Universitext

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Mathematical Analysis I

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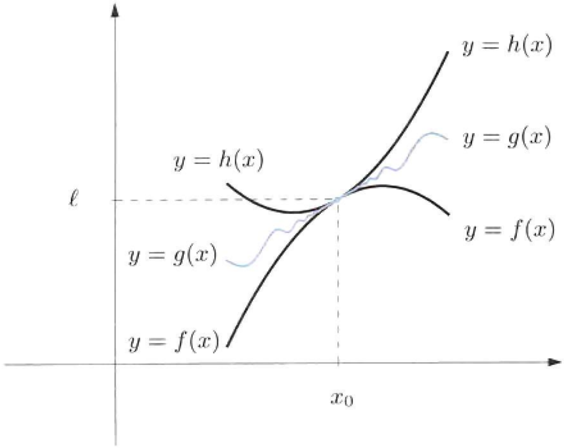

Figure 4.3. The squeeze rule

$$\forall x \in \operatorname{dom} h, \qquad x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad \ell - \varepsilon < h(x) < \ell + \varepsilon. \qquad (4.4)$$

Define then $I'''(c) = I(c) \cap I'(c) \cap I''(c)$. On $I'''(c) \setminus \{c\}$ the constraints (4.2), (4.3) and (4.4) all hold, hence in particular

$$x \in I'''(c) \setminus \{c\} \quad \Rightarrow \quad \ell - \varepsilon < f(x) \leq g(x) \leq h(x) < \ell + \varepsilon.$$

This means $g(x) \in I_\varepsilon(\ell)$, concluding the proof. □

Examples 4.6

i) Let us prove the fundamental limit

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.} \qquad (4.5)$$

Observe first that $y = \dfrac{\sin x}{x}$ is even, for $\dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x}$. It is thus sufficient to consider a positive x tending to 0, i.e., prove that $\lim\limits_{x \to 0^+} \dfrac{\sin x}{x} = 1$. Recalling (3.13), for all $x > 0$ we have $\sin x < x$, or $\dfrac{\sin x}{x} < 1$. To find a lower bound, suppose $x < \frac{\pi}{2}$ and consider points on the unit circle: let A have coordinates $(1, 0)$, P coordinates $(\cos x, \sin x)$ and let Q be defined by $(1, \tan x)$ (Fig. 4.4). The circular sector OAP is a proper subset of the triangle OAQ, so

$$\text{Area } OAP < \text{ Area } OAQ.$$

Since

$$\text{Area } OAP = \frac{\overline{OA} \cdot \widehat{AP}}{2} = \frac{x}{2} \qquad \text{and} \qquad \text{Area } OAQ = \frac{\overline{OA} \cdot \overline{AQ}}{2} = \frac{\tan x}{2},$$

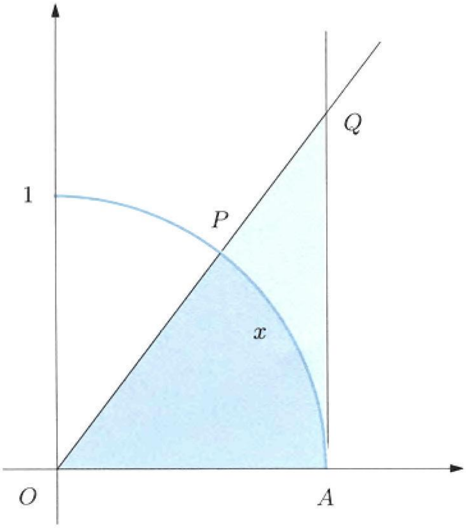

Figure 4.4. The sector OAP is properly contained in OAQ

it follows

$$\frac{x}{2} < \frac{\sin x}{2\cos x}, \qquad \text{i.e.,} \qquad \cos x < \frac{\sin x}{x}.$$

Eventually, on $0 < x < \frac{\pi}{2}$ one has

$$\cos x < \frac{\sin x}{x} < 1.$$

The continuity of the cosine ensures $\lim_{x\to 0^+} \cos x = 1$. Now the claim follows from the Second comparison theorem.

ii) We would like to study how the function $g(x) = \dfrac{\sin x}{x}$ behaves for x tending to $+\infty$. Remember that

$$-1 \le \sin x \le 1 \tag{4.6}$$

for any real x. Dividing by $x > 0$ will not alter the inequalities, so in every neighbourhood $I_A(+\infty)$ of $+\infty$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Now set $f(x) = -\dfrac{1}{x}$, $h(x) = \dfrac{1}{x}$ and note $\lim_{x\to+\infty} \dfrac{1}{x} = 0$. By the previous theorem

$$\lim_{x\to+\infty} \frac{\sin x}{x} = 0.$$

□

The latter example is part of a more general result which we state next (and both are consequences of Theorem 4.5).

Corollary 4.7 *Let f be a* **bounded** *function around c, i.e., there exist a neighbourhood $I(c)$ and a constant $C > 0$ such that*

$$|f(x)| \leq C, \qquad \forall x \in I(c) \setminus \{c\}. \tag{4.7}$$

Let g be such that

$$\lim_{x \to c} g(x) = 0.$$

Then it follows

$$\lim_{x \to c} f(x)g(x) = 0.$$

Proof. By definition $\lim_{x \to c} g(x) = 0$ if and only if $\lim_{x \to c} |g(x)| = 0$, and (4.7) implies

$$0 \leq |f(x)g(x)| \leq C|g(x)|, \qquad \forall x \in I(c) \setminus \{c\}.$$

The claim follows by applying Theorem 4.5. □

Theorem 4.8 (Second comparison theorem – infinite case) *Let f, g be given functions and*

$$\lim_{x \to c} f(x) = +\infty.$$

If there exists a neighbourhood $I(c)$ of c, where both functions are defined (except possibly at c), such that

$$f(x) \leq g(x), \qquad \forall x \in I(c) \setminus \{c\}, \tag{4.8}$$

then

$$\lim_{x \to c} g(x) = +\infty.$$

A result of the same kind for f holds when the limit of g is $-\infty$.

Proof. The proof is, with the necessary changes, like that of Theorem 4.5, hence left to the reader. □

Example 4.9

Compute the limit of $g(x) = x + \sin x$ when $x \to +\infty$. Using (4.6) we have

$$x - 1 \leq x + \sin x, \qquad \forall x \in \mathbb{R}.$$

Set $f(x) = x - 1$; since $\lim_{x \to +\infty} f(x) = +\infty$, the theorem tells us

$$\lim_{x \to +\infty} (x + \sin x) = +\infty.$$

□

4.1.3 Algebra of limits. Indeterminate forms of algebraic type

This section is devoted to the interaction of limits with the algebraic operations of sum, difference, product and quotient of functions.

First though, we must extend arithmetic operations to treat the symbols $+\infty$ and $-\infty$. Let us set:

$$
\begin{aligned}
+\infty + s &= +\infty && (\text{if } s \in \mathbb{R} \text{ or } s = +\infty)\\
-\infty + s &= -\infty && (\text{if } s \in \mathbb{R} \text{ or } s = -\infty)\\
\pm\infty \cdot s &= \pm\infty && (\text{if } s > 0 \text{ or } s = +\infty)\\
\pm\infty \cdot s &= \mp\infty && (\text{if } s < 0 \text{ or } s = -\infty)\\
\frac{\pm\infty}{s} &= \pm\infty && (\text{if } s > 0)\\
\frac{\pm\infty}{s} &= \mp\infty && (\text{if } s < 0)\\
\frac{s}{0} &= \infty && (\text{if } s \in \mathbb{R} \setminus \{0\} \text{ or } s = \pm\infty)\\
\frac{s}{\pm\infty} &= 0 && (\text{if } s \in \mathbb{R})
\end{aligned}
$$

Instead, the following expressions are *not* defined

$$
\pm\infty + (\mp\infty), \qquad \pm\infty - (\pm\infty), \qquad \pm\infty \cdot 0, \qquad \frac{\pm\infty}{\pm\infty}, \qquad \frac{0}{0}.
$$

A result of the foremost importance comes next.

Theorem 4.10 *Suppose f admits limit ℓ (finite or infinite) and g admits limit m (finite or infinite) for $x \to c$. Then*

$$
\lim_{x\to c} \big(f(x) \pm g(x)\big) = \ell \pm m,
$$

$$
\lim_{x\to c} \big(f(x)\, g(x)\big) = \ell\, m,
$$

$$
\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\ell}{m},
$$

provided the right-hand-side expressions make sense. (In the last case one assumes $g(x) \neq 0$ on some $I(c)\setminus\{c\}$.)

Proof. We shall prove two relations only, referring the reader to ⇝ `Limits` for the ones left behind. The first we concentrate upon is

$$\lim_{x\to c} \big(f(x)+g(x)\big) = \ell + m$$

when ℓ and m are finite. Fix $\varepsilon > 0$, and consider the neighbourhood of ℓ of radius $\varepsilon/2$. By assumption there is a neighbourhood $I'(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon/2.$$

For the same reason there is also an $I''(c)$ with

$$\forall x \in \operatorname{dom} g, \qquad x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon/2.$$

Put $I(c) = I'(c) \cap I''(c)$. Then if $x \in \operatorname{dom} f \cap \operatorname{dom} g$ belongs to $I(c) \setminus \{c\}$, both inequalities hold; the triangle inequality (1.1) yields

$$\begin{aligned} |(f(x)+g(x)) - (\ell+m)| &= |(f(x)-\ell) + (g(x)-m)| \\ &\le |f(x)-\ell| + |g(x)-m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

proving the assertion.

The second relation is

$$\lim_{x\to c} \big(f(x)\,g(x)\big) = +\infty$$

with $\ell = +\infty$ and $m > 0$ finite. For a given real $A > 0$, consider the neighbourhood of $+\infty$ with end-point $B = 2A/m > 0$. We know there is a neighbourhood $I'(c)$ such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) > B.$$

On the other hand, considering the neighbourhood of m of radius $m/2$, there exists an $I''(c)$ such that

$$\forall x \in \operatorname{dom} g, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < m/2,$$

i.e., $m/2 < g(x) < 3m/2$. Set $I(c) = I'(c) \cap I''(c)$. If $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is in $I(c) \setminus \{c\}$, the previous relations will be both fulfilled, whence

$$f(x)\,g(x) > f(x)\,\frac{m}{2} > B\frac{m}{2} = A.$$

□

Corollary 4.11 *If f and g are continuous maps at a point $x_0 \in \mathbb{R}$, then also $f(x) \pm g(x)$, $f(x)\,g(x)$ and $\dfrac{f(x)}{g(x)}$ (provided $g(x_0) \neq 0$) are continuous at x_0.*

Proof. The condition that f and g are continuous at x_0 is equivalent to $\lim_{x\to x_0} f(x) = f(x_0)$ and $\lim_{x\to x_0} g(x) = g(x_0)$ (recall (3.9)). The previous theorem allows to conclude. □

Corollary 4.12 *Rational functions are continuous on their domain. In particular, polynomials are continuous on* $\mathbb{R}$.

Proof. We verified in Example 3.17, part i), that the constants $y = a$ and the linear function $y = x$ are continuous on $\mathbb{R}$. Consequently, maps like $y = ax^n$ ($n \in \mathbb{N}$) are continuous. But then so are polynomials, being sums of the latter. Rational functions, as quotients of polynomials, inherit the property wherever the denominator does not vanish. □

Examples 4.13

i) Calculate

$$\lim_{x\to 0} \frac{2x - 3\cos x}{5 + x\sin x} = \ell.$$

The continuity of numerator and denominator descends from algebraic operations on continuous maps, and the denominator is not zero at $x = 0$. The substitution of 0 to x produces $\ell = -3/5$.

ii) Discuss the limit behaviour of $y = \tan x$ when $x \to \frac{\pi}{2}$. Since

$$\lim_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1 \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}} \cos x = \cos\frac{\pi}{2} = 0,$$

the above theorem tells

$$\lim_{x\to\frac{\pi}{2}} \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} = \frac{1}{0} = \infty.$$

But one can be more precise by looking at the sign of the tangent around $\frac{\pi}{2}$. Since $\sin x > 0$ in a neighbourhood of $\frac{\pi}{2}$, while $\cos x > 0$ (< 0) in a left (resp. right) neighbourhood of $\frac{\pi}{2}$, it follows

$$\lim_{x\to\frac{\pi}{2}^{\pm}} \tan x = \mp\infty.$$

iii) Let $R(x) = \dfrac{P(x)}{Q(x)}$ be rational and reduced, meaning the polynomials P, Q have no common factor. Call $x_0 \in \mathbb{R}$ a zero of Q, i.e., a point such that $Q(x_0) = 0$. Clearly $P(x_0) \neq 0$, otherwise P and Q would be both divisible by $(x - x_0)$. Then

$$\lim_{x\to x_0} R(x) = \infty$$

follows. In this case too, the sign of $R(x)$ around of x_0 retains some information. For instance, $y = \dfrac{x^2 - 3x + 1}{x^2 - x}$ is positive on a left neighbourhood of $x_0 = 1$ and negative on a right neighbourhood, so

$$\lim_{x\to 1^{\pm}} \frac{x^2-3x+1}{x^2-x} = \mp\infty.$$

In contrast, the function $y = \dfrac{x-2}{x^2-2x+1}$ is negative in a whole neighbourhood of $x_0 = 1$, hence

$$\lim_{x\to 1} \frac{x-2}{x^2-2x+1} = -\infty.$$

□

Theorem 4.10 gives no indication about the limit behaviour of an algebraic expression in three cases, listed below. The expressions in question are called **indeterminate forms** of algebraic type.

i) Consider $f(x)+g(x)$ (resp. $f(x)-g(x)$) when both f, g tend to ∞ with different (resp. same) signs. This gives rise to the indeterminate form denoted by the symbol

$$\infty - \infty.$$

ii) The product $f(x)\,g(x)$, when one function tends to ∞ and the other to 0, is the indeterminate form with symbol

$$\infty \cdot 0.$$

iii) Relatively to $\dfrac{f(x)}{g(x)}$, in case both functions tend to ∞ or 0, the indeterminate forms are denoted with

$$\frac{\infty}{\infty} \quad \text{or} \quad \frac{0}{0}.$$

In presence of an indeterminate form, the limit behaviour cannot be told a priori, and there are examples for each possible limit: infinite, finite non-zero, zero, even non-existing limit. Every indeterminate form should be treated singularly and requires often a lot of attention.

Later we shall find the actual limit behaviour of many important indeterminate forms. With those and this section's theorems we will discuss more complicated indeterminate forms. Additional tools to analyse this behaviour will be provided further on: they are the local comparison of functions by means of the Landau symbols (Sect. 5.1), de l'Hôpital's Theorem (Sect. 6.11), the Taylor expansion (Sect. 7.1).

Examples 4.14

i) Let x tend to $+\infty$ and define functions $f_1(x) = x + x^2$, $f_2(x) = x + 1$, $f_3(x) = x + \frac{1}{x}$, $f_4(x) = x + \sin x$. Set $g(x) = x$. Using Theorem 4.10, or Example 4.9, one verifies easily that all maps tend to $+\infty$. One has

$$\lim_{x\to+\infty} [f_1(x) - g(x)] = \lim_{x\to+\infty} x^2 = +\infty,$$

$$\lim_{x\to+\infty} [f_2(x) - g(x)] = \lim_{x\to+\infty} 1 = 1,$$

$$\lim_{x\to+\infty} [f_3(x) - g(x)] = \lim_{x\to+\infty} \frac{1}{x} = 0,$$

whereas the limit of $f_4(x) - g(x) = \sin x$ does not exist: the function $\sin x$ is periodic and assumes each value between -1 and 1 infinitely many times as $x \to +\infty$.

ii) Consider now $x \to 0$. Let $f_1(x) = x^3$, $f_2(x) = x^2$, $f_3(x) = x$, $f_4(x) = x^2 \sin \frac{1}{x}$, and $g(x) = x^2$. All functions converge to 0 (for f_4 apply Corollary 4.7). Now

$$\lim_{x\to 0} \frac{f_1(x)}{g(x)} = \lim_{x\to 0} x = 0,$$
$$\lim_{x\to 0} \frac{f_2(x)}{g(x)} = \lim_{x\to 0} 1 = 1,$$
$$\lim_{x\to 0} \frac{f_3(x)}{g(x)} = \lim_{x\to 0} \frac{1}{x}, = \infty,$$

but $\dfrac{f_4(x)}{g(x)} = \sin \dfrac{1}{x}$ does not admit limit for $x \to 0$ (Remark 4.19 furnishes a proof of this).

iii) Let us consider a polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0 \qquad (a_n \neq 0)$$

for $x \to \pm\infty$. A function of this sort can give rise to an indeterminate form $\infty - \infty$ according to the coefficients' signs and the degree of the monomials involved. The problem is sorted by factoring out the leading term (monomial of maximal degree) x^n

$$P(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

The part in brackets converges to a_n when $x \to \pm\infty$, so

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n = \infty$$

The sign of the limit is easily found. For instance,

$$\lim_{x\to -\infty} (-5x^3 + 2x^2 + 7) = \lim_{x\to -\infty} (-5x^3) = +\infty.$$

Take now a reduced rational function

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} \qquad (a_n, b_m \neq 0,\ m > 0).$$

When $x \to \pm\infty$, an indeterminate form $\frac{\infty}{\infty}$ arises. With the same technique as before,

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m} = \begin{cases} \infty & \text{if } n > m, \\ \dfrac{a_n}{b_m} & \text{if } n = m, \\ 0 & \text{if } n < m. \end{cases}$$

For example:

$$\lim_{x\to+\infty} \frac{3x^3-2x+1}{x-x^2} = \lim_{x\to+\infty} \frac{3x^3}{-x^2} = -\infty\,,$$

$$\lim_{x\to-\infty} \frac{-4x^5+2x^3-7}{8x^5-x^4+5x} = \lim_{x\to-\infty} \frac{-4x^5}{8x^5} = -\frac{1}{2}\,,$$

$$\lim_{x\to-\infty} \frac{6x^2-x+5}{-x^3+9} = \lim_{x\to-\infty} \frac{6x^2}{-x^3} = 0.$$

iv) The function $y = \dfrac{\sin x}{x}$ becomes indeterminate $\frac{0}{0}$ for $x \to 0$; we proved in part i), Examples 4.6 that y converges to 1. From this, we can deduce the behaviour of $y = \dfrac{1-\cos x}{x^2}$ as $x \to 0$, another indeterminate form of the type $\frac{0}{0}$. In fact,

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2} \cdot \lim_{x\to 0} \frac{1}{1+\cos x}.$$

The fundamental trigonometric equation $\cos^2 x + \sin^2 x = 1$ together with Theorem 4.10 gives

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 1.$$

The same theorem tells also that the second limit is $\frac{1}{2}$, so we conclude

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.}$$

□

With these examples we have taken the chance to look at the behaviour of elementary functions at the boundary points of their domains. For completeness we gather the most significant limits relative to the elementary functions of Sect. 2.6. For explanations ⇝ `Elementary functions.`

$\lim_{x\to+\infty} x^\alpha = +\infty\,,$	$\lim_{x\to 0^+} x^\alpha = 0$	$\alpha > 0$
$\lim_{x\to+\infty} x^\alpha = 0\,,$	$\lim_{x\to 0^+} x^\alpha = +\infty$	$\alpha < 0$
$\lim_{x\to\pm\infty} \dfrac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$	$= \dfrac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$	
$\lim_{x\to+\infty} a^x = +\infty\,,$	$\lim_{x\to-\infty} a^x = 0$	$a > 1$
$\lim_{x\to+\infty} a^x = 0\,,$	$\lim_{x\to-\infty} a^x = +\infty$	$a < 1$
$\lim_{x\to+\infty} \log_a x = +\infty\,,$	$\lim_{x\to 0^+} \log_a x = -\infty$	$a > 1$
$\lim_{x\to+\infty} \log_a x = -\infty\,,$	$\lim_{x\to 0^+} \log_a x = +\infty$	$a < 1$

$$\lim_{x\to\pm\infty} \sin x\,, \qquad \lim_{x\to\pm\infty} \cos x\,, \qquad \lim_{x\to\pm\infty} \tan x \qquad \text{do not exist}$$

$$\lim_{x\to(\frac{\pi}{2}+k\pi)^{\pm}} \tan x = \mp\infty\,, \quad \forall k \in \mathbb{Z}$$

$$\lim_{x\to\pm1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm 1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

4.1.4 Substitution theorem

The so-called Substitution theorem is important in itself for theoretical reasons, besides providing a very useful method to compute limits.

Theorem 4.15 *Suppose a map f admits limit*

$$\lim_{x\to c} f(x) = \ell, \tag{4.9}$$

finite or not. Let g be defined on a neighbourhood of ℓ (excluding possibly the point ℓ) and such that

i) if $\ell \in \mathbb{R}$, g is continuous at ℓ;

ii) if $\ell = +\infty$ or $\ell = -\infty$, the limit $\lim\limits_{y\to\ell} g(y)$ exists, finite or not.

Then the composition $g \circ f$ admits limit for $x \to c$ and

$$\lim_{x\to c} g(f(x)) = \lim_{y\to\ell} g(y). \tag{4.10}$$

Proof. Set $m = \lim\limits_{y\to\ell} g(y)$ (noting that under *i)*, $m = g(\ell)$). Given any neighbourhood $I(m)$ of m, by *i)* or *ii)* there will be a neighbourhood $I(\ell)$ of ℓ such that

$$\forall y \in \operatorname{dom} g, \qquad y \in I(\ell) \quad \Rightarrow \quad g(y) \in I(m).$$

Note that in case *i)* we can use $I(\ell)$ instead of $I(\ell) \setminus \{\ell\}$ because g is continuous at ℓ (recall (3.7)), while ℓ does not belong to $I(\ell)$ for case *ii)*. With such $I(\ell)$, assumption (4.9) implies the existence of a neighbourhood $I(c)$ of c with

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell).$$

Since $x \in \operatorname{dom} g \circ f$ means $x \in \operatorname{dom} f$ plus $y = f(x) \in \operatorname{dom} g$, the previous two implications now give

$$\forall x \in \operatorname{dom} g \circ f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad g(f(x)) \in I(m).$$

But $I(m)$ was arbitrary, so

$$\lim_{x \to c} g(f(x)) = m. \qquad \square$$

Remark 4.16 An alternative condition that yields the same conclusion is the following:

i') if $\ell \in \mathbb{R}$, there is a neighbourhood $I(c)$ of c where $f(x) \neq \ell$ for all $x \neq c$, and the limit $\lim_{y \to \ell} g(y)$ exists, finite or infinite.

The proof is analogous. $\square$

In case $\ell \in \mathbb{R}$ and g is continuous at ℓ (case *i)*), then $\lim_{y \to \ell} g(y) = g(\ell)$, so (4.10) reads

$$\boxed{\lim_{x \to c} g(f(x)) = g(\lim_{x \to c} f(x)).} \tag{4.11}$$

An imprecise but effective way to put (4.11) into words is to say that a continuous function *commutes* (exchanges places) with the symbol of limit.

Theorem 4.15 implies that continuity is inherited by composite functions, as we discuss hereby.

Corollary 4.17 *Let f be continuous at x_0, and define $y_0 = f(x_0)$. Let furthermore g be defined around y_0 and continuous at y_0. Then the composite $g \circ f$ is continuous at x_0.*

Proof. From (4.11)

$$\lim_{x \to x_0} (g \circ f)(x) = g(\lim_{x \to x_0} f(x)) = g(f(x_0)) = (g \circ f)(x_0),$$

which is equivalent to the claim. $\square$

A few practical examples will help us understand how the Substitution theorem and its corollary are employed.

Examples 4.18

i) The map $h(x) = \sin(x^2)$ is continuous on $\mathbb{R}$, being the composition of the continuous functions $f(x) = x^2$ and $g(y) = \sin y$.

ii) Let us determine

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2}.$$

Set $f(x) = x^2$ and

$$g(y) = \begin{cases} \dfrac{\sin y}{y} & \text{if } y \neq 0, \\ 1 & \text{if } y = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 0$, and we know that g is continuous at the origin. Thus

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

iii) We study the behaviour of $h(x) = \arctan\left(\dfrac{1}{x-1}\right)$ around the point 1. Defining $f(x) = \dfrac{1}{x-1}$, we have $\lim_{x\to 1^\pm} f(x) = \pm\infty$. If we call $g(y) = \arctan y$, $\lim_{y\to\pm\infty} g(y) = \pm\dfrac{\pi}{2}$ (see the Table on page 101). Therefore

$$\lim_{x\to 1^\pm} \arctan\left(\frac{1}{x-1}\right) = \lim_{y\to\pm\infty} g(y) = \pm\frac{\pi}{2}.$$

iv) Determine

$$\lim_{x\to+\infty} \log\sin\frac{1}{x}.$$

Setting $f(x) = \sin\frac{1}{x}$ has the effect that $\ell = \lim_{x\to+\infty} f(x) = 0$. Note that $f(x) > 0$ for all $x > \frac{1}{\pi}$. With $g(y) = \log y$ we have $\lim_{y\to 0^+} g(y) = -\infty$, so Remark 4.16 yields

$$\lim_{x\to+\infty} \log\sin\frac{1}{x} = \lim_{y\to 0^+} g(y) = -\infty.$$

□

Remark 4.19 Theorem 4.15 extends easily to cover the case where the role of f is played by a sequence $a : n \mapsto a_n$ with limit

$$\lim_{n\to\infty} a_n = \ell.$$

Namely, under the same assumptions on g,

$$\lim_{n\to\infty} g(a_n) = \lim_{y\to\ell} g(y).$$

This result is often used to disprove the existence of a limit, in that it provides a **Criterion of non-existence for limits** : *if two sequences $a : n \mapsto a_n$, $b : n \mapsto b_n$ have the same limit ℓ and*

$$\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n),$$

then g does not admit limit when its argument tends to ℓ.

For example we can prove, with the aid of the criterion, that $y = \sin x$ has no limit when $x \to +\infty$: define the sequences $a_n = 2n\pi$ and $b_n = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} 0 = 0, \qquad \text{and at the same time} \qquad \lim_{n\to\infty} \sin b_n = \lim_{n\to\infty} 1 = 1.$$

Similarly, the function $y = \sin \frac{1}{x}$ has neither left nor right limit for $x \to 0$. □

4.2 More fundamental limits. Indeterminate forms of exponential type

Consider the paramount limit (3.3). Instead of the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$, we look now at the function of real variable

$$h(x) = \left(1 + \frac{1}{x}\right)^x.$$

It is defined when $1 + \frac{1}{x} > 0$, hence on $(-\infty, -1) \cup (0, +\infty)$. The following result states that h and the sequence resemble each other closely when x tends to infinity.

Property 4.20 *The following limit holds*

$$\lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{e}.$$

Proof. ⇝ The number e. □

By manipulating this formula we achieve a series of new fundamental limits. The substitution $y = \frac{x}{a}$, with $a \neq 0$, gives

$$\lim_{x\to\pm\infty} \left(1 + \frac{a}{x}\right)^x = \lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^{ay} = \left[\lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^y\right]^a = \mathrm{e}^a.$$

In terms of the variable $y = \frac{1}{x}$ then,

$$\lim_{x\to 0} (1 + x)^{1/x} = \lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^y = \mathrm{e}.$$

The continuity of the logarithm together with (4.11) furnish

$$\lim_{x\to 0} \frac{\log_a(1 + x)}{x} = \lim_{x\to 0} \log_a (1 + x)^{1/x} = \log_a \lim_{x\to 0} (1 + x)^{1/x} = \log_a \mathrm{e} = \frac{1}{\log a}$$

for any $a > 0$. In particular, taking $a = \mathrm{e}$:

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

Note by the way $a^x - 1 = y$ is equivalent to $x = \log_a(1+y)$, and $y \to 0$ if $x \to 0$. With this substitution,

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{y\to 0} \frac{y}{\log_a(1+y)} = \left[\lim_{y\to 0} \frac{\log_a(1+y)}{y}\right]^{-1} = \log a. \tag{4.12}$$

Taking $a = \mathrm{e}$ produces

$$\lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = 1.$$

Eventually, let us set $1 + x = \mathrm{e}^y$. Since $y \to 0$ when $x \to 0$,

$$\begin{aligned} \lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} &= \lim_{y\to 0} \frac{\mathrm{e}^{\alpha y} - 1}{\mathrm{e}^y - 1} = \lim_{y\to 0} \frac{\mathrm{e}^{\alpha y} - 1}{y} \frac{y}{\mathrm{e}^y - 1} \\ &= \lim_{y\to 0} \frac{(\mathrm{e}^\alpha)^y - 1}{y} \lim_{y\to 0} \frac{y}{\mathrm{e}^y - 1} = \log \mathrm{e}^\alpha = \alpha \end{aligned} \tag{4.13}$$

for any $\alpha \in \mathbb{R}$.

For the reader's conveniency, all fundamental limits found so far are gathered below.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to \pm\infty} \left(1 + \frac{a}{x}\right)^x = \mathrm{e}^a \qquad (a \in \mathbb{R})$$

$$\lim_{x\to 0} (1+x)^{1/x} = \mathrm{e}$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a} \quad (a > 0); \quad \text{in particular,} \quad \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a \quad (a > 0); \qquad \text{in particular,} \quad \lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = 1$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad (\alpha \in \mathbb{R}).$$

Let us return to the map $h(x) = \left(1 + \dfrac{1}{x}\right)^x$. By setting $f(x) = \left(1 + \dfrac{1}{x}\right)$ and $g(x) = x$, we can write

$$h(x) = [f(x)]^{g(x)}.$$

In general such an expression may give rise to **indeterminate forms** for x tending to a certain c. Suppose f, g are functions defined in a neighbourhood of c, except possibly at c, and that they admit limit for $x \to c$. Assume moreover $f(x) > 0$ around c, so that h is well defined in a neighbourhood of c (except possibly at c). To understand h it is convenient to use the identity

$$f(x) = \mathrm{e}^{\log f(x)}.$$

From this in fact we obtain

$$h(x) = \mathrm{e}^{g(x)\log f(x)}.$$

By continuity of the exponential and (4.11), we have

$$\lim_{x\to c}[f(x)]^{g(x)} = \exp\left(\lim_{x\to c}[g(x)\log f(x)]\right).$$

In other words, $h(x)$ can be studied by looking at the exponent $g(x)\log f(x)$. An indeterminate form of the latter will thus develop an **indeterminate form** of exponential type for $h(x)$. Namely, we might find ourselves in one of these situations:

i) g tends to ∞ and f to 1 (so $\log f$ tends to 0): the exponent is an indeterminate form of type $\infty \cdot 0$, whence we say that h presents an indeterminate form of type
$$1^\infty.$$

ii) g and f both tend to 0 (so $\log f$ tends to $-\infty$): once again the exponent is of type $\infty \cdot 0$, and the function h is said to have an indeterminate form of type
$$0^0.$$

iii) g tends to 0 and f tends to $+\infty$ ($\log f \to +\infty$): the exponent is of type $\infty \cdot 0$, and h becomes indeterminate of type
$$\infty^0.$$

Examples 4.21

i) The map $h(x) = \left(1 + \dfrac{1}{x}\right)^x$ is an indeterminate form of type 1^∞ when $x \to \pm\infty$, whose limit equals e.

ii) The function $h(x) = x^x$, for $x \to 0^+$, is an indeterminate form of type 0^0. We shall prove in Chap. 6 that $\lim\limits_{x\to 0^+} x\log x = 0$, therefore $\lim\limits_{x\to 0^+} h(x) = 1$.

iii) The function $h(x) = x^{1/x}$ is for $x \to +\infty$ an indeterminate form of type ∞^0. Substituting $y = \frac{1}{x}$, and recalling that $\log \frac{1}{y} = -\log y$, we obtain $\lim\limits_{x\to+\infty} \frac{\log x}{x} = -\lim\limits_{y\to 0^+} y \log y = 0$, hence $\lim\limits_{x\to+\infty} h(x) = 1$. □

When dealing with $h(x) = [f(x)]^{g(x)}$, a rather common mistake – with tragic consequences – is to calculate first the limit of f and/or g, substitute the map with this value and compute the limit of the expression thus obtained. This is to emphasize that **it might be incorrect** to calculate the limit for $x \to c$ of the indeterminate form $h(x) = [f(x)]^{g(x)}$ by finding first

$$m = \lim_{x\to c} g(x), \qquad \text{and from this proceed to} \quad \lim_{x\to c} [f(x)]^m.$$

Equally incorrect might be to determine

$$\lim_{x\to c} \ell^{g(x)}, \qquad \text{already knowing} \quad \ell = \lim_{x\to c} f(x).$$

For example, suppose we are asked to find the limit of $h(x) = \left(1 + \frac{1}{x}\right)^x$ for $x \to \pm\infty$; we might think of finding first $\ell = \lim\limits_{x\to\pm\infty} \left(1 + \frac{1}{x}\right) = 1$ and from this $\lim\limits_{x\to\pm\infty} 1^x = \lim\limits_{x\to\pm\infty} 1 = 1$. This would lead us to believe, wrongly, that h converges to 1, in spite of the fact the correct limit is e.

4.3 Global features of continuous maps

Hitherto the focus has been on several local properties of functions, whether in the neighbourhood of a real point or a point at infinity, and limits have been discussed in that respect. Now we turn our attention to continuous functions defined on a real interval, and establish properties of global nature, i.e., those relative to the behaviour on the entire domain.

Let us start with a plain definition.

Definition 4.22 *A* **zero** *of a real-valued function f is a point $x_0 \in \operatorname{dom} f$ at which the function vanishes.*

For instance, the zeroes of $y = \sin x$ are the multiples of π, i.e., the elements of the set $\{m\pi \mid m \in \mathbb{Z}\}$.

The problem of solving an equation like

$$f(x) = 0$$