Crowell and Slesnick's
Calculus with Analytic Geometry

The Dartmouth CHANCE Project [1]

Version 3.0.3, 5 January 2008

Chapter 10

Geometry in the Plane

Suppose that we are concerned with the motion of a particle as it moves in a plane. At any time t during the motion, the position of the particle is given by its two coordinates, which depend on time, and may therefore be denoted by $x(t)$ and $y(t)$, respectively. The set of points traced out by the particle as it moves during a given interval of time is a curve. The function which describes the position of the particle is called a parametrization, and a curve described by such a function is said to be parametrized. In the first sections of this chapter we shall develop the mathematical theory of parametrized curves, abstracting from the picture of a physical particle in motion. Later we shall return to this application and define the notions of velocity and acceleration of such particles.

Parametrized curves represent an important generalization of the curves encountered thus far as the graphs of functions. As we shall see, a parametrized curve is not necessarily the graph of an equation $y = f(x)$.

10.1 Parametrically Defined Curves.

When we speak of the plane in this book, we assume, unless otherwise stated, that a pair of coordinate axes has been chosen. As a result, we identify the set of points in the plane with the set R^2 of all ordered pairs of real numbers. A convenient notation for a function P whose domain is an interval I of real numbers and whose range is a subset of the plane is $P : I \to R^2$. Every function $P : I \to R^2$ defines two **coordinate functions,** the functions which assign to every t in I the two coordinates of the point $P(t)$. If we denote the first coordinate function by f, and the second one by g, then they are defined by the equation

$$P(t) = (f(t), g(t)), \quad \text{for every } t \text{ in } I. \tag{10.1}$$

Conversely, of course, every ordered pair of real-valued functions f and g with an interval I as common domain defines a function $P : I \to R^2$ by equation (1).

Since the first and second coordinates of an element of R^2 are usually the x- and y-coordinates, respectively, we may alternatively define a function $P : I \to R^2$ by a pair of equations

$$\begin{cases} x = f(t), \\ y = g(t), \end{cases}$$

where f and g are real-valued functions with domain I. Then, for every t in I, we have $P(t) = (x, y) = (f(t), g(t))$. It is also common practice to denote the coordinate functions themselves by x and y. When this is done, we do not hesitate to write the equations $x = x(t)$ and $y = y(t)$, and the function $P : I \to R^2$ is defined by

$$P(t) = (x(t), y(t)), \quad \text{for every } t \text{ in } I.$$

A function $P : I \to R^2$ is said to be **continuous at** t_0 if both coordinate functions are continuous at t_0. If the coordinate functions are denoted by x and y, then we define

$$\lim_{t \to t_0} P(t) = (\lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t)).$$

As a result, the definition of continuity for P is entirely analogous to that for a real-valued function: P is continuous at t_0 if t_0 is in the domain of P and if $\lim_{t \to t_0} P(t) = P(t_0)$. As before, the function P is simply said to be **continuous** if it is continuous at every number in its domain.

A **curve** in the plane is by definition a subset of R^2 which is the range of some continuous function $P : I \to R^2$. Every curve is the range of many such functions, and, as a result, it is necessary to choose our terminology carefully. We shall call a continuous function $P : I \to R^2$, a **parametrization** of the curve C which is the range of P, and we shall say that C is **parametrically defined** by $P : I \to R^2$. The points of the curve C obviously consist of the set of all points $P(t)$, for every t in I. By a **parametrized curve** we shall mean the range of a specified continuous function $P : I \to R^2$. Speaking more casually, we shall refer to the curve defined parametrically by

$$P(t) = (x(t), y(t)),$$

or, equivalently, to the curve defined parametrically by the equations

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases}$$

for every t in some interval I which is the common domain of the continuous functions x and y. If t is regarded as an independent variable, it is called the **parameter** of the parametrized curve.

Example 197. Draw the curve defined parametrically by

$$P(t) = (t^2, t), \quad -\infty < t < \infty.$$

This is, of course, also the curve defined by the equations

$$\begin{cases} x = t^2, \\ y = t, \quad -\infty < t < \infty. \end{cases}$$

It is plotted in Figure 1. Since the set of all points (x, y) which satisfy the above two equations is equal to the set of all points (x, y) such that $x = y^2$, we recognize the curve as a parabola.

seconds	0	1	2	3	4
meters	80	75.1	60.4	35.9	1.6

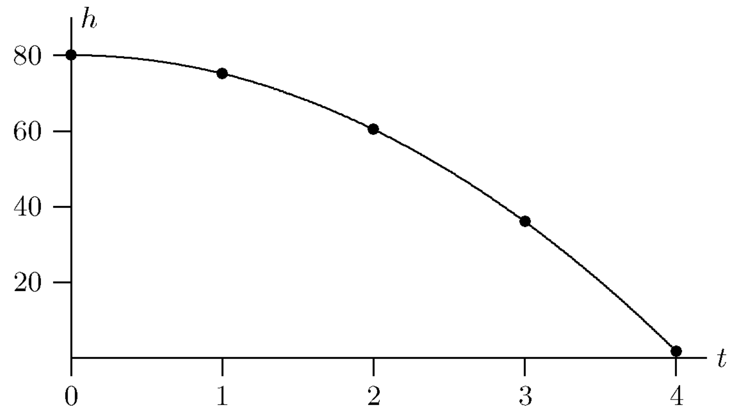

Figure 1.0.1 A data plot, height versus time.

is often written Δx, where the meaning of Δ (a capital delta in the Greek alphabet) is "change in". (Thus, Δx can be read as "change in x" although it usually is read as "delta x". The point is that Δx denotes a single number, and should not be interpreted as "delta times x".) For example, if $A = (2, 1)$ and $B = (3, 3)$, $\Delta x = 3 - 2 = 1$. Similarly, the "change in y" is written Δy. In our example, $\Delta y = 3 - 1 = 2$, the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B. The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) are:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that either or both of these might be negative.

1.1 Lines

If we have two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then we can draw one and only one line through both points. By the *slope* of this line we mean the ratio of Δy to Δx. The slope is often denoted m: $m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. For example, the line joining the points $(1, -2)$ and $(3, 5)$ has slope $(5 + 2)/(3 - 1) = 7/2$.

EXAMPLE 1.1.1 According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to \$26050. If taxable income was between \$26050 and \$134930, then, in addition, 28% was to be paid on the amount between \$26050 and \$67200, and 33% paid on the amount over \$67200 (if any). Interpret the tax bracket information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the y-axis against the taxable income on the x-axis.

The percentages, when converted to decimal values 0.15, 0.28, and 0.33, are the *slopes* of the straight lines which form the graph of the tax for the corresponding tax brackets. The tax graph is what's called a *polygonal line*, i.e., it's made up of several straight line segments of different slopes. The first line starts at the point (0,0) and heads upward with slope 0.15 (i.e., it goes upward 15 for every increase of 100 in the x-direction), until it reaches the point above $x = 26050$. Then the graph "bends upward," i.e., the slope changes to 0.28. As the horizontal coordinate goes from $x = 26050$ to $x = 67200$, the line goes upward 28 for each 100 in the x-direction. At $x = 67200$ the line turns upward again and continues with slope 0.33. See figure 1.1.1. □

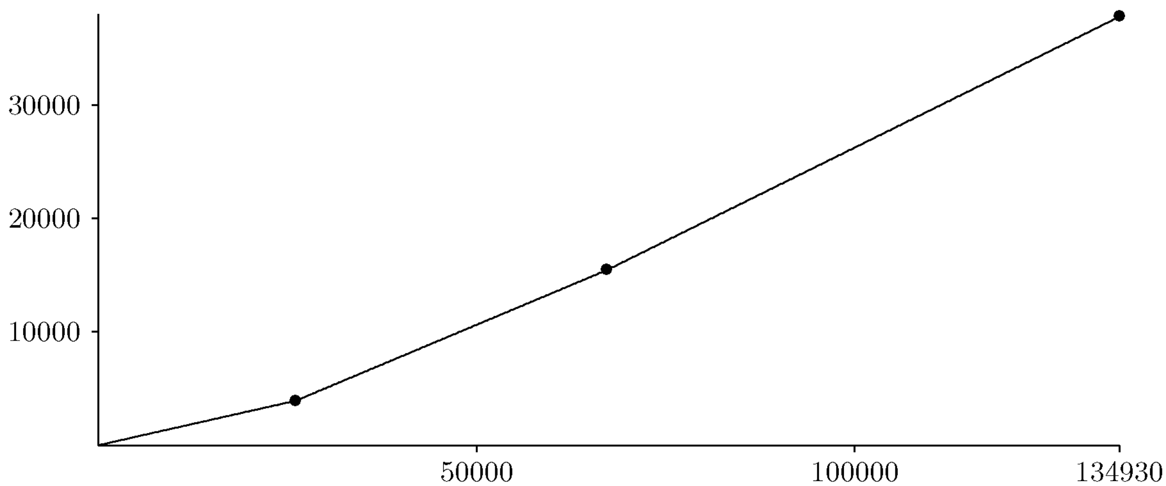

Figure 1.1.1 Tax vs. income.

The most familiar form of the equation of a straight line is: $y = mx + b$. Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m. If you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the **y-intercept**, because it is where the line crosses the y-axis. If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of either point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the "point-slope" form of the equation of a straight line: start with $(y - y_1)/(x - x_1) = m$ and then multiply to get $(y - y_1) = m(x - x_1)$, the point-slope form. Of course, this may be further manipulated to get $y = mx - mx_1 + y_1$, which is essentially the "$mx + b$" form.

It is possible to find the equation of a line between two points directly from the relation $(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$, which says "the slope measured between the point (x_1, y_1) and the point (x_2, y_2) is the same as the slope measured between the point (x_1, y_1)

and any other point (x, y) on the line." For example, if we want to find the equation of the line joining our earlier points $A(2, 1)$ and $B(3, 3)$, we can use this formula:

$$\frac{y-1}{x-2} = \frac{3-1}{3-2} = 2, \qquad \text{so that} \qquad y - 1 = 2(x - 2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

Of course, this is really just the point-slope formula, except that we are not computing m in a separate step.

The slope m of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline, while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward; while if m is negative but near zero, then it points only a little downward. These four possibilities are illustrated in figure 1.1.2.

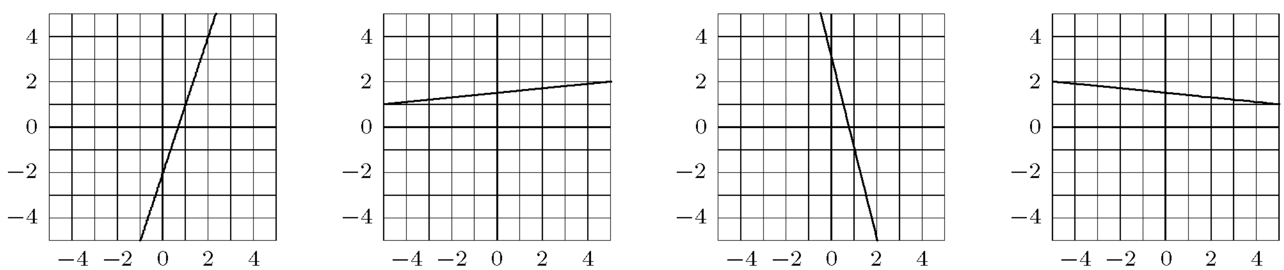

Figure 1.1.2 Lines with slopes 3, 0.1, −4, and −0.1.

If $m = 0$, then the line is horizontal: its equation is simply $y = b$.

There is one type of line that cannot be written in the form $y = mx + b$, namely, vertical lines. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

Sometimes it is useful to find the x-intercept of a line $y = mx + b$. This is the x-value when $y = 0$. Setting $mx + b$ equal to 0 and solving for x gives: $x = -b/m$. For example, the line $y = 2x - 3$ through the points $A(2, 1)$ and $B(3, 3)$ has x-intercept $3/2$.

EXAMPLE 1.1.2 Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., $t = 1$), you pass a sign saying it is 110 miles to Seattle, and after driving another half-hour you pass a sign saying it is 85 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from Seattle, graph and find the equation $y = mt + b$ for your distance from Seattle. Find the slope, y-intercept, and t-intercept, and describe the practical meaning of each.

The graph of y versus t is a straight line because you are traveling at constant speed. The line passes through the two points $(1, 110)$ and $(1.5, 85)$, so its slope is $m = (85 -$

110)/(1.5 − 1) = −50. The meaning of the slope is that you are traveling at 50 mph; m is negative because you are traveling *toward* Seattle, i.e., your distance y is *decreasing*. The word "velocity" is often used for $m = -50$, when we want to indicate direction, while the word "speed" refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

$$\frac{y-110}{t-1} = -50, \qquad \text{so that} \qquad y = -50(t-1) + 110 = -50t + 160.$$

The meaning of the y-intercept 160 is that when $t = 0$ (when you started the trip) you were 160 miles from Seattle. To find the t-intercept, set $0 = -50t + 160$, so that $t = 160/50 = 3.2$. The meaning of the t-intercept is the duration of your trip, from the start until you arrive in Seattle. After traveling 3 hours and 12 minutes, your distance y from Seattle will be 0.

□

Exercises 1.1.

1. Find the equation of the line through $(1, 1)$ and $(-5, -3)$ in the form $y = mx + b$. ⇒

2. Find the equation of the line through $(-1, 2)$ with slope -2 in the form $y = mx + b$. ⇒

3. Find the equation of the line through $(-1, 1)$ and $(5, -3)$ in the form $y = mx + b$. ⇒

4. Change the equation $y - 2x = 2$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

5. Change the equation $x + y = 6$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

6. Change the equation $x = 2y - 1$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

7. Change the equation $3 = 2y$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

8. Change the equation $2x + 3y + 6 = 0$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

9. Determine whether the lines $3x + 6y = 7$ and $2x + 4y = 5$ are parallel. ⇒

10. Suppose a triangle in the x, y–plane has vertices $(-1, 0)$, $(1, 0)$ and $(0, 2)$. Find the equations of the three lines that lie along the sides of the triangle in $y = mx + b$ form. ⇒

11. Suppose that you are driving to Seattle at constant speed. After you have been traveling for an hour you pass a sign saying it is 130 miles to Seattle, and after driving another 20 minutes you pass a sign saying it is 105 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from your starting point, graph and find the equation $y = mt + b$ for your distance from your starting point. How long does the trip to Seattle take? ⇒

12. Let x stand for temperature in degrees Celsius (centigrade), and let y stand for temperature in degrees Fahrenheit. A temperature of 0°C corresponds to 32°F, and a temperature of 100°C corresponds to 212°F. Find the equation of the line that relates temperature Fahrenheit y to temperature Celsius x in the form $y = mx + b$. Graph the line, and find the point at which this line intersects $y = x$. What is the practical meaning of this point? ⇒