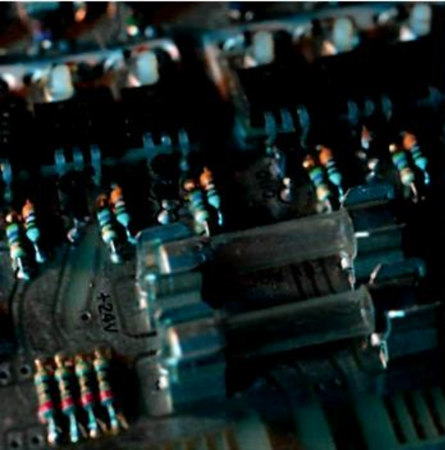
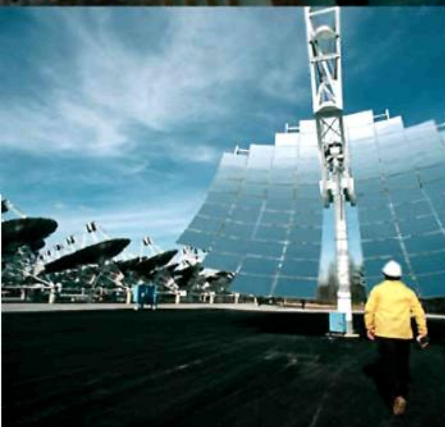


Fifth edition



HIGHER ENGINEERING MATHEMATICS

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Newnes

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Integration by parts

43.1 Introduction

From the product rule of differentiation:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx},$$

where u and v are both functions of x.

Rearranging gives: $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

Integrating both sides with respect to x gives:

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

i.e. $$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

or $$\boxed{\int \mathbf{u\,dv = uv} - \int \mathbf{v\,du}}$$

This is known as the **integration by parts formula** and provides a method of integrating such products of simple functions as $\int xe^x\,dx$, $\int t\sin t\,dt$, $\int e^\theta \cos\theta\,d\theta$ and $\int x\ln x\,dx$.

Given a product of two terms to integrate the initial choice is: 'which part to make equal to u' and 'which part to make equal to v'. The choice must be such that the 'u part' becomes a constant after successive differentiation and the 'dv part' can be integrated from standard integrals. Invariable, the following rule holds: If a product to be integrated contains an algebraic term (such as x, t^2 or 3θ) then this term is chosen as the u part. The one exception to this rule is when a '$\ln x$' term is involved; in this case $\ln x$ is chosen as the 'u part'.

43.2 Worked problems on integration by parts

Problem 1. Determine $\int x\cos x\,dx$.

From the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

Let $u = x$, from which $\frac{du}{dx} = 1$, i.e. $du = dx$ and let $dv = \cos x\,dx$, from which $v = \int \cos x\,dx = \sin x$.

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\int u\;\;dv \;=\; u\;\;v \;-\; \int v\;\;du$$

$$\int x\;\;\cos x\,dx \;=\; (x)\;(\sin x) \;-\; \int (\sin x)\;(dx)$$

i.e. $$\int x\cos x\,dx = x\sin x - (-\cos x) + c$$

$$= \mathbf{x\sin x + \cos x + c}$$

[This result may be checked by differentiating the right hand side,

i.e. $\frac{d}{dx}(x\sin x + \cos x + c)$

$= [(x)(\cos x) + (\sin x)(1)] - \sin x + 0$ using the product rule

$= x\cos x$, which is the function being integrated]

Problem 2. Find $\int 3te^{2t}\,dt$.

Let $u = 3t$, from which, $\frac{du}{dt} = 3$, i.e. $du = 3\,dt$ and let $dv = e^{2t}\,dt$, from which, $v = \int e^{2t}\,dt = \frac{1}{2}e^{2t}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int 3te^{2t}\,dt = (3t)\left(\frac{1}{2}e^{2t}\right) - \int \left(\frac{1}{2}e^{2t}\right)(3\,dt)$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\int e^{2t}\,dt$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\left(\frac{e^{2t}}{2}\right) + c$$

Hence

$$\int \mathbf{3t\,e^{2t}\,dt = \tfrac{3}{2}e^{2t}\left(t - \tfrac{1}{2}\right) + c},$$

which may be checked by differentiating.

Problem 3. Evaluate $\int_0^{\frac{\pi}{2}} 2\theta \sin\theta\,d\theta$.

Let $u = 2\theta$, from which, $\frac{du}{d\theta} = 2$, i.e. $du = 2\,d\theta$ and let $dv = \sin\theta\,d\theta$, from which,

$$v = \int \sin\theta\,d\theta = -\cos\theta$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int 2\theta\sin\theta\,d\theta = (2\theta)(-\cos\theta) - \int(-\cos\theta)(2\,d\theta)$$

$$= -2\theta\cos\theta + 2\int\cos\theta\,d\theta$$

$$= -2\theta\cos\theta + 2\sin\theta + c$$

Hence $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$

$$= [-2\theta\cos\theta + 2\sin\theta]_0^{\frac{\pi}{2}}$$

$$= \left[-2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}\right] - [0 + 2\sin 0]$$

$$= (-0 + 2) - (0 + 0) = \mathbf{2}$$

since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$

Problem 4. Evaluate $\int_0^1 5xe^{4x}\,dx$, correct to 3 significant figures.

Let $u = 5x$, from which $\frac{du}{dx} = 5$, i.e. $du = 5\,dx$ and let $dv = e^{4x}\,dx$, from which, $v = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$.

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int 5xe^{4x}\,dx = (5x)\left(\frac{e^{4x}}{4}\right) - \int\left(\frac{e^{4x}}{4}\right)(5\,dx)$$

$$= \frac{5}{4}xe^{4x} - \frac{5}{4}\int e^{4x}\,dx$$

$$= \frac{5}{4}xe^{4x} - \frac{5}{4}\left(\frac{e^{4x}}{4}\right) + c$$

$$= \frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right) + c$$

Hence $\int_0^1 5xe^{4x}\,dx$

$$= \left[\frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right)\right]_0^1$$

$$= \left[\frac{5}{4}e^4\left(1 - \frac{1}{4}\right)\right] - \left[\frac{5}{4}e^0\left(0 - \frac{1}{4}\right)\right]$$

$$= \left(\frac{15}{16}e^4\right) - \left(-\frac{5}{16}\right)$$

$$= 51.186 + 0.313 = 51.499 = \mathbf{51.5},$$

correct to 3 significant figures

Problem 5. Determine $\int x^2\sin x\,dx$.

Let $u = x^2$, from which, $\frac{du}{dx} = 2x$, i.e. $du = 2x\,dx$, and let $dv = \sin x\,dx$, from which,

$$v = \int \sin x\,dx = -\cos x$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x^2\sin x\,dx = (x^2)(-\cos x) - \int(-\cos x)(2x\,dx)$$

$$= -x^2\cos x + 2\left[\int x\cos x\,dx\right]$$

The integral, $\int x\cos x\,dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.

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From Problem 1, $\int x\cos x\,dx = x\sin x + \cos x$

Hence $\int x^2 \sin x\,dx$

$$= -x^2\cos x + 2\{x\sin x + \cos x\} + c$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$
$$= \mathbf{(2 - x^2)\cos x + 2x\sin x + c}$$

In general, if the algebraic term of a product is of power n, then the integration by parts formula is applied n times.

Now try the following exercise.

Exercise 168 Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int x e^{2x}\,dx$ $\quad\left[\frac{e^{2x}}{2}\left(x - \frac{1}{2}\right) + c\right]$

2. $\int \frac{4x}{e^{3x}}\,dx$ $\quad\left[-\frac{4}{3}e^{-3x}\left(x + \frac{1}{3}\right) + c\right]$

3. $\int x\sin x\,dx$ $\quad[-x\cos x + \sin x + c]$

4. $\int 5\theta\cos 2\theta\,d\theta$ $\quad\left[\frac{5}{2}\left(\theta\sin 2\theta + \frac{1}{2}\cos 2\theta\right) + c\right]$

5. $\int 3t^2 e^{2t}\,dt$ $\quad\left[\frac{3}{2}e^{2t}\left(t^2 - t + \frac{1}{2}\right) + c\right]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_0^2 2xe^x\,dx$ $\quad[16.78]$

7. $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$ $\quad[0.2500]$

8. $\int_0^{\frac{\pi}{2}} t^2\cos t\,dt$ $\quad[0.4674]$

9. $\int_1^2 3x^2 e^{\frac{x}{2}}\,dx$ $\quad[15.78]$

43.3 Further worked problems on integration by parts

Problem 6. Find $\int x\ln x\,dx$.

The logarithmic function is chosen as the 'u part'.

Thus when $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$, i.e. $du = \frac{dx}{x}$

Letting $dv = x\,dx$ gives $v = \int x\,dx = \frac{x^2}{2}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x\ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\frac{dx}{x}$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

Hence $\int x\ln x\,dx = \mathbf{\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c}$ **or**

$$\mathbf{\frac{x^2}{4}(2\ln x - 1) + c}$$

Problem 7. Determine $\int \ln x\,dx$.

$\int \ln x\,dx$ is the same as $\int (1)\ln x\,dx$

Let $u = \ln x$, from which, $\frac{du}{dx} = \frac{1}{x}$, i.e. $du = \frac{dx}{x}$

and let $dv = 1dx$, from which, $v = \int 1\,dx = x$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int \ln x\,dx = (\ln x)(x) - \int x\frac{dx}{x}$$
$$= x\ln x - \int dx = x\ln x - x + c$$

Hence $\int \mathbf{\ln x\,dx = x(\ln x - 1) + c}$

Problem 8. Evaluate $\int_1^9 \sqrt{x}\ln x\,dx$, correct to 3 significant figures.

Let $u = \ln x$, from which $\mathrm{d}u = \dfrac{\mathrm{d}x}{x}$

and let $\mathrm{d}v = \sqrt{x}\,\mathrm{d}x = x^{\frac{1}{2}}\,\mathrm{d}x$, from which,

$$v = \int x^{\frac{1}{2}}\,\mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}}$$

Substituting into $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$ gives:

$$\int \sqrt{x}\ln x\,\mathrm{d}x = (\ln x)\left(\frac{2}{3}x^{\frac{3}{2}}\right) - \int\left(\frac{2}{3}x^{\frac{3}{2}}\right)\left(\frac{\mathrm{d}x}{x}\right)$$
$$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,\mathrm{d}x$$

$$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + c$$

$$= \frac{2}{3}\sqrt{x^3}\left[\ln x - \frac{2}{3}\right] + c$$

Hence $\int_1^9 \sqrt{x}\ln x\,\mathrm{d}x$

$$= \left[\frac{2}{3}\sqrt{x^3}\left(\ln x - \frac{2}{3}\right)\right]_1^9$$

$$= \left[\frac{2}{3}\sqrt{9^3}\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\sqrt{1^3}\left(\ln 1 - \frac{2}{3}\right)\right]$$

$$= \left[18\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\left(0 - \frac{2}{3}\right)\right]$$

$= 27.550 + 0.444 = 27.994 =$ **28.0**,

correct to 3 significant figures

Problem 9. Find $\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x$.

When integrating a product of an exponential and a sine or cosine function it is immaterial which part is made equal to 'u'.

Let $u = \mathrm{e}^{ax}$, from which $\dfrac{\mathrm{d}u}{\mathrm{d}x} = a\mathrm{e}^{ax}$,

i.e. $\mathrm{d}u = a\mathrm{e}^{ax}\,\mathrm{d}x$ and let $\mathrm{d}v = \cos bx\,\mathrm{d}x$, from which,

$$v = \int \cos bx\,\mathrm{d}x = \frac{1}{b}\sin bx$$

Substituting into $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$ gives:

$$\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x$$
$$= (\mathrm{e}^{ax})\left(\frac{1}{b}\sin bx\right) - \int\left(\frac{1}{b}\sin bx\right)(a\mathrm{e}^{ax}\,\mathrm{d}x)$$
$$= \frac{1}{b}\mathrm{e}^{ax}\sin bx - \frac{a}{b}\left[\int \mathrm{e}^{ax}\sin bx\,\mathrm{d}x\right] \quad (1)$$

$\int \mathrm{e}^{ax}\sin bx\,\mathrm{d}x$ is now determined separately using integration by parts again:

Let $u = \mathrm{e}^{ax}$ then $\mathrm{d}u = a\mathrm{e}^{ax}\,\mathrm{d}x$, and let $\mathrm{d}v = \sin bx\,\mathrm{d}x$, from which

$$v = \int \sin bx\,\mathrm{d}x = -\frac{1}{b}\cos bx$$

Substituting into the integration by parts formula gives:

$$\int \mathrm{e}^{ax}\sin bx\,\mathrm{d}x = (\mathrm{e}^{ax})\left(-\frac{1}{b}\cos bx\right) - \int\left(-\frac{1}{b}\cos bx\right)(a\mathrm{e}^{ax}\,\mathrm{d}x)$$
$$= -\frac{1}{b}\mathrm{e}^{ax}\cos bx + \frac{a}{b}\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x$$

Substituting this result into equation (1) gives:

$$\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x = \frac{1}{b}\mathrm{e}^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}\mathrm{e}^{ax}\cos bx + \frac{a}{b}\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x\right]$$
$$= \frac{1}{b}\mathrm{e}^{ax}\sin bx + \frac{a}{b^2}\mathrm{e}^{ax}\cos bx - \frac{a^2}{b^2}\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x$$

The integral on the far right of this equation is the same as the integral on the left hand side and thus they may be combined.

$$\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x + \frac{a^2}{b^2}\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x = \frac{1}{b}\mathrm{e}^{ax}\sin bx + \frac{a}{b^2}\mathrm{e}^{ax}\cos bx$$

i.e. $\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

i.e. $\left(\frac{b^2+a^2}{b^2}\right)\int e^{ax}\cos bx\,dx$

$$= \frac{e^{ax}}{b^2}(b\sin bx + a\cos bx)$$

Hence $\int e^{ax}\cos bx\,dx$

$$= \left(\frac{b^2}{b^2+a^2}\right)\left(\frac{e^{ax}}{b^2}\right)(b\sin bx + a\cos bx)$$

$$= \mathbf{\frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c}$$

Using a similar method to above, that is, integrating by parts twice, the following result may be proved:

$$\int \mathbf{e^{ax}\sin bx\,dx}$$

$$= \mathbf{\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c} \qquad (2)$$

Problem 10. Evaluate $\int_0^{\frac{\pi}{4}} e^t \sin 2t\,dt$, correct to 4 decimal places.

Comparing $\int e^t \sin 2t\,dt$ with $\int e^{ax}\sin bx\,dx$ shows that $x=t$, $a=1$ and $b=2$.

Hence, substituting into equation (2) gives:

$$\int_0^{\frac{\pi}{4}} e^t \sin 2t\,dt$$

$$= \left[\frac{e^t}{1^2+2^2}(1\sin 2t - 2\cos 2t)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}\left(\sin 2\left(\frac{\pi}{4}\right) - 2\cos 2\left(\frac{\pi}{4}\right)\right)\right]$$

$$- \left[\frac{e^0}{5}(\sin 0 - 2\cos 0)\right]$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}(1-0)\right] - \left[\frac{1}{5}(0-2)\right] = \frac{e^{\frac{\pi}{4}}}{5} + \frac{2}{5}$$

$= \mathbf{0.8387}$, correct to 4 decimal places

Now try the following exercise.

Exercise 169 Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int 2x^2 \ln x\,dx$ $\quad\left[\frac{2}{3}x^3\left(\ln x - \frac{1}{3}\right) + c\right]$

2. $\int 2\ln 3x\,dx$ $\quad[2x(\ln 3x - 1) + c]$

3. $\int x^2 \sin 3x\,dx$
$\left[\frac{\cos 3x}{27}(2-9x^2) + \frac{2}{9}x\sin 3x + c\right]$

4. $\int 2e^{5x}\cos 2x\,dx$
$\left[\frac{2}{29}e^{5x}(2\sin 2x + 5\cos 2x) + c\right]$

5. $\int 2\theta \sec^2\theta\,d\theta$ $\quad[2[\theta\tan\theta - \ln(\sec\theta)] + c]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_1^2 x\ln x\,dx$ $\quad[0.6363]$

7. $\int_0^1 2e^{3x}\sin 2x\,dx$ $\quad[11.31]$

8. $\int_0^{\frac{\pi}{2}} e^t \cos 3t\,dt$ $\quad[-1.543]$

9. $\int_1^4 \sqrt{x^3}\ln x\,dx$ $\quad[12.78]$

10. In determining a Fourier series to represent $f(x) = x$ in the range $-\pi$ to π, Fourier coefficients are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

where n is a positive integer. Show by using integration by parts that $a_n = 0$ and $b_n = -\frac{2}{n}\cos n\pi$.

11. The equation $C = \int_0^1 e^{-0.4\theta} \cos 1.2\theta \, d\theta$

and $$S = \int_0^1 e^{-0.4\theta} \sin 1.2\theta \, d\theta$$

are involved in the study of damped oscillations. Determine the values of C and S.

[$C = 0.66$, $S = 0.41$]

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