Mavzu:Parametrik ko'rinishda berilgan va oshkormas funksiyalar hosilasi. Yuqori tartibli hosilalar

Reja:

1. **Parametrik ko'rinishda berilgan funksiyaning hosilasi.**
2. **Oshkormas funksiyaning hosilasi**
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Parametrik ko'rinishda berilgan funksiyaning hosilasi.

Faraz qilaylik, x ning funksiyasi y ushbu

$$\begin{cases} x = f(t) \\ y = \varphi(t) \end{cases}, \quad (t_0 \leq t \leq T) \qquad (1)$$

parametrik tenglamalar bilan berilgan bo'lsin. Bu funksiyalar hosilalarga ega va $x = f(t)$ funksiya hosilaga ega bo'lgan $t = \Phi(x)$ teskari funksiyaga ega deb faraz qilamiz. U holda parametrik tenglamalar bilan aniqlangan $y = F(x)$ funksiyani murakkab funksiya deb qarash mumkun:

$$y = \varphi(t) \quad , \quad t = \Phi(x)$$

t – oraliqdagi argument.

Murakkab funksiyani differensiallash qoidasiga muvofiq

$$y'_x = y'_t t'_x = \varphi'_x(t)\Phi'_x(x) \qquad (2)$$

Teskari funksiyani differensiallash haqidagi teoremaga asosan

$$\Phi'_x(x) = \frac{1}{\varphi'_x(t)}.$$

Oxirgi ifodani (2) tenglikka qo'yib, shuni hosil qilamiz:

$$y'_x = \frac{\varphi'(t)}{f'(t)}$$

yoki

$$y'_x = \frac{y'_t}{x'_t} \qquad (3)$$

Chiqarilgan formula y ning x ga bevosita bog'lanishining ifodasini topmay turib, parametrik berilgan funksiyaning y'_x hosilasini topishga imkon beradi.

1 – misol. x ning funksiyasi y ushbu parametrik tenglamalar bilan berilgan

$$\begin{cases} x = acost \\ y = asint' \end{cases} \qquad (0 \le t \le \pi).$$

1) t ning istalgan qiymatida, 2) $t = \frac{\pi}{4}$ qiymatida $\frac{dy}{dx}$ hosila topilsin.

Yechish.

$$1)\ y'_x = \frac{(a \sin t)'}{(a \cos t)'} = \frac{a \cos t}{-a \sin t} = -\operatorname{ctg} t;$$

$$2)\ (y'_x)_{t=\frac{\pi}{4}} = -ctg\,\frac{\pi}{4} = -1.$$

2 – misol. Ihtiyoriy $(0 \le t \le 2\pi)$ nuqtada ushbu

$$x = a(t - \sin t),$$
$$y = a(1 - \cos t)$$

tsikloidaga urinmaning burchak koeffitsienti topilsin.

Yechish. Urinmaning har bir nuqtadagi burchak koeffitsienti y'_x hosilaning bu nuqtadagi qiymatiga teng, ya'ni

$$y'_x = \frac{y'_t}{x'_t}$$

Ammo

$$x'_t = a(1 - cost), \qquad y'_t = asint.$$

Demak,

$$y'_x = \frac{asint}{a(1-cost)} = \frac{2sin\frac{t}{2}cos\frac{t}{2}}{2sin^2\frac{t}{2}} = ctg\,\frac{t}{2} = tg\left(\frac{\pi}{2} - \frac{t}{2}\right).$$

Demak, tsikloidaning har bir nuqtasidagi urinmaning burchak koeffitsienti $tg\left(\frac{\pi}{2} - \frac{t}{2}\right)$ ga teng, bu yerda t – parametrning shu nuqtaga mos qiymati. Ammo buning ma'nosi urinmaning OX o'qiga a og'ish burchagi $\frac{\pi}{2} - \frac{t}{2}$ ga teng demakdir$(-\pi \le t \le \pi)$.

Oshkormas funksiyaning hosilasi

Ikkita x va y o'zgaruvchilarning qiymatlari o'zaro biror tenglama bilan bog'langan bo'lsin, biz uni simvolik tarzda bunday belgilaymiz:

$$F(x, y) = 0. \qquad (1)$$

Agar $y = f(x)$ funksiya biror (a,b) intervalda aniqlangan bo'lib, (1) tenglamada y o'rniga $f(x)$ ifoda qo'yilganda tenglama x ga nisbatan ainiyatga aylansa, u holda $y = f(x)$ funksiya (1) tenglama bilan aniqlangan oshkormas funksiya bo'ladi.

Masalan,

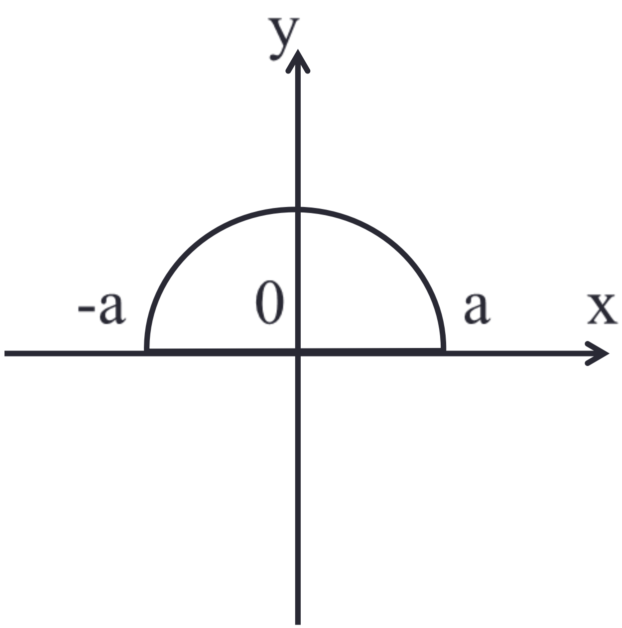
$$x^2 + y^2 - a^2 = 0 \qquad (2)$$

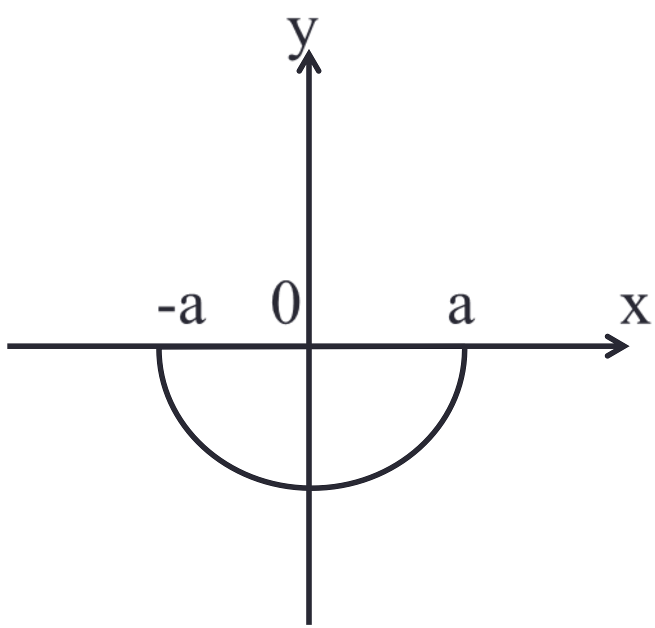
tenglama mana bu

$$y = \sqrt{a^2 - x^2}, \qquad (3)$$

$$y = -\sqrt{a^2 - x^2} \qquad (4)$$

Elementar funksiyalalrni nooshkor tarzda aniqlaydi.





Haqiqatan, bu qiymatlarni (2) tenglamaga qo'ygandan so'ng ayniyat hosil bo'ladi:

$$x^2 + (a^2 - x^2) - a^2 = 0.$$

(3) va (4) ifodalar ikki tenglamani *y* ga nisbatan yechish yo'li bilan hosil qilingan. Ammo, har qanday oshkormas berilgan funksiyani ham oshkor shaklda bermoq, ya'ni $y = f(x)$ shaklga qo'yish mumkin bo'lavermaydi, bu yerda *f(x)* elementar funksiya.

Masalan,

$$y^6 - y - x^2 = 0$$

yoki

$$y - x - \frac{1}{4} siny = 0$$

tenglamalar bilan berilgan funksiyalar elementar funksiyalar bilan ifodalanmaydi, ya'ni bu tenglamalarni elementar funksiyalar orqali *y* ga nisbatan yechish mumkin emas.

1 – izoh. "Oshkor funksiya" va "oshkormas funksiya" terminlari funksiyaning tabiatini emas, balki berilish usulini xarakterlaydi. Har bir oshkor funksiya $y = f(x)$ ni oshkormas funksiya $y - f(x) = 0$ shaklida berish mumkin.

Endi oshkormas funksiyani oshkor ko'rinishka keltirmastan, ya'ni $y = f(x)$ shaklga almashtirmastan, uning hosilasini topish qoidasini ko'rsatamiz.

Faraz qilaylik, funksiya ushbu

$$x^2 + y^2 - a^2 = 0$$

tenglama bilan berilgan bo'lsin. Agar *y* bu yerda *x* ning shu tenglik bilan aniqlanadigan funksiyasi bo'lsa, u holda bu tenglik ayniyatdir.

y ni *x* ning funksiyasi deb hisoblab, bu ayniyatning ikkala tomonini *x* bo'yicha differensiallab (murakkab funksiyani differensiallash qoidasidan foydalangan holda), shuni topamiz:

$$2x + 2yy' = 0,$$

bundan

$$y' = -\frac{x}{y}.$$

Agar biz ushbu

$$y = \sqrt{a^2 - x^2}$$

oshkor funksiyani differensiallagan bo'lsak,

$$y' = -\frac{x}{\sqrt{a^2-x^2}} = -\frac{x}{y},$$

ya'ni xuddi osha natijani hosil qilgan bo'lar edik.

y x ning oshkormas funksiyasi bo'lgan holga yana bir misol ko'ramiz:

$$y^6 - y - x^2 = 0.$$

Buni *x* bo'yicha differensiallaymiz:

$$6y^5\ y' - y' - 2x = 0,$$

bundan

$$y' = \frac{2x}{6y^5-1}.$$

2 – izoh. Argument *x* ning berilgan qiymatida oshkormas funksiya hosilasining qiymatini topish uchun *x* ning berilgan qiymatida *y* funksiyaning qiymatini ham bilish zarurligi keltirilgan misollardan kelib chiqadi.

Yuqori tartibli hosilalar

$y = f(x)$ funksiya biror $[a, b]$ kesmada differensiallanuvchi bo'lsin. $f'(x)$ hosilaning qiymatlari, umuman aytkanda, x ga bog'liq, ya'ni $f'(x)$ hosila ham x ning funksiyasidan iborat. Bu funksiyani differensiallab, $f(x)$ funksiyaning ikkinchi tartibli hosilasi deb ataladigan hosilani topamiz.

Birinchi hosiladan olingan hosila *ikkinchi tartibli hosila* yoki boshlang'ich funksiyaning *ikkinchi hosilasi* deyiladi va y'' yoki $f''(x)$ simvol bilan belgilanadi:

$$y'' = (y')' = f''(x).$$

Masalan, $y = x^5$ bo'lsa, u holda

$$y' = 5x^4,$$
$$y'' = (5x^4)' = 20x^3.$$

Ikkinchi hosilaning hosilasi *uchinchi tartibli hosila* yoki *uchinchi hosila* deyiladi va y''' yoki $f'''(x)$ bilan belgilanadi.

Umuman *f(x)* funksiyaning *n-tartibli hosilasi* deb, uning (n-1)-tartibli hosilasining (birinchi tartibli) hosilasiga aytiladi va $y^{(n)}$ yoki $f^{(n)}(x)$ simvol bilan belgilanadi:

$$y^{(n)} = \left(y^{(n-1)}\right)' = f^{(n)}(x).$$

(Hosilaning tartibi daraja ko'rsatkichi deb tushunmasligi uchun qavs ichiga olinadi.)

To'rtinchi, beshinchi va undan yuqori tartibli hosilalar rim raqamlari bilan belgilanadi: $y^{\mathrm{IV}}, y^{\mathrm{V}}, y^{\mathrm{VI}}, \ldots$. Bunday holda hosilaning tartibini qavssiz yozish mumkin. Masalan, agar $y = x^5$ bo'lsa, u holda

$$y' = 5x^4, \qquad y'' = 20x^3, \qquad y''' = 60x^2, \qquad y^{\mathrm{IV}} = y^{(4)} = 120x,$$
$$y^{\mathrm{V}} = y^{(5)} = 120, \; y^{(6)} = y^{(7)} = \cdots = 0.$$

Hosilalar tuzish qonuni istalgan tartibli hosilalar uchun saqlanadi va quyidagidan iborat:

$(u+v)^n$ ifodani N'yuton binomi formulasiga muvofiq yoyish va hosil bo'lgan yoyilmada *u va v* ning daraja ko'rsatkichlari bilan almashtirish kerak, shu bilan birga yoyilmaning chetki hadlariga kiruvchi nol darajalarni $(u^0 = v^0 = 1)$ funksiyalarning o'zlari bilan almashtirish kerak:

$$y^{(n)} = (uv)^{(n)} = (u)^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{1\,.2}u^{(n-2)}v'' + \cdots + uv^{(n)}.$$

Bu esa *Leybnits formulasidir*.

Oshkormas funksiyalarning ba parametrik berilgan funksiyalarning turli tartibli hosilalari.

1.Oshkormas funksiyalardan turli tartibli hosilalarni topish usulini misolda ko'rsatamiz.

Faraz qilaylik, x ning oshkormas funksiyasi y ushbu

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \qquad (1)$$

Tenglik bilan aniqlansin. Bu tenglikning barcha hadlarini, y bunda x ning funksiyasi ekanligini esda tutgan holda, x bo'yicha differensiallaymiz:

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0;$$

bundan

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y.} \qquad (2)$$

Oxirgi tenglikni yana x bo'yicha (x ning funksiyasi y ekanligini nazarda tutib) differensiallaymiz:

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{y - x\frac{dy}{dx}}{y^2}.$$

Bu yerda $\frac{dy}{dx}$ hosila o'rniga uning ifodasini (2) tenglikdan olib qo'ysak,

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{y + x\frac{b^2}{a^2}\frac{x}{y}}{y^2},$$

Yoki soddalashtirgandan so'ng

$$\frac{d^2y}{dx^2} = -\frac{b^2(a^2y^2 + b^2x^2)}{a^4y^3}.$$

tenglamadan

$$a^2y^2 + b^2x^2 = a^2b^2$$

chiqadi, shuning uchun ikkinchi hosilani

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$$

ko'rinishda yozish mumkin. Oxirgi tenglikni x bo'yicha differensiallab, $\frac{d^3y}{dx^3}$ ni topamiz va hokazo.

2. Endi parametrik berilgan funksiyadan yuqori tartibli hosilalar topish masalasini qarab chiqamiz.

Faraz qilaylik, x ning funksiyasi y ushbu parametrik tenglamalar bilan berilgan bo'lsin:

$$\begin{cases} x = f(t) \\ y = \varphi(t) \end{cases}, \quad (t_0 \leq t \leq T), \tag{3}$$

bunda $x = f(t)$ funksiya $[t_0, T]$ kesmada $t=\Phi(x)$ teskari funksiyaga ega bo'lsin.

Bu holda $\frac{dy}{dx}$ hosila

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{4}$$

tenglik bilan aniqlanishi yuqorida isbotlangan.

Ikkinchi hosila $\frac{d^2y}{dx^2}$ ni topish uchun x ning funksiyasi t ekanligini nazarda tutib, (4) tenglikni x bo'yicha differensiallaymiz:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}, \qquad (5)$$

ammo

$$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right) - \frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2},$$

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}.$$

Oxirgi ifodani (5) formulaga qo'yib, shuni hosil qilamiz:

$$\frac{d^2y}{dx^2} = \frac{f'(t)\varphi''(t) - \varphi'(t)f''(t)}{[f'(t)]^3}.$$

Shunga o'xshash boshqa yuqori tartibli hosilalarni ham topish mumkin.

Funksiya hosilasini topishga doir aralash misollar

1. $x = acost, \quad y = bsint.$

2. $x = a(t - sint),$
$y = a(1 - cost).$

3. $x = acos^3 t, \quad y = bsin^3 t.$

4. $x = \frac{3at}{1+t^2}, \quad y = \frac{3at^2}{1+t^2}.$

5. $x = 2\ln ctgt, \quad y = tgt + ctgt.$

6. $y^2 = 4px.$

7. $x^2 + y^2 = a^2.$

8. $b^2x^2 + a^2y^2 = a^2b^2.$

9. $y^3 - 3y + 2ax = 0.$

10. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$

11. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

12. $y^2 - 2xy + b^2 = 0.$

13. $x^3 + y^3 - 3axy = 0.$

14. $y = cos(x + y).$

15. $\cos(xy) = x.$

16. $y = \frac{x^3 - x^2 + 1}{5}.$

17. $y = 2ax^3 - \frac{x^2}{b} + c\,.$

18. $y = 6x^{\frac{7}{2}} + 4x^{\frac{5}{2}} + 2x\,.$

19. $y = \sqrt[3]{x^2} - \sqrt{x} + 5\,.$

20. $y = (1 + 4x^3)(1 + 2x^2).$