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Mathematical Analysis I

Springer

and ask ourselves whether there are real numbers a and b rendering f differentiable at the origin. The continuity at the origin (recall: differentiable implies continuous) forces the two values

$$\lim_{x\to 0^-} f(x) = -4, \qquad \lim_{x\to 0^+} f(x) = f(0) = -b+1$$

to agree, hence $b = 5$. With b fixed, we may impose the equality of the right and left limits of $f'(x)$ for $x \to 0$, to the effect that $f'(x)$ admits finite limit for $x \to 0$. Then we use Theorem 6.15, which prescribes that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2a\cos 2x = 2a, \qquad \text{and} \qquad \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (5+e^x) = 6$$

are the same, so $a = 3$. □

Remark 6.17 In using Theorem 6.15 one should not forget to impose continuity at the point x_0. The mere existence of the limit for f' is not enough to guarantee f will be differentiable at x_0. For example, $f(x) = x + \operatorname{sign} x$ is differentiable at every $x \neq 0$: since $f'(x) = 1$, it necessarily follows $\lim\limits_{x\to 0} f'(x) = 1$. The function is nonetheless not differentiable, because not continuous, at $x = 0$. □

6.4 Extrema and critical points

Definition 6.18 *One calls* $x_0 \in \operatorname{dom} f$ *a* **relative** (*or* **local**) **maximum point for** f *if there is a neighbourhood* $I_r(x_0)$ *of* x_0 *such that*

$$\forall x \in I_r(x_0) \cap \operatorname{dom} f, \qquad f(x) \le f(x_0).$$

Then $f(x_0)$ *is a* **relative** (*or* **local**) **maximum of** f.
One calls x_0 *an* **absolute maximum point** (*or* **global maximum point**) **for** f *if*

$$\forall x \in \operatorname{dom} f, \qquad f(x) \le f(x_0),$$

and $f(x_0)$ *becomes the* (**absolute**) **maximum of** f. *In either case, the maximum is said* **strict** *if* $f(x) < f(x_0)$ *when* $x \neq x_0$.

Exchanging the symbols $\le$ with $\ge$ one obtains the definitions of **relative** and **absolute minimum point**. A minimum or maximum point shall be referred to generically as an **extremum** (**point**) of f.

Examples 6.19

i) The parabola $f(x) = 1+2x-x^2 = 2-(x-1)^2$ has a strict absolute maximum point at $x_0 = 1$, and 2 is the function's absolute maximum. Notice the derivative $f'(x) = 2(1-x)$ is zero at that point. There are no minimum points (relative or absolute).

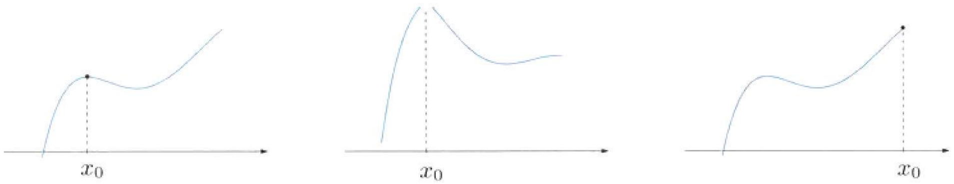

Figure 6.3. Types of maxima

ii) For $g(x) = \arcsin x$ (see Fig. 2.24), $x_0 = 1$ is a strict absolute maximum point, with maximum value $\frac{\pi}{2}$. The point $x_1 = -1$ is a strict absolute minimum, with value $-\frac{\pi}{2}$. At these extrema g is not differentiable. □

We are interested in finding the extremum points of a given function. Provided the latter is differentiable, it might be useful to look for the points where the first derivative vanishes.

Definition 6.20 *A* **critical point** (*or* **stationary point**) *of f is a point x_0 at which f is differentiable with derivative $f'(x_0) = 0$.*

The tangent at a critical point is horizontal.

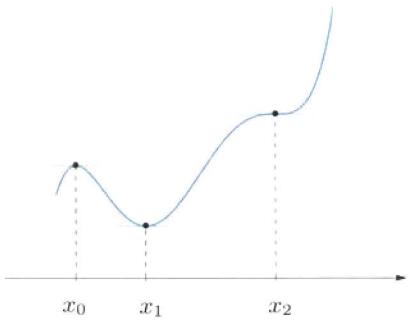

Figure 6.4. Types of critical points

Theorem 6.21 (Fermat) *Suppose f is defined in a full neighbourhood of a point x_0 and differentiable at x_0. If x_0 is an extremum point, then it is critical for f, i.e.,*

$$f'(x_0) = 0.$$

Proof. To fix ideas, assume x_0 is a relative maximum point and that $I_r(x_0)$ is a neighbourhood where $f(x) \leq f(x_0)$ for all $x \in I_r(x_0)$. On such neighbourhood then $\Delta f = f(x) - f(x_0) \leq 0$.

If $x > x_0$, hence $\Delta x = x - x_0 > 0$, the difference quotient $\dfrac{\Delta f}{\Delta x}$ is non-positive. Corollary 4.3 implies

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Vice versa, if $x < x_0$, i.e., $\Delta x < 0$, then $\dfrac{\Delta f}{\Delta x}$ is non-negative, so

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

By Property 6.14,

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0},$$

so $f'(x_0)$ is simultaneously ≤ 0 and ≥ 0, hence zero.
A similar argument holds for relative minima. □

Fermat's Theorem 6.21 ensures that the extremum points of a differentiable map which belong to the *interior* of the domain should be searched for among critical points.

A function can nevertheless have critical points that are not extrema, as in Fig. 6.4. The map $f(x) = x^3$ has the origin as a critical point ($f'(x) = 3x^2 = 0$ if and only if $x = 0$), but admits no extremum since it is strictly increasing on the whole $\mathbb{R}$.

At the same time though, a function may have non-critical extremum point (Fig. 6.3); this happens when a function is not differentiable at an extremum that lies inside the domain (e.g. $f(x) = |x|$, whose absolute minimum is attained at the origin), or when the extremum point is on the boundary (as in Example 6.19 ii)). The upshot is that in order to find *all* extrema of a function, browsing through the critical points might not be sufficient.

To summarise, extremum points are contained among the points of the domain at which either

i) the first derivative vanishes,

ii) or the function is not differentiable,

iii) or among the domain's boundary points (inside $\mathbb{R}$).

6.5 Theorems of Rolle and of the Mean Value

The results we present in this section, called Theorem of Rolle and Mean Value Theorem, are fundamental for the study of differentiable maps on an interval.

Theorem 6.22 (Rolle) *Let f be a function defined on a closed bounded interval $[a, b]$, continuous on $[a, b]$ and differentiable on (a, b) (at least). If $f(a) = f(b)$, there exists an $x_0 \in (a, b)$ such that*

$$f'(x_0) = 0.$$

In other words, f admits at least one critical point in (a, b).

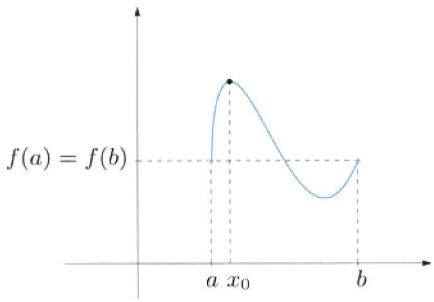

Figure 6.5. Rolle's Theorem

Proof. By the Theorem of Weierstrass the range $f([a, b])$ is the closed interval $[m, M]$ bounded by the minimum and maximum values m, M of the map:

$$m = \min_{x \in [a,b]} f(x) = f(x_m), \qquad M = \max_{x \in [a,b]} f(x) = f(x_M),$$

for suitable $x_m, x_M \in [a, b]$.
In case $m = M$, f is constant on $[a, b]$, so in particular $f'(x) = 0$ for any $x \in (a, b)$ and the theorem follows.
Suppose then $m < M$. Since $m \le f(a) = f(b) \le M$, one of the strict inequalities $f(a) = f(b) < M$, $m < f(a) = f(b)$ will hold.
If $f(a) = f(b) < M$, the absolute maximum point x_M cannot be a nor b; thus, $x_M \in (a, b)$ is an interior extremum point at which f is differentiable. By Fermat's Theorem 6.21 we have that $x_M = x_0$ is a critical point.
If $m < f(a) = f(b)$, one proves analogously that x_m is the critical point x_0 of the claim. □

The theorem proves the existence of one critical point in (a, b); Fig. 6.5 shows that there could actually be more.

Theorem 6.23 (Mean Value Theorem or **Lagrange Theorem)** *Let f be defined on the closed and bounded interval $[a,b]$, continuous on $[a,b]$ and differentiable (at least) on (a,b). Then there is a point $x_0 \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(x_0). \tag{6.9}$$

Every such point x_0 we shall call **Lagrange point for f in (a,b)**.

Proof. Introduce an auxiliary map

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$$

defined on $[a,b]$. It is continuous on $[a,b]$ and differentiable on (a,b), as difference of f and an affine map, which is differentiable on all of $\mathbb{R}$. Note

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

It is easily seen that

$$g(a) = f(a), \qquad g(b) = f(a),$$

so Rolle's Theorem applies to g, with the consequence that there is a point $x_0 \in (a,b)$ satisfying

$$g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0.$$

But this is exactly (6.9). □

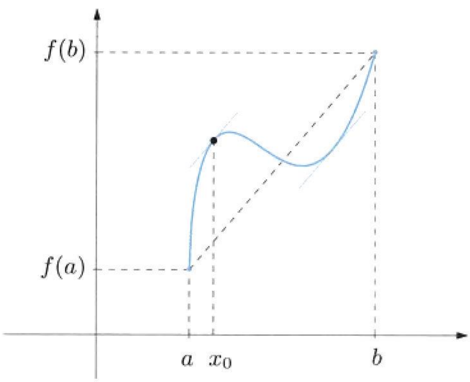

Figure 6.6. Lagrange point for f in (a,b)

The meaning of the Mean Value Theorem is clarified in Fig. 6.6. At each Lagrange point, the tangent to the graph of f is *parallel* to the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.

Example 6.24

Consider $f(x) = 1 + x + \sqrt{1-x^2}$, a continuous map on its domain $[-1, 1]$ as composite of elementary continuous functions. It is also differentiable on the open interval $(-1, 1)$ (not at the end-points), in fact

$$f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}.$$

Thus f fulfills the Mean Value Theorem's hypotheses, and must admit a Lagrange point in $(-1, 1)$. Now (6.9) becomes

$$1 = \frac{f(1) - f(-1)}{1-(-1)} = f'(x_0) = 1 - \frac{x_0}{\sqrt{1-x_0^2}},$$

satisfied by $x_0 = 0$. □

6.6 First and second finite increment formulas

We shall discuss a couple of useful relations to represent how a function varies when passing from one point to another of its domain.

Let us begin by assuming f is differentiable at x_0. By definition

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

that is to say

$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0.$$

Using the Landau symbols of Sect. 5.1, this becomes

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0), \quad x \to x_0.$$

An equivalent formulation is

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0), \quad x \to x_0, \tag{6.10}$$

or

$$\Delta f = f'(x_0)\Delta x + o(\Delta x), \quad \Delta x \to 0, \tag{6.11}$$

by putting $\Delta x = x - x_0$ and $\Delta f = f(x) - f(x_0)$.

Equations (6.10)-(6.11) are equivalent writings of what we call the **first formula of the finite increment** , the geometric interpretation of which can be found in

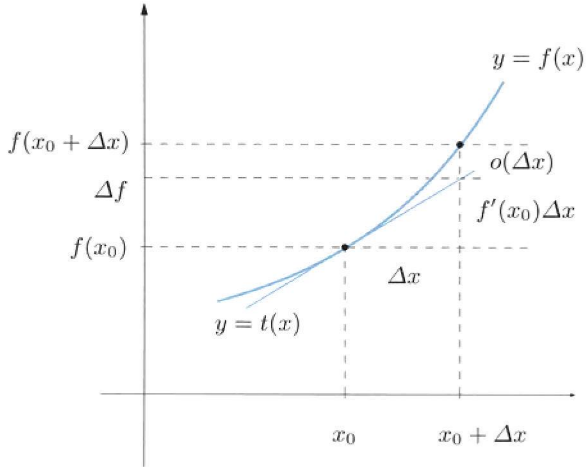

Figure 6.7. First formula of the finite increment

Fig. 6.7. It tells that if $f'(x_0) \neq 0$, the increment Δf, corresponding to a change Δx, is proportional to Δx itself, if one disregards an infinitesimal which is negligible with respect to Δx. For Δx small enough, in practice, Δf can be treated as $f'(x_0)\Delta x$.

Now take f continuous on an interval I of $\mathbb{R}$ and differentiable on the interior points. Fix $x_1 < x_2$ in I and note that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore f, restricted to $[x_1, x_2]$, satisfies the Mean Value Theorem, so there is $\bar{x} \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\bar{x}),$$

that is, a point $\bar{x} \in (x_1, x_2)$ with

$$\boxed{f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1).} \tag{6.12}$$

We shall refer to this relation as the **second formula of the finite increment.** It has to be noted that the point $\bar{x}$ depends upon the choice of x_1 and x_2, albeit this dependency is in general not explicit. The formula's relevance derives from the possibility of gaining information about the increment $f(x_2) - f(x_1)$ from the behaviour of f' on the interval $[x_1, x_2]$.

The second formula of the finite increment may be used to describe the local behaviour of a map in the neighbourhood of a certain x_0 with more precision than that permitted by the first formula. Suppose f is continuous at x_0 and differentiable around x_0 except possibly at the point itself. If x is a point in the neighbourhood of x_0, (6.12) can be applied to the interval bounded by x_0 and x, to the effect that

$$\Delta f = f'(\bar{x})\Delta x, \tag{6.13}$$

where $\bar{x}$ lies between x_0 and x. This alternative formulation of (6.12) expresses the increment of the dependent variable Δf as if it were a multiple of Δx; at closer look though, one realises that the proportionality coefficient, i.e., the derivative evaluated at a point near x_0, depends upon Δx (and on x_0), besides being usually not known.

A further application of (6.12) is described in the next result. This will be useful later.

Property 6.25 *A function defined on a real interval I and everywhere differentiable is constant on I if and only if its first derivative vanishes identically.*

Proof. Let f be the map. Suppose first f is constant, therefore for every $x_0 \in I$, the difference quotient $\dfrac{f(x) - f(x_0)}{x - x_0}$, with $x \in I$, $x \neq x_0$, is zero. Then $f'(x_0) = 0$ by definition of derivative.
Vice versa, suppose f has zero derivative on I and let us prove that f is constant on I. This would be equivalent to demanding

$$f(x_1) = f(x_2), \qquad \forall x_1, x_2 \in I.$$

Take $x_1, x_2 \in I$ and use formula (6.12) on f. For a suitable $\bar{x}$ between x_1, x_2, we have

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1) = 0,$$

thus $f(x_1) = f(x_2)$. □

6.7 Monotone maps

In the light of the results on differentiability, we tackle the issue of monotonicity.

Theorem 6.26 *Let I be an interval upon which the map f is differentiable. Then:*

a) If f is increasing on I, then $f'(x) \geq 0$ for all $x \in I$.

b1) If $f'(x) \geq 0$ for any $x \in I$, then f is increasing on I;

b2) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.

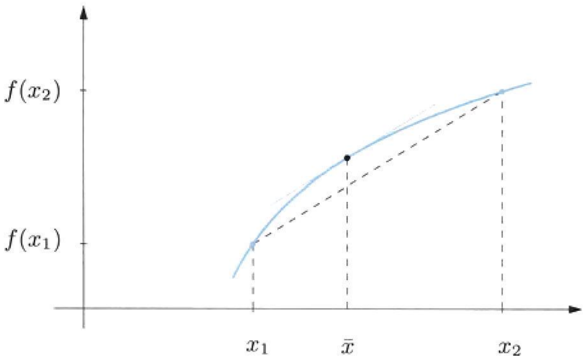

Figure 6.8. Proof of Theorem 6.26, *b*)

Proof. Let us prove claim *a*). Suppose f increasing on I and consider an interior point x_0 of I. For all $x \in I$ such that $x < x_0$, we have

$$f(x) - f(x_0) \le 0 \qquad \text{and} \qquad x - x_0 < 0.$$

Thus, the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x is non-negative. On the other hand, for any $x \in I$ with $x > x_0$,

$$f(x) - f(x_0) \ge 0 \qquad \text{and} \qquad x - x_0 > 0.$$

Here too the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x is positive or zero. Altogether,

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} \ge 0, \qquad \forall x \ne x_0;$$

Corollary 4.3 on

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

yields $f'(x_0) \ge 0$. As for the possible extremum points in I, we arrive at the same conclusion by considering one-sided limits of the difference quotient, which is always ≥ 0.

Now to the implications in parts *b*). Take f with $f'(x) \ge 0$ for all $x \in I$. The idea is to fix points $x_1 < x_2$ in I and prove that $f(x_1) \le f(x_2)$. For that we use (6.12) and note that $f'(\bar{x}) \ge 0$ by assumption. But since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1) \ge 0,$$

proving *b1*). Considering f such that $f'(x) > 0$ for all $x \in I$ instead, (6.12) implies $f(x_2) - f(x_1) > 0$, hence also *b2*) holds. □