2-Mavzu: Algebraik to'ldiruvchilar va minorlar. Matritsalar va ular ustida amallar. Teskari matritsa.

Reja:

1. Algebraik to'ldiruvchilar va minorlar.
2. n-tartibli determinant haqida tushuncha.
3. Matritsa tushunchasi
4. Matritsalar ustida amallar
5. Teskari matritsa

Determinant biror elementining minori deb, shu determinantdan bu element turgan satr va ustunni o’chirishdan hosil bo’lgan determinantga aytiladi. Soddalik uchun quyidagi uchinchi tartibli determinantni olamiz:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant a_{ik} elementining minori M_{ik} $(i, k = 1,2,3)$ bilan belgilanadi. Masalan a_{11} elementning minori

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ son,

a_{32} elementning minori esa $M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ son bo'ladi va h.k.

Determinant biror elementining algebraik to'ldiruvchisi deb musbat yoki manfiy ishora bilan olingan minoriga aytiladi. a_{ik} elementning algebraik to'ldiruvchisi A_{ik} bilan belgilanadi, bunda $A_{ik} = (-1)^{i+k} \times M_{ik}$. Masalan,

$A_{11} = (-1)^{1+1} \times M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ son bo'ladi,

a_{32} elementning algebraik to'ldiruvchisi

$A_{32} = (-1)^{3+2} \times M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ son bo'ladi va h.k.

Determinant, biror satr (ustun) elementlari bilan ularning algebraik to'ldiruvchilariga ko'paytmalari yig'indisiga teng.

Shunday qilib, ushbu tenglik o'rinli:

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}, \quad \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \qquad (6)$$

$$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}, \Delta = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

$$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}, \ \Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

Determinantning (6) formulalarning biri bo'yicha yozilishi uning satr (ustun) elementlari bo'yicha *yoyilmasi* deb ataladi. Bu tengliklarning birinchisini isbotlaymiz. Buning uchun (6) formulaning o'ng qismini ushbu ko'rinishda yozib olamiz.

$$\Delta = (a_{11}a_{22}a_{33} - a_{32}a_{23}a_{11}) - (a_{12}a_{21}a_{33} - -a_{12}a_{23}a_{31}) + (a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13}).$$

Har bir qavsdan umumiy ko'paytuvchini chiqaramiz:

$$\Delta = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + +a_{13}(a_{21}a_{32} - a_{31}a_{22}). \qquad (7)$$

Qavslarda turgan miqdorlar a_{11}, a_{12}, a_{13} elementlarning algebraik to'ldiruvchilaridir, ya'ni

$$a_{22}a_{33} - a_{32}a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = A_{11}$$

$$-(a_{21}a_{33} - a_{23}a_{31}) = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = A_{12}$$

$$a_{21}a_{32} - a_{31}a_{22} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = A_{13} \qquad (8)$$

(7) formulani (8) formulani hisobga olgan holda bunday yozamiz:

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13},$$

shuni isbotlash kerak edi.

3-misol. Ushbu determinantni hisoblang.

$$\begin{vmatrix} 1 & 0 & -1 \\ 3 & 2 & -2 \\ 1 & 4 & 5 \end{vmatrix}$$

Yechish. Bunda birinchi satrda nol bo'lganligi uchun birinchi satr elementlari bo'yicha yoyish formulasidan foydalanish qulaydir. Quyidagini topamiz:

$$\Delta=1\cdot\begin{vmatrix} 2 & -2 \\ 4 & 5 \end{vmatrix} - 1\cdot\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 1\cdot(10+8)-1\cdot(12-2)=8$$

n-tartibli determinant haqida tushuncha.

n -tartibli matritsani, ya'ni $n \times n$ ta sondan iborat ushbu jadvalni qaraymiz.

$$\begin{pmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \dots & \dots \dots \dots \dots \dots . \\ a_{n1} & a_{n2} \dots a_{nn} \end{pmatrix}$$

Bu matritsaning n $-$tartibli determinant deb ushbu songa aytiladi

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{11} & a_{12} \dots a_{1n} \\ \dots & \dots \dots \dots \dots \dots \\ a_{n1} & a_{n2} \dots a_{nn} \end{vmatrix}$$

n −tartibli determinant uchun yuqoridagi barcha xossalar, jumladan determinantni biror satr (ustun) bo'yicha yoyish formulasi bu yerda ham o'rinli. Istalgan tartibli determinantni hisoblashda ayni shu formuladan foydalaniladi.

Misol. Ushbu to'rtinchi tartibli determinantni ikkinchi satr elementlari bo'yicha yoyish yo'li bilan hisoblang:

$$\Delta = \begin{vmatrix} 2 & 1 & 4 & 3 \\ 5 & 0 & -1 & 0 \\ 2 & -1 & 6 & 0 \\ 1 & 5 & -1 & 2 \end{vmatrix}$$

Yechish: Quyidagiga egamiz:

$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24} =$$

$$= -5 \cdot \begin{vmatrix} 1 & 4 & 3 \\ -1 & 6 & 3 \\ 5 & -1 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 4 & 3 \\ 2 & 6 & 3 \\ 1 & -1 & 2 \end{vmatrix} +$$

$$+1 \cdot \begin{vmatrix} 2 & 1 & 3 \\ 2 & -1 & 3 \\ 1 & 5 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 1 & 4 \\ 2 & -1 & 6 \\ 1 & 5 & -1 \end{vmatrix} =$$

$$= -5 \cdot \begin{vmatrix} 1 & 4 & 3 \\ -1 & 6 & 3 \\ 5 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 2 & -1 & 3 \\ 1 & 5 & 2 \end{vmatrix} = 18$$

Determinantni biror qator elementlari bo’yicha yoyish formulasi bu qatordagi elementlarning bittasidan boshqalari nolga teng bo’lganda ayniqsa sodda ko’rinishga ega bo’ladi.

Matritsalar va ular ustida amallar.

m ta satrli va n ta ustunli to'g'ri burchakli jadval shaklida yozilgan $m \cdot n$ ta son berilgan bo'lsin.

$$A = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} \dots & a_{mn} \end{pmatrix} \qquad (1)$$

Bunday jadval $m \times n$ o'lchamli *to'g'ri burchakli matritsa* deb ataladi. Bu jadvaldagi a_{ij} sonlar uning *elementlari* deb ataladi.

Elementlar satrlar va ustunlar hosil qiladi. $i$va j indekslar a_{ij} element turadigan satr va ustunning tartib raqamini ko’rsatadi. Yozuvni qisqartirish maqsadida (1) matritsa ko’pincha ushbu ko’rinishda yoziladi;

$$A = (a_{ij}), \qquad (i = 1,2, \dots, m; j = 1,2, \dots, n)$$

Agar $n = 1$ bo’lsa, u holda ustun matritsaga ega bo’lamiz:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ a_{m1} \end{pmatrix}$$

Satrlari soni ustunlari soniga teng, ya'ni $m = n$ bo'lgan ushbu matritsa n –tartibli *kvadrat matritsa* deyiladi.

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots \cdots & \cdots \cdots \cdots & \cdots \cdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{pmatrix}$$

Har bir n –tartibli A kvadrat matritsa uchun shu matritsalar elementlaridan tuzilgan n –tartibli determinantni hisoblash mumkin. Bu determinant $detA$ orqali belgilanadi. Agar $detA \neq 0$ bo'lsa, u holda A kvadrat matritsa *xosmas* deb ataladi.

Agar $detA = 0$ bo'lsa, u holda A kvadrat matritsa *xos* deb ataladi.

Kvadrat matritsaning $a_{11}, a_{22}, \dots, a_{nn}$ elementlar joylashgan diagonali *bosh diagonal*, $a_{1n}, a_{2n-1}, \dots, a_{n1}$ elementlari joyla shgan diagonal *yordamchi diagonal* deyiladi. Bosh diagonalidagi elementlaridan farqli barcha elementlari 0 ga teng kvadrat matritsa *diagonal matritsa* deyiladi.

$$A = \begin{pmatrix} a_{11} & 0 \dots & 0 \\ 0 & a_{22} \dots & 0 \\ \dots & \dots \dots \dots & \dots \\ 0 & 0 \dots & a_{nn} \end{pmatrix}$$

Bunda $detA = a_{11} \cdot a_{22} \ldots \cdot a_{nn}$. Bosh diagonalidagi barcha elementlari $a \neq 0$ bo’lgan kvadrat matritsa *skalyar matritsa* deb ataladi:

$$A = \begin{pmatrix} a & 0 \ldots & 0 \\ 0 & a \ldots & 0 \\ \ldots & \ldots \ldots \ldots & \ldots \ldots \\ 0 & 0 \ldots & a \end{pmatrix}$$

Ravshanki, $detA = a^n$.

Bosh diagonalidagi barcha elementlari 1 ga teng diagonal matritsa *birlik matritsa* deyiladi va E bilan belgilanadi.

$$A = \begin{pmatrix} 1 & 0 \ldots & 0 \\ 0 & 1 \ldots & 0 \\ \ldots & \ldots \ldots \ldots & \ldots \ldots \\ 0 & 0 \ldots & 1 \end{pmatrix}$$

Birlik matritsaning determinanti birga teng. $detE = 1.$
Barcha elementlari nolga teng matritsa *nol matritsa* deyiladi va Q bilan belgilanadi.

$$A = \begin{pmatrix} 0 & 0 \dots & 0 \\ 0 & 0 \dots & 0 \\ \dots & \dots \dots \dots & \dots .. \\ 0 & 0 \dots & 0 \end{pmatrix}$$

A matritsada barcha satrlarni mos ustunlar bilan almashtirishdan hosil bo'lgan A^* matritsa A matritsaga nisbatan *transponirlangan matritsa* deb ataladi. Agar A kvadrat matritsa bo'lsa, u holda $detA = detA^*.$

Agar $A = A^*$ bajarilsa A ga *simmetrik matritsa* deyiladi.

Matritsalar ustida amallar.

Agar ikkita $A = (a_{ij})$ va $B = (b_{ij})$ matritsa bir xil o’lchamli hamda i va j indekslarining barcha qiymatlari uchun $a_{ij} = b_{ij}$ bo’lsa, bu matritsalar *teng* deb ataladi. Matritsalarni qo’shish, songa ko’paytirish va bir biriga ko’paytirish mumkin.

Bir xil o’lchamli $A = (a_{ij})$ va B $= (b_{ij})$ matritsalarning *yig’indisi* deb, elementlari quyidagicha aniqlanadigan o’sha o’lchamli $C = (c_{ij})$ matritsaga aytiladi:

$$c_{ij} + a_{ij} + b_{ij} \quad (i = 1,..,m; j = 1,\dots,n)$$

Matritsalar yig’indisi $C = A + B$ kabi belgilanadi.

Shunday qilib, bir xildagi matritsalarni qo'shishda bu matritsalarning mos elementlarini qo'shish lozim.

1-misol. Ushbu

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -3 \\ 0 & 1 & 4 \end{pmatrix} \text{ va } B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ 0 & 3 & -2 \end{pmatrix}$$

matritsalar yig'indisini toping.

$$A + B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -3 \\ 0 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ 0 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 1+2 & 0+1 & 1-1 \\ 2+3 & 1-2 & -3+1 \\ 0+0 & 1+3 & 4-2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 0 \\ 5 & -1 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$

Ikki matritsaning *ayirmsi* ham shunday aniqlanadi.

$A = (a_{ij})$ matritsaning λ songa *ko'paytmasi* deb, $C = (c_{ij})$ matritsaga aytiladi:

$$c_{ij} = \lambda a_{ij}.$$

Shunday qilib, matritsani songa ko'paytirishda shu songa matritsaning barcha elementlarini ko'paytirish lozim.

2-misol. Ushbu

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 5 & 0 & 1 \end{pmatrix}$$

matritsani 2 soniga ko'paytiring.

Yechish.

$$2A = 2 \cdot \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 & 2 \cdot 3 & 2 \cdot 2 \\ 2 \cdot 4 & 2 \cdot 2 & 2 \cdot 1 \\ 2 \cdot 5 & 2 \cdot 0 & 2 \cdot 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 6 & 4 \\ 8 & 4 & 2 \\ 10 & 0 & 2 \end{pmatrix}$$

Matritsalarni qo'shish va songa ko'paytirish aamallari chiziqli amallar bo'lib, quyidagi xossalarga ega:

1) $A + B = B + A$;
2) $(A + B) + C = A + (B + C)$;
3) $A + Q = Q + A = A$;
4) $\mu(\lambda A) = \lambda(\mu A)$;
5) $\lambda(A + B) = \lambda A + \lambda B$
6) $(\lambda + \mu)A = \lambda A + \mu A$

Bu yerda λ , μ- sonlar, A, B, C −matritsalar, Q −nol matritsa.

3-misol.

$A = \begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix}$ va $B = \begin{pmatrix} 0 & -3 \\ 4 & 6 \end{pmatrix}$ matritsalar berilgan.

$2A - B$ matritsani toping.

Yechish. $2A$ matritsani tuzamiz;

$$2A = 2\begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ -6 & 10 \end{pmatrix}.$$

Bu $2A$ matritsadan B matritsani ayiramiz:

$$2A - B = \begin{pmatrix} 8 & 4 \\ -6 & 10 \end{pmatrix} - \begin{pmatrix} 0 & -3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 8-0 & 4-(-3) \\ -6-4 & 10-6 \end{pmatrix} =$$

$$= \begin{pmatrix} 8 & 7 \\ -10 & 4 \end{pmatrix}.$$

Navbatdagi amal matritsalarni ko’paytirish amaliga o’tamiz.

$i \times j$ o'lchamli $A = (a_{ij})$ matritsaning $j \times k$ o'lchamli $B = (b_{jk})$ matritsaga *ko'paytmasi* deb, $i \times k$ o'lchamli shunday $C = (c_{ik})$ matritsaga aytiladiki,

uning c_{ik} elementi A matritsa i −satri elementlarini B matritsa j −ustunining mos elementlariga ko'paytmalari yig'indisiga teng, ya'ni

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{ij}b_{jk}.$$

Matritsalar ko'paytmasi bunday belgilanadi $C = A \cdot B$.

5-misol. Ushbu matritsalarni ko'paytiring:

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ va } B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Yechish. AB ko'paytma mavjud, chunki A matritsaning ustunlari soni 2 ga teng, B matritsaning satrlari soni ham 2 ga teng. Bu ko'paytmani tuzamiz:

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot 2 - 1 \cdot 1 & 1 \cdot 1 - 1 \cdot 1 \\ 2 \cdot 2 + 1 \cdot 1 & 2 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 2 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 3 \\ 3 & 2 \end{pmatrix}$$

BA matritsa mavjud emas, chunki B matritsaning ustunlari soni 2 ga teng. A matritsaning satrlari soni esa 3 ga teng. Agar A va B matritsalar bir xil tartibli bo'lsa, u holda AB va BA ko'paytmalar tartibi bir xil bo'ladi.

6-misol. Ushbu matritsalarni ko'paytiring:

$$A = \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & -3 \\ 2 & 6 \end{pmatrix}.$$

Yechish. Matritsalarni ko'paytirish uchun asosiy talab bajariladi. Shuning uchun

$$AB = \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 2\cdot 4 - 3\cdot 2 & 2\cdot(-3) - 3\cdot 6 \\ 5\cdot 4 + 1\cdot 2 & 5\cdot(-3) + 1\cdot 6 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -24 \\ 22 & -9 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & -3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 \cdot 2 - 3 \cdot 5 & 4 \cdot (-3) - 3 \cdot 1 \\ 2 \cdot 2 + 6 \cdot 5 & 2 \cdot (-3) + 6 \cdot 1 \end{pmatrix} = \begin{pmatrix} -7 & -15 \\ 34 & 0 \end{pmatrix}$$

Bundan $AB \neq BA$ ekanligi ko’rinib turibdi.