\}-\left\{\frac{81}{4}-18-\frac{45}{2}+18\right\}\right]-\left[\left\{4+\frac{16}{3}-10-12\right\}-\left\{\frac{1}{4}-\frac{2}{3}-\frac{5}{2}+6\right\}\right]$$

$$=\left[\left\{3\frac{1}{12}\right\}-\left\{-2\frac{1}{4}\right\}\right]-\left[\left\{-12\frac{2}{3}\right\}-\left\{3\frac{1}{12}\right\}\right]$$

$$=\left[5\frac{1}{3}\right]-\left[-15\frac{3}{4}\right]$$

$=\mathbf{21\frac{1}{12}}$ or **21.08 square units**

Problem 4. Determine the area enclosed by the curve $y=3x^2+4$, the x-axis and ordinates $x=1$ and $x=4$ by (a) the trapezoidal rule, (b) the mid-ordinate rule, (c) Simpson's rule, and (d) integration

x	0	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	4	7	10.75	16	22.75	31	40.75	52

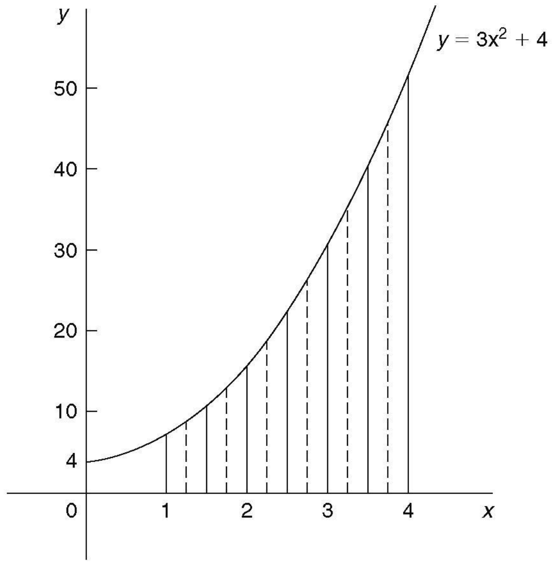

Figure 55.6

The curve $y=3x^2+4$ is shown plotted in Fig. 55.6.

(a) **By the trapezoidal rule**

$$\text{Area}=\begin{pmatrix}\text{width of}\\\text{interval}\end{pmatrix}\left[\frac{1}{2}\begin{pmatrix}\text{first}+\text{last}\\\text{ordinate}\end{pmatrix}+\begin{pmatrix}\text{sum of}\\\text{remaining}\\\text{ordinates}\end{pmatrix}\right]$$

Selecting 6 intervals each of width 0.5 gives:

$$\text{Area}=(0.5)\left[\frac{1}{2}(7+52)+10.75+16+22.75+31+40.75\right]$$

$$=\mathbf{75.375\ square\ units}$$

(b) **By the mid-ordinate rule**,
area = (width of interval) (sum of mid-ordinates). Selecting 6 intervals, each of width 0.5 gives the mid-ordinates as shown by the broken lines in Fig. 55.6.

$$\text{Thus, area}=(0.5)(8.5+13+19+26.5+35.5+46)$$

$$=\mathbf{74.25\ square\ units}$$

(c) **By Simpson's rule**,

$$\text{area}=\frac{1}{3}\begin{pmatrix}\text{width of}\\\text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first}+\text{last}\\\text{ordinates}\end{pmatrix}+4\begin{pmatrix}\text{sum of even}\\\text{ordinates}\end{pmatrix}+2\begin{pmatrix}\text{sum of remaining}\\\text{odd ordinates}\end{pmatrix}\right]$$

Selecting 6 intervals, each of width 0.5, gives:

$$\text{area}=\frac{1}{3}(0.5)[(7+52)+4(10.75+22.75+40.75)+2(16+31)]$$

$$=\mathbf{75\ square\ units}$$

(d) **By integration**, shaded area

$$=\int_1^4 y\,dx$$

$$=\int_1^4 (3x^2+4)\,dx$$

$$=\left[x^3+4x\right]_1^4$$

$$=\mathbf{75\ square\ units}$$

Integration gives the precise value for the area under a curve. In this case Simpson's rule is seen to be the most accurate of the three approximate methods.

Problem 5. Find the area enclosed by the curve $y=\sin 2x$, the x-axis and the ordinates $x=0$ and $x=\pi/3$

A sketch of $y=\sin 2x$ is shown in Fig. 55.7.

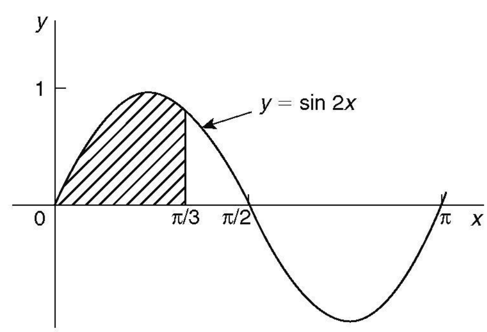

Figure 55.7

(Note that $y=\sin 2x$ has a period of $\dfrac{2\pi}{2}$, i.e. π radians.)

$$\text{Shaded area} = \int_0^{\pi/3} y\,dx$$
$$= \int_0^{\pi/3} \sin 2x\,dx$$
$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/3}$$
$$= \left\{-\frac{1}{2}\cos\frac{2\pi}{3}\right\} - \left\{-\frac{1}{2}\cos 0\right\}$$
$$= \left\{-\frac{1}{2}\left(-\frac{1}{2}\right)\right\} - \left\{-\frac{1}{2}(1)\right\}$$
$$= \frac{1}{4} + \frac{1}{2} = \mathbf{\frac{3}{4}}\ \textbf{square units}$$

Now try the following exercise

Exercise 191 Further problems on area under curves

Unless otherwise stated all answers are in square units.

1. Shown by integration that the area of the triangle formed by the line $y = 2x$, the ordinates $x = 0$ and $x = 4$ and the x-axis is 16 square units.

2. Sketch the curve $y = 3x^2 + 1$ between $x = -2$ and $x = 4$. Determine by integration the area enclosed by the curve, the x-axis and ordinates $x = -1$ and $x = 3$. Use an approximate method to find the area and compare your result with that obtained by integration. [32]

In Problems 3 to 8, find the area enclosed between the given curves, the horizontal axis and the given ordinates.

3. $y = 5x$; $x = 1, x = 4$ [37.5]

4. $y = 2x^2 - x + 1$; $x = -1, x = 2$ [7.5]

5. $y = 2\sin 2\theta$; $\theta = 0, \theta = \frac{\pi}{4}$ [1]

6. $\theta = t + e^t$; $t = 0, t = 2$ [8.389]

7. $y = 5\cos 3t$; $t = 0, t = \frac{\pi}{6}$ [1.67]

8. $y = (x-1)(x-3)$; $x = 0, x = 3$ [2.67]

55.3 Further worked problems on the area under a curve

Problem 6. A gas expands according to the law $pv = \text{constant}$. When the volume is $3\,\text{m}^3$ the pressure is 150 kPa. Given that work done $= \int_{v_1}^{v_2} p\,dv$, determine the work done as the gas expands from $2\,\text{m}^3$ to a volume of $6\,\text{m}^3$

$pv = \text{constant}$. When $v = 3\,\text{m}^3$ and $p = 150$ kPa the constant is given by $(3 \times 150) = 450\,\text{kPa m}^3$ or 450 kJ.

Hence $pv = 450$, or $p = \frac{450}{v}$

$$\text{Work done} = \int_2^6 \frac{450}{v}\,dv$$
$$= [450\ln v]_2^6 = 450[\ln 6 - \ln 2]$$
$$= 450\ln\frac{6}{2} = 450\ln 3 = \mathbf{494.4\ kJ}$$

Problem 7. Determine the area enclosed by the curve $y = 4\cos\left(\frac{\theta}{2}\right)$, the θ-axis and ordinates $\theta = 0$ and $\theta = \frac{\pi}{2}$

The curve $y = 4\cos(\theta/2)$ is shown in Fig. 55.8.

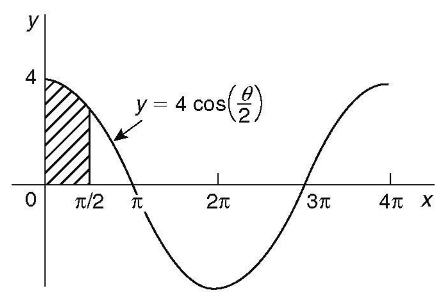

Figure 55.8

(Note that $y = 4\cos\left(\frac{\theta}{2}\right)$ has a maximum value of 4 and period $2\pi/(1/2)$, i.e. 4π rads.)

$$\text{Shaded area} = \int_0^{\pi/2} y\,d\theta = \int_0^{\pi/2} 4\cos\frac{\theta}{2}\,d\theta$$
$$= \left[4\left(\frac{1}{\frac{1}{2}}\right)\sin\frac{\theta}{2}\right]_0^{\pi/2}$$

$$= \left(8 \sin \frac{\pi}{4}\right) - (8 \sin 0)$$

$$= \mathbf{5.657 \text{ square units}}$$

Problem 8. Determine the area bounded by the curve $y = 3e^{t/4}$, the t-axis and ordinates $t = -1$ and $t = 4$, correct to 4 significant figures

A table of values is produced as shown.

t	-1	0	1	2	3	4
$y = 3e^{t/4}$	2.34	3.0	3.85	4.95	6.35	8.15

Since all the values of y are positive the area required is wholly above the t-axis.

$$\text{Hence area} = \int_1^4 y\, dt$$

$$= \int_1^4 3e^{t/4} dt = \left[\frac{3}{\left(\frac{1}{4}\right)} e^{t/4}\right]_{-1}^4$$

$$= 12\left[e^{t/4}\right]_{-1}^4 = 12(e^1 - e^{-1/4})$$

$$= 12(2.7183 - 0.7788)$$

$$= 12(1.9395) = \mathbf{23.27 \text{ square units}}$$

Problem 9. Sketch the curve $y = x^2 + 5$ between $x = -1$ and $x = 4$. Find the area enclosed by the curve, the x-axis and the ordinates $x = 0$ and $x = 3$. Determine also, by integration, the area enclosed by the curve and the y-axis, between the same limits

A table of values is produced and the curve $y = x^2 + 5$ plotted as shown in Fig. 55.9.

x	-1	0	1	2	3
y	6	5	6	9	14

$$\text{Shaded area} = \int_0^3 y\, dx = \int_0^3 (x^2 + 5)\, dx$$

$$= \left[\frac{x^3}{5} + 5x\right]_0^3$$

$$= \mathbf{24 \text{ square units}}$$

When $x = 3$, $y = 3^2 + 5 = 14$, and when $x = 0$, $y = 5$.

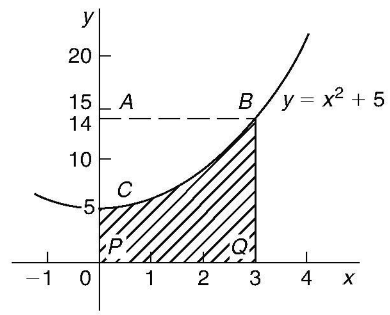

Figure 55.9

Since $y = x^2 + 5$ then $x^2 = y - 5$ and $x = \sqrt{y - 5}$
The area enclosed by the curve $y = x^2 + 5$ (i.e. $x = \sqrt{y - 5}$), the y-axis and the ordinates $y = 5$ and $y = 14$ (i.e. area ABC of Fig. 55.9) is given by:

$$\text{Area} = \int_{y=5}^{y=14} x\, dy = \int_5^{14} \sqrt{y - 5}\, dy$$

$$= \int_5^{14} (y - 5)^{1/2}\, dy$$

Let $u = y - 5$, then $\frac{du}{dy} = 1$ and $dy = du$

Hence $\int (y - 5)^{1/2} dy = \int u^{1/2} du = \frac{2}{3} u^{3/2}$
(for algebraic substitutions, see Chapter 49)
Since $u = y - 5$ then

$$\int_5^{14} \sqrt{y - 5}\, dy = \frac{2}{3}\left[(y - 5)^{3/2}\right]_5^{14}$$

$$= \frac{2}{3}\left[\sqrt{9^3} - 0\right]$$

$$= \mathbf{18 \text{ square units}}$$

(Check: From Fig. 55.9, area BCPQ + area $ABC = 24 + 18 = 42$ square units, which is the area of rectangle $ABQP$.)

Problem 10. Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis

$$y = x^3 - 2x^2 - 8x = x(x^2 - 2x - 8)$$

$$= x(x + 2)(x - 4)$$

When $y = 0$, then $x = 0$ or $(x + 2) = 0$ or $(x - 4) = 0$, i.e. when $y = 0$, $x = 0$ or -2 or 4, which means that the curve crosses the x-axis at 0, -2 and 4. Since the curve is a continuous function, only one other co-ordinate value needs to be calculated before a sketch of the curve can be produced. When $x = 1$, $y = -9$, showing that the part of the curve between $x = 0$ and $x = 4$ is negative. A sketch of $y = x^3 - 2x^2 - 8x$ is shown in Fig. 55.10. (Another method of sketching Fig. 55.10 would have been to draw up a table of values.)

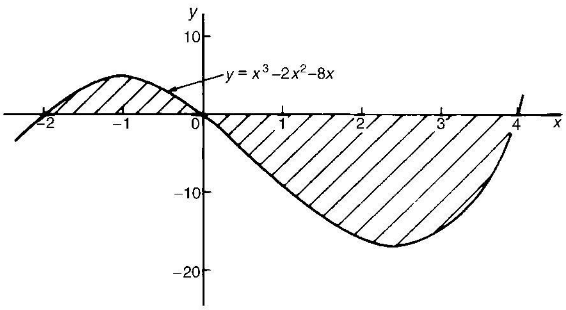

Figure 55.10

$$\text{Shaded area} = \int_{-2}^{0} (x^3 - 2x^2 - 8x)\,dx - \int_{0}^{4} (x^3 - 2x^2 - 8x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{0}^{4}$$

$$= \left(6\frac{2}{3}\right) - \left(-42\frac{2}{3}\right)$$

$$= \mathbf{49\frac{1}{3}\ square\ units}$$

Now try the following exercise

Exercise 192 Further problems on areas under curves

In Problems 1 and 2, find the area enclosed between the given curves, the horizontal axis and the given ordinates.

1. $y = 2x^3$; $x = -2, x = 2$ [16 square units]
2. $xy = 4$; $x = 1, x = 4$ [5.545 square units]
3. The force F newtons acting on a body at a distance x metres from a fixed point is given by: $F = 3x + 2x^2$. If work done $= \int_{x_1}^{x_2} F\,dx$, determine the work done when the body moves from the position where $x = 1$ m to that where $x = 3$ m. [29.33 Nm]
4. Find the area between the curve $y = 4x - x^2$ and the x-axis. [10.67 square units]
5. Determine the area enclosed by the curve $y = 5x^2 + 2$, the x-axis and the ordinates $x = 0$ and $x = 3$. Find also the area enclosed by the curve and the y-axis between the same limits. [51 sq. units, 90 sq. units]
6. Calculate the area enclosed between $y = x^3 - 4x^2 - 5x$ and the x-axis. [73.83 sq. units]
7. The velocity v of a vehicle t seconds after a certain instant is given by: $v = (3t^2 + 4)$ m/s. Determine how far it moves in the interval from $t = 1$ s to $t = 5$ s. [140 m]
8. A gas expands according to the law $pv =$ constant. When the volume is $2\,m^3$ the pressure is 250 kPa. Find the work done as the gas expands from $1\,m^3$ to a volume of $4\,m^3$ given that work done $= \int_{v_1}^{v_2} p\,dv$ [693.1 kJ]

55.4 The area between curves

The area enclosed between curves $y = f_1(x)$ and $y = f_2(x)$ (shown shaded in Fig. 55.11) is given by:

$$\text{shaded area} = \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

$$= \int_a^b [f_2(x) - f_2(x)]\,dx$$

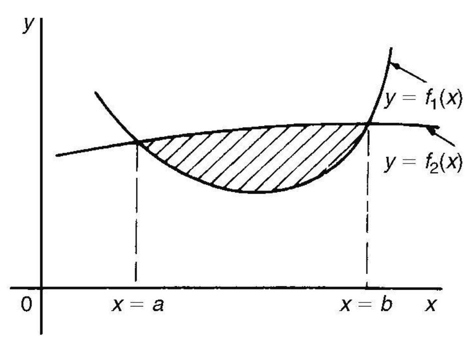

Figure 55.11

Problem 11. Determine the area enclosed between the curves $y = x^2 + 1$ and $y = 7 - x$

At the points of intersection, the curves are equal. Thus, equating the y-values of each curve gives: $x^2 + 1 = 7 - x$, from which $x^2 + x - 6 = 0$. Factorising gives $(x - 2)(x + 3) = 0$, from which, $x = 2$ and $x = -3$. By firstly determining the points of intersection the range of x-values has been found. Tables of values are produced as shown below.

x	−3	−2	−1	0	1	2
$y = x^2 + 1$	10	5	2	1	2	5

x	−3	0	2
$y = 7 - x$	10	7	5

A sketch of the two curves is shown in Fig. 55.12.

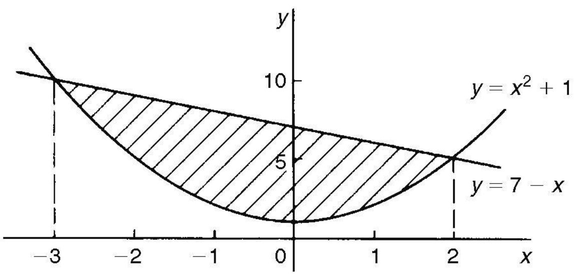

Figure 55.12

$$\text{Shaded area} = \int_{-3}^{2} (7 - x)dx - \int_{-3}^{2} (x^2 + 1)dx$$

$$= \int_{-3}^{2} [(7 - x) - (x^2 + 1)]dx$$

$$= \int_{-3}^{2} (6 - x - x^2)dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$$

$$= \left(7\frac{1}{3}\right) - \left(-13\frac{1}{2}\right)$$

$$= \mathbf{20\frac{5}{6}\ square\ units}$$

Problem 12. (a) Determine the coordinates of the points of intersection of the curves $y = x^2$ and $y^2 = 8x$. (b) Sketch the curves $y = x^2$ and $y^2 = 8x$ on the same axes. (c) Calculate the area enclosed by the two curves

(a) At the points of intersection the coordinates of the curves are equal. When $y = x^2$ then $y^2 = x^4$.

Hence at the points of intersection $x^4 = 8x$, by equating the y^2 values.

Thus $x^4 - 8x = 0$, from which $x(x^3 - 8) = 0$, i.e. $x = 0$ or $(x^3 - 8) = 0$.

Hence at the points of intersection $x = 0$ or $x = 2$.

When $x = 0$, $y = 0$ and when $x = 2$, $y = 2^2 = 4$.

Hence the points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are (0, 0) and (2, 4)

(b) A sketch of $y = x^2$ and $y^2 = 8x$ is shown in Fig. 55.13

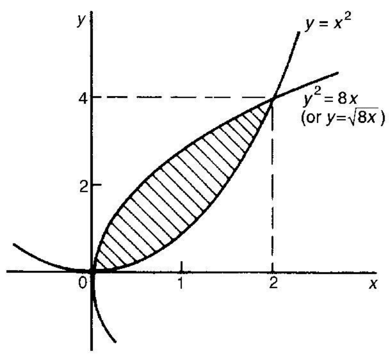

Figure 55.13

(c) $$\textbf{Shaded area} = \int_0^2 \{\sqrt{8x} - x^2\}dx$$

$$= \int_0^2 \{(\sqrt{8})x^{1/2} - x^2\}dx$$

$$= \left[(\sqrt{8})\frac{x^{3/2}}{(\frac{3}{2})} - \frac{x^3}{3}\right]_0^2$$

$$= \left\{\frac{\sqrt{8}\sqrt{8}}{(\frac{3}{2})} - \frac{8}{3}\right\} - \{0\}$$

$$= \frac{16}{3} - \frac{8}{3} = \frac{8}{3}$$

$$= \mathbf{2\frac{2}{3}\ square\ units}$$

Problem 13. Determine by integration the area bounded by the three straight lines $y=4-x$, $y=3x$ and $3y=x$

Each of the straight lines is shown sketched in Fig. 55.14.

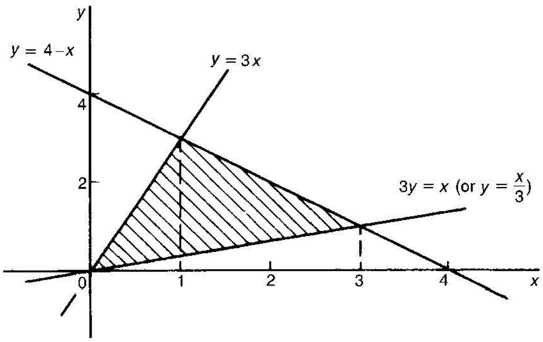

Figure 55.14

$$\text{Shaded area} = \int_0^1 \left(3x - \frac{x}{3}\right)dx + \int_1^3 \left[(4-x) - \frac{x}{3}\right]dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^2}{6}\right]_0^1 + \left[4x - \frac{x^2}{2} - \frac{x^2}{6}\right]_1^3$$

$$= \left[\left(\frac{3}{2} - \frac{1}{6}\right) - (0)\right] + \left[\left(12 - \frac{9}{2} - \frac{9}{6}\right) - \left(4 - \frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \left(1\frac{1}{3}\right) + \left(6 - 3\frac{1}{3}\right)$$

$$= \mathbf{4\ square\ units}$$

Now try the following exercise

Exercise 193 Further problems on areas between curves

1. Determine the coordinates of the points of intersection and the area enclosed between the parabolas $y^2=3x$ and $x^2=3y$.
[(0, 0) and (3, 3), 3 sq. units]

2. Sketch the curves $y=x^2+3$ and $y=7-3x$ and determine the area enclosed by them.
[20.83 square units]

3. Determine the area enclosed by the curves $y=\sin x$ and $y=\cos x$ and the y-axis.
[0.4142 square units]

4. Determine the area enclosed by the three straight lines $y=3x$, $2y=x$ and $y+2x=5$
[2.5 sq. units]