CHAPTER 8

Analytic Geometry in Two and Three Dimensions

The oval-shaped lawn behind the White House in Washington, D.C. is called *the Ellipse*. It has views of the Washington Monument, the Jefferson Memorial, the Department of Commerce, and the Old Post Office Building. The Ellipse is 616 ft long, 528 ft wide, and is in the shape of a conic section. Its shape can be modeled using the methods of this chapter. See page 652.

HISTORY OF CONIC SECTIONS

Parabolas, ellipses, and hyperbolas had been studied for many years when Apollonius (c. 262–190 B.C.) wrote his eight-volume *Conic Sections*. Apollonius, born in northwestern Asia Minor, was the first to unify these three curves as cross sections of a cone and to view the hyperbola as having two branches. Interest in conic sections was renewed in the 17th century when Galileo proved that projectiles follow parabolic paths and Johannes Kepler (1571–1630) discovered that planets travel in elliptical orbits.

Chapter 8 Overview

Analytic geometry combines number and form. It is the marriage of algebra and geometry that grew from the works of Frenchmen René Descartes (1596–1650) and Pierre de Fermat (1601–1665). Their achievements allowed geometry problems to be solved algebraically and algebra problems to be solved geometrically—two major themes of this book. Analytic geometry opened the door for Newton and Leibniz to develop calculus.

In Sections 8.1–8.4, we will learn that parabolas, ellipses, and hyperbolas are all conic sections and can all be expressed as second-degree equations. We will investigate their uses, including the reflective properties of parabolas and ellipses and how hyperbolas are used in long-range navigation. In Section 8.5, we will see how parabolas, ellipses, and hyperbolas are unified in the polar-coordinate setting. In Section 8.6, we will move from the two-dimensional plane to revisit the concepts of point, line, midpoint, distance, and vector in three-dimensional space.

8.1 Conic Sections and Parabolas

What you'll learn about

- Conic Sections
- Geometry of a Parabola
- Translations of Parabolas
- Reflective Property of a Parabola

. . . and why

Conic sections are the paths of nature: Any free-moving object in a gravitational field follows the path of a conic section.

Conic Sections

Imagine two nonperpendicular lines intersecting at a point V. If we fix one of the lines as an *axis* and rotate the other line (the *generator*) around the axis, then the generator sweeps out a **right circular cone** with **vertex** V, as illustrated in Figure 8.1. Notice that V divides the cone into two parts called **nappes**, with each nappe of the cone resembling a pointed ice-cream cone.

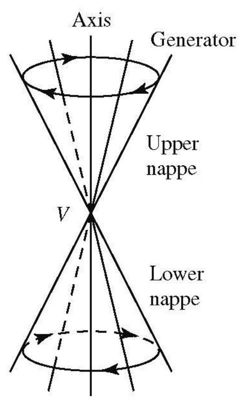


FIGURE 8.1 A right circular cone (of two nappes).

A **conic section** (or **conic**) is a cross section of a cone, in other words, the intersection of a plane with a right circular cone. The three basic conic sections are the *parabola*, the *ellipse*, and the *hyperbola* (Figure 8.2a).

Some atypical conics, known as **degenerate conic sections**, are shown in Figure 8.2b. Because it is atypical and lacks some of the features usually associated with an ellipse, a circle is considered to be a degenerate ellipse. Other degenerate conic sections can be obtained from cross sections of a degenerate cone; such cones occur when the generator and axis of the cone are parallel or perpendicular. (See Exercise 73.)

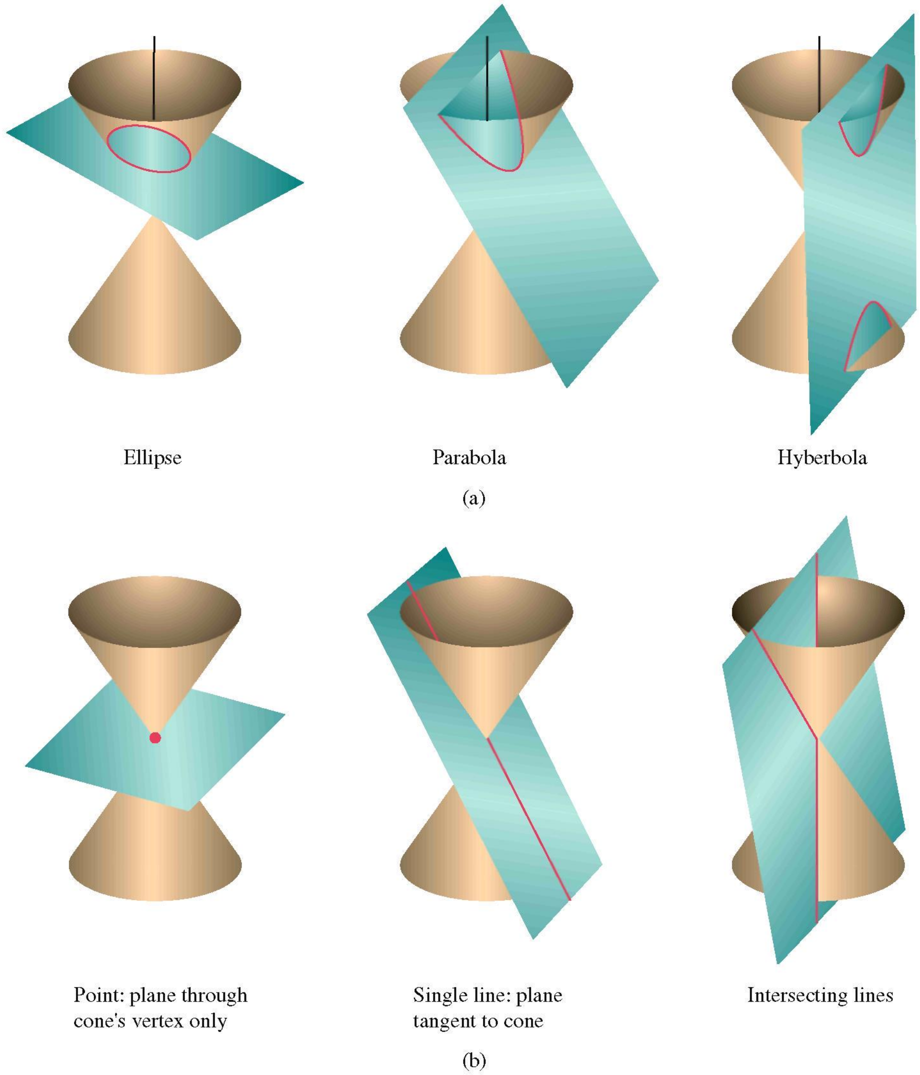


FIGURE 8.2 (a) The three standard types of conic sections and (b) three degenerate conic sections.

The conic sections can be defined algebraically as the graphs of **second-degree (quadratic) equations in two variables**, that is, equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not all zero.

OBJECTIVE

Students will be able to find the equation, focus, and directrix of a parabola.

MOTIVATE

Ask students if all parabolas are similar (in the geometric sense). **(Yes)**

LESSON GUIDE

Day 1: Conic Sections; Geometry of a Parabola
Day 2: Translations of Parabolas; Reflective Property of a Parabola

BIBLIOGRAPHY

For students: *Practical Conic Sections,* J. W. Downs. Dale Seymour Publications, 1993.

Comet, Carl Sagan and Ann Druyan. Ballentine Books, 1997.

For teachers: *Astronomy: From the Earth to the Universe (5th ed.),* J. M. Pasachoff. Saunders College Publishing, 1998.

Multicultural and Gender Equity in the Mathematics Classroom: The Gift of Diversity (1997 Yearbook), Janet Trentacosta (Ed.) National Council of Teachers of Mathematics, 1997.

Posters:

Conic Sections. Dale Seymour Publications

Locating Satellites in Elliptical Orbits, National Council of Teachers of Mathematics

TEACHING NOTES

You may wish to construct the nappe of a cone by rolling up a piece of paper in order to illustrate various conic sections as cross sections of a cone.

Note that a circle is a special case of an ellipse and thus considered a degenerate conic.

Geometry of a Parabola

In Section 2.1 we learned that the graph of a quadratic function is an upward or downward opening parabola. We have seen the role of the parabola in free-fall and projectile motion. We now investigate the geometric properties of parabolas.

A DEGENERATE PARABOLA

If the focus F lies on the directrix l, the parabola "degenerates" to the line through F perpendicular to l. Henceforth, we will assume F does not lie on l.

DEFINITION Parabola

A **parabola** is the set of all points in a plane equidistant from a particular line (the **directrix**) and a particular point (the **focus**) in the plane. (See Figure 8.3.)

LOCUS OF A POINT

Before the word *set* was used in mathematics, the Latin word *locus*, meaning "place," was often used in geometric definitions. The locus of a point was the set of possible places a point could be and still fit the conditions of the definition. Sometimes, conics are still defined in terms of loci.

The line passing through the focus and perpendicular to the directrix is the **(focal) axis** of the parabola. The axis is the line of symmetry for the parabola. The point where the parabola intersects its axis is the **vertex** of the parabola. The vertex is located midway between the focus and the directrix and is the point of the parabola that is closest to both the focus and the directrix. See Figure 8.3.

TEACHING NOTE

Exercise 71 explains how Figure 8.3 was created, and this exercise can be used instead of (or in addition to) Exploration 1.

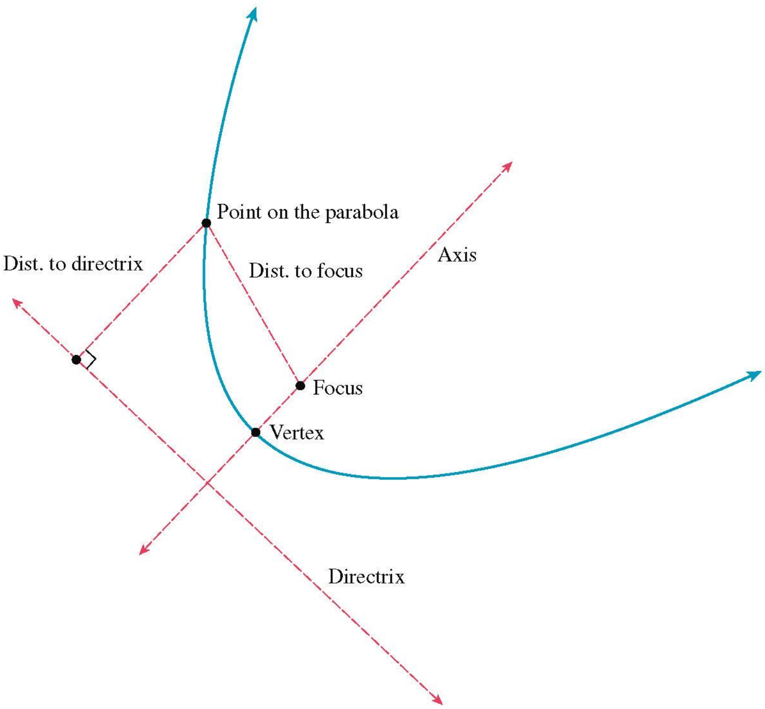


FIGURE 8.3 *Structure of a Parabola.* The distance from each point on the parabola to both the focus and the directrix is the same.

EXPLORATION EXTENSIONS

Find the y-coordinate for a point on the parabola that is a distance of 25 units from the focus. **Ans. 24**

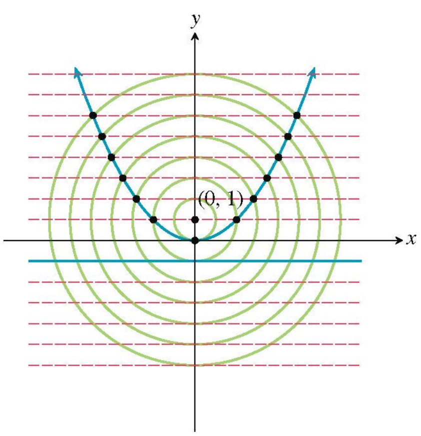


FIGURE 8.4 The geometry of a parabola.

EXPLORATION 1 Understanding the Definition of Parabola

1. Prove that the vertex of the parabola with focus (0, 1) and directrix $y = -1$ is (0, 0). (See Figure 8.4.)
2. Find an equation for the parabola shown in Figure 8.4. $x^2 = 4y$
3. Find the coordinates of the points of the parabola that are highlighted in Figure 8.4.

We can generalize the situation in Exploration 1 to show that an equation for the parabola with focus $(0, p)$ and directrix $y = -p$ is $x^2 = 4py$. (See Figure 8.5.)

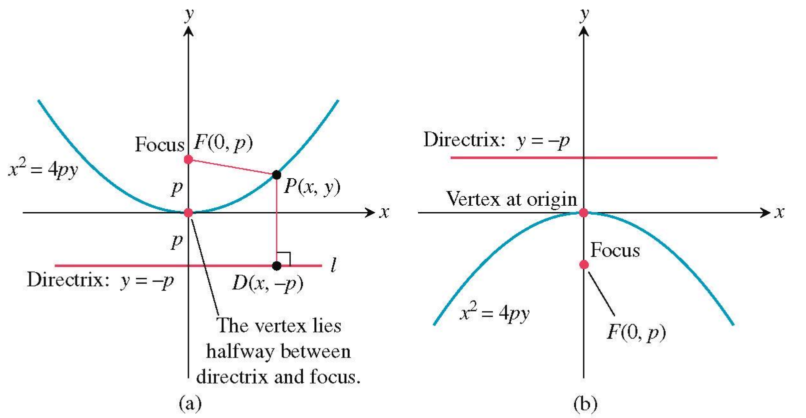


FIGURE 8.5 Graphs of $x^2 = 4py$ with (a) $p > 0$ and (b) $p < 0$.

ALERT

Note that the standard form of the equation of an upward or downward opening parabola is not the same as the standard form of a quadratic function.

We must show first that a point $P(x, y)$ that is equidistant from $F(0, p)$ and the line $y = -p$ satisfies the equation $x^2 = 4py$, and then that a point satisfying the equation $x^2 = 4py$ is equidistant from $F(0, p)$ and the line $y = -p$:

Let $P(x, y)$ be equidistant from $F(0, p)$ and the line $y = -p$. Notice that

$$\sqrt{(x - 0)^2 + (y - p)^2} = \text{distance from } P(x, y) \text{ to } F(0, p), \text{ and}$$

$$\sqrt{(x - x)^2 + (y - (-p))^2} = \text{distance from } P(x, y) \text{ to } y = -p.$$

Equating these distances and squaring yields:

$$(x - 0)^2 + (y - p)^2 = (x - x)^2 + (y - (-p))^2$$

$$x^2 + (y - p)^2 = 0 + (y + p)^2 \quad \text{Simplify.}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \quad \text{Expand.}$$

$$x^2 = 4py \quad \text{Combine like terms.}$$

By reversing the above steps, we see that a solution (x, y) of $x^2 = 4py$ is equidistant from $F(0, p)$ and the line $y = -p$.

The equation $x^2 = 4py$ is the **standard form** of the equation of an upward or downward opening parabola with vertex at the origin. If $p > 0$, the parabola opens upward; if $p < 0$, it opens downward. An alternative algebraic form for such a parabola is $y = ax^2$, where $a = 1/(4p)$. So the graph of $x^2 = 4py$ is also the graph of the quadratic function $f(x) = ax^2$.

NAME GAME

Additional features of a parabola are defined in Exercises 74–76. The focal width is the length of the *latus rectum*.

When the equation of an upward or downward opening parabola is written as $x^2 = 4py$, the value p is interpreted as the **focal length** of the parabola—the *directed* distance from the vertex to the focus of the parabola. A line segment with endpoints on a parabola is a **chord** of the parabola. The value $|4p|$ is the **focal width** of the parabola—the length of the chord through the focus and perpendicular to the axis.

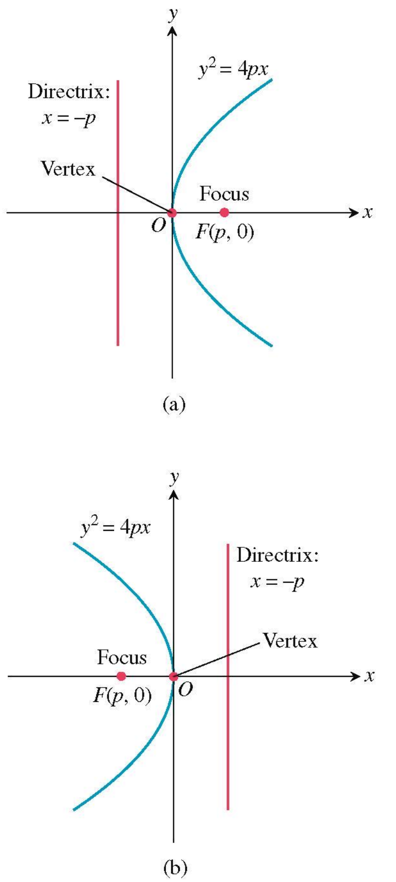


FIGURE 8.6 Graph of $y^2 = 4px$ with (a) $p > 0$ and (b) $p < 0$.

Parabolas that open to the right or to the left are *inverse relations* of upward or downward opening parabolas. So equations of parabolas with vertex (0, 0) that open to the right or to the left have the standard form $y^2 = 4px$. If $p > 0$, the parabola opens to the right, and if $p < 0$, to the left. (See Figure 8.6.)

Parabolas with Vertex (0, 0)

• **Standard equation**	$x^2 = 4py$	$y^2 = 4px$
• **Opens**	Upward or downward	To the right or to the left
• **Focus**	$(0, p)$	$(p, 0)$
• **Directrix**	$y = -p$	$x = -p$
• **Axis**	y-axis	x-axis
• **Focal length**	p	p
• **Focal width**	$\|4p\|$	$\|4p\|$

See Figures 8.5 and 8.6.

EXAMPLE 1 Finding the Focus, Directrix, and Focal Width

Find the focus, the directrix, and the focal width of the parabola $y = -(1/2)x^2$.

SOLUTION Multiplying both sides of the equation by -2 yields the standard form $x^2 = -2y$. The coefficient of y is $4p = -2$, and $p = -1/2$. So the focus is $(0, p) = (0, -1/2)$. Because $-p = -(-1/2) = 1/2$, the directrix is the line $y = 1/2$. The focal width is $|4p| = |-2| = 2$.

Now try Exercise 1.

NOTES ON EXAMPLES

When solving a problem such as Example 2, students should be strongly encouraged to draw a sketch of the given information. You may wish to draw a sketch for the students to model this behavior.

EXAMPLE 2 Finding an Equation of a Parabola

Find an equation in standard form for the parabola whose directrix is the line $x = 2$ and whose focus is the point $(-2, 0)$.

SOLUTION Because the directrix is $x = 2$ and the focus is $(-2, 0)$, the focal length is $p = -2$ and the parabola opens to the left. The equation of the parabola in standard form is $y^2 = 4px$, or more specifically, $y^2 = -8x$.

Now try Exercise 15.

Translations of Parabolas

When a parabola with the equation $x^2 = 4py$ or $y^2 = 4px$ is translated horizontally by h units and vertically by k units, the vertex of the parabola moves from (0, 0) to (h, k). (See Figure 8.7.) Such a translation does not change the focal length, the focal width, or the direction the parabola opens.

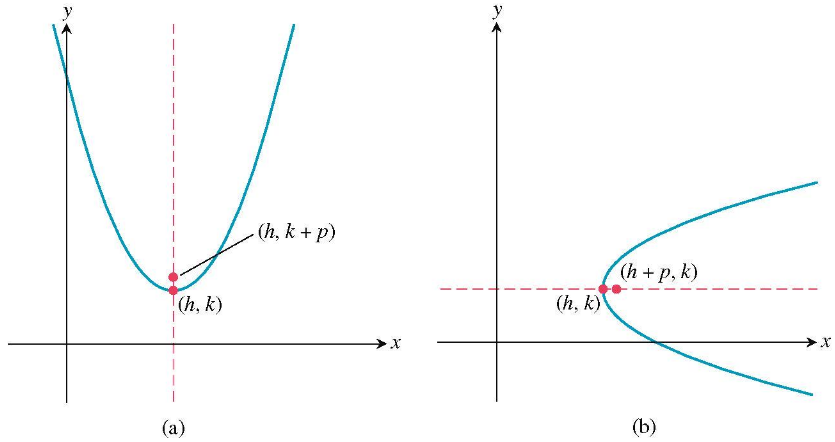


FIGURE 8.7 Parabolas with vertex (h, k) and focus on (a) $x = h$ and (b) $y = k$.

TEACHING NOTE

Students should be able to relate the transformations of parabolas (and other conic sections) to the transformations that were studied beginning in Section 1.5.

Parabolas with Vertex (h, k)

• **Standard equation**	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
• **Opens**	Upward or downward	To the right or to the left
• **Focus**	$(h, k + p)$	$(h + p, k)$
• **Directrix**	$y = k - p$	$x = h - p$
• **Axis**	$x = h$	$y = k$
• **Focal length**	p	p
• **Focal width**	$\lvert 4p \rvert$	$\lvert 4p \rvert$

See Figure 8.7.

EXAMPLE 3 Finding an Equation of a Parabola

Find the standard form of the equation for the parabola with vertex (3, 4) and focus (5, 4).

SOLUTION The axis of the parabola is the line passing through the vertex (3, 4) and the focus (5, 4). This is the line $y = 4$. So the equation has the form

$$(y - k)^2 = 4p(x - h).$$

Because the vertex $(h, k) = (3, 4)$, $h = 3$ and $k = 4$. The directed distance from the vertex (3, 4) to the focus (5, 4) is $p = 5 - 3 = 2$, so $4p = 8$. Thus the equation we seek is

$$(y - 4)^2 = 8(x - 3).$$

Now try Exercise 21.

When solving a problem like Example 3, it is a good idea to sketch the vertex, the focus, and other features of the parabola as we solve the problem. This makes it easy to see whether the axis of the parabola is horizontal or vertical and the relative positions of its features. Exploration 2 "walks us through" this process.

EXPLORATION EXTENSIONS

Check your work by using a grapher to graph the equation you found in step 8.

EXPLORATION 2 Building a Parabola

Carry out the following steps using a sheet of rectangular graph paper.

1. Let the focus F of a parabola be $(2, -2)$ and its directrix be $y = 4$. Draw the x- and y-axes on the graph paper. Then sketch and label the focus and directrix of the parabola.
2. Locate, sketch, and label the axis of the parabola. What is the equation of the axis?
3. Locate and plot the vertex V of the parabola. Label it by name and coordinates.
4. What are the focal length and focal width of the parabola?
5. Use the focal width to locate, plot, and label the endpoints of a chord of the parabola that parallels the directrix.
6. Sketch the parabola.
7. Which direction does it open? downward
8. What is its equation in standard form? $(x - 2)^2 = -12(y - 1)$

Sometimes it is best to sketch a parabola by hand, as in Exploration 2; this helps us see the structure and relationships of the parabola and its features. At other times, we may want or need an accurate, active graph. If we wish to graph a parabola using a function grapher, we need to solve the equation of the parabola for y, as illustrated in Example 4.

EXAMPLE 4 Graphing a Parabola

Use a function grapher to graph the parabola $(y - 4)^2 = 8(x - 3)$ of Example 3.

SOLUTION

$$(y - 4)^2 = 8(x - 3)$$

$$y - 4 = \pm\sqrt{8(x - 3)} \qquad \text{Extract square roots.}$$

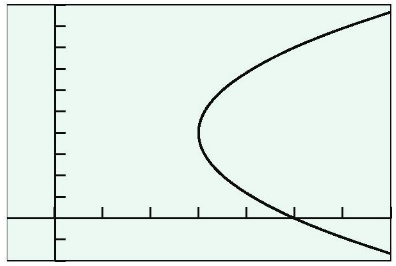
$$y = 4 \pm \sqrt{8(x - 3)} \qquad \text{Add 4.}$$

Let $Y_1 = 4 + \sqrt{8(x - 3)}$ and $Y_2 = 4 - \sqrt{8(x - 3)}$, and graph the two equations in a window centered at the vertex, as shown in Figure 8.8.

Now try Exercise 45.

CLOSING THE GAP

In Figure 8.8, we centered the graphing window at the vertex (3, 4) of the parabola to ensure that this point would be plotted. This avoids the common grapher error of a gap between the two upper and lower parts of the conic section being plotted.



[–1, 7] by [–2, 10]

FIGURE 8.8 The graphs of $Y1 = 4 + \sqrt{8(x-3)}$ and $Y2 = 4 - \sqrt{8(x-3)}$ together form the graph of $(y-4)^2 = 8(x-3)$. (Example 4)

EXAMPLE 5 Using Standard Forms with a Parabola

Prove that the graph of $y^2 - 6x + 2y + 13 = 0$ is a parabola, and find its vertex, focus, and directrix.

SOLUTION Because this equation is quadratic in the variable y, we complete the square with respect to y to obtain a standard form.

$$y^2 - 6x + 2y + 13 = 0$$

$$y^2 + 2y = 6x - 13 \quad \text{Isolate the } y\text{-terms.}$$

$$y^2 + 2y + 1 = 6x - 13 + 1 \quad \text{Complete the square.}$$

$$(y+1)^2 = 6x - 12$$

$$(y+1)^2 = 6(x-2)$$

This equation is in the standard form $(y-k)^2 = 4p(x-h)$, where $h = 2$, $k = -1$, and $p = 6/4 = 3/2 = 1.5$. It follows that

- the vertex (h, k) is $(2, -1)$;
- the focus $(h + p, k)$ is $(3.5, -1)$, or $(7/2, -1)$;
- the directrix $x = h - p$ is $x = 0.5$, or $x = 1/2$.

Now try Exercise 49.

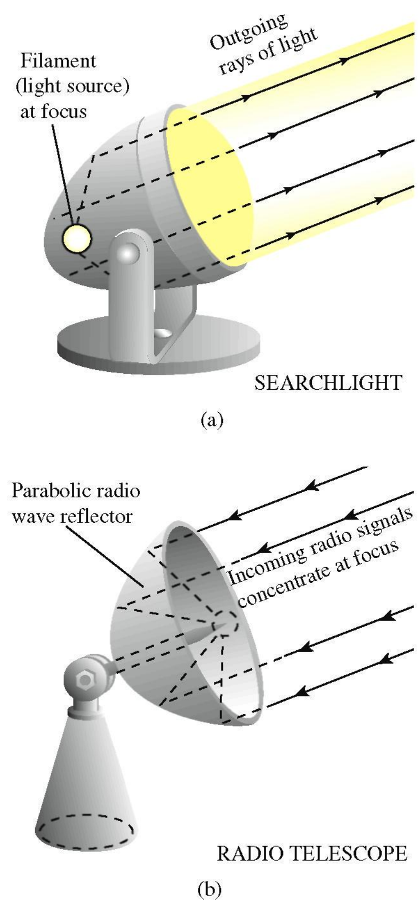


FIGURE 8.9 Examples of parabolic reflectors.

Reflective Property of a Parabola

The main applications of parabolas involve their use as reflectors of sound, light, radio waves, and other electromagnetic waves. If we rotate a parabola in three-dimensional space about its axis, the parabola sweeps out a **paraboloid of revolution**. If we place a signal source at the focus of a reflective paraboloid, the signal reflects off the surface in lines parallel to the axis of symmetry, as illustrated in Figure 8.9a. This property is used by flashlights, headlights, searchlights, microwave relays, and satellite up-links.

The principle works for signals traveling in the reverse direction as well; signals arriving parallel to a parabolic reflector's axis are directed toward the reflector's focus. This property is used to intensify signals picked up by radio telescopes and television satellite dishes, to focus arriving light in reflecting telescopes, to concentrate heat in solar ovens, and to magnify sound for sideline microphones at football games. See Figure 8.9b.

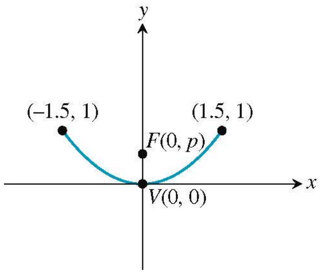


FIGURE 8.10 Cross section of parabolic reflector in Example 6.

EXAMPLE 6 Studying a Parabolic Microphone

On the sidelines of each of its televised football games, the FBTV network uses a parabolic reflector with a microphone at the reflector's focus to capture the conversations among players on the field. If the parabolic reflector is 3 ft across and 1 ft deep, where should the microphone be placed?

SOLUTION

We draw a cross section of the reflector as an upward opening parabola in the Cartesian plane, placing its vertex V at the origin (see Figure 8.10). We let the focus F have coordinates $(0, p)$ to yield the equation

$$x^2 = 4py.$$

Because the reflector is 3 ft across and 1 ft deep, the points $(\pm 1.5, 1)$ must lie on the parabola. The microphone should be placed at the focus, so we need to find the value of p. We do this by substituting the values we found into the equation:

$$x = 4py$$

$$(\pm 1.5)^2 = 4p(1)$$

$$2.25 = 4p$$

$$p = 2.25/4 = 0.5625$$

Because $p = 0.5625$ ft, or 6.75 inches, the microphone should be placed inside the reflector along its axis and 6.75 inches from its vertex.

Now try Exercise 59.

FOLLOW-UP

Ask students to explain why parabolic mirrors are used in flashlights.

ASSIGNMENT GUIDE

Day 1: Ex. 1, 5–19 odds, 31–32, 37, 40, 57
Day 2: Ex. 3, 21–54 multiples of 3, 59, 61

COOPERATIVE LEARNING

Group Activity: Ex. 63–64

NOTES ON EXERCISES

Ex. 59–64 are applications of parabolas.
Ex. 65–70 provide practice for standardized tests.
Ex. 74–76 are quite challenging and could be used for a group project.

ONGOING ASSESSMENT

Self-Assessment: Ex. 1, 15, 21, 45, 49, 59
Embedded Assessment: Ex. 57–58

QUICK REVIEW 8.1 *(For help, go to Sections P.2, P.5, and 2.1.)*

In Exercises 1 and 2, find the distance between the given points.

1. $(-1, 3)$ and $(2, 5)$ $\sqrt{13}$ **2.** $(2, -3)$ and (a, b)

In Exercises 3 and 4, solve for y in terms of x.

3. $2y^2 = 8x$ $y = \pm 2\sqrt{x}$ **4.** $3y^2 = 15x$ $y = \pm\sqrt{5x}$

In Exercises 5 and 6, complete the square to rewrite the equation in vertex form.

5. $y = -x^2 + 2x - 7$ **6.** $y = 2x^2 + 6x - 5$

In Exercises 7 and 8, find the vertex and axis of the graph of f. Describe how the graph of f can be obtained from the graph of $g(x) = x^2$, and graph f.

7. $f(x) = 3(x - 1)^2 + 5$ **8.** $f(x) = -2x^2 + 12x + 1$

In Exercises 9 and 10, write an equation for the quadratic function whose graph contains the given vertex and point.

9. Vertex $(-1, 3)$, point $(0, 1)$ $f(x) = -2(x + 1)^2 + 3$

10. Vertex $(2, -5)$, point $(5, 13)$ $f(x) = 2(x - 2)^2 - 5$