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Mathematical Analysis I

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Corollary 4.7 *Let f be a* **bounded** *function around c, i.e., there exist a neighbourhood $I(c)$ and a constant $C > 0$ such that*

$$|f(x)| \leq C, \qquad \forall x \in I(c) \setminus \{c\}. \tag{4.7}$$

Let g be such that

$$\lim_{x \to c} g(x) = 0.$$

Then it follows

$$\lim_{x \to c} f(x)g(x) = 0.$$

Proof. By definition $\lim_{x \to c} g(x) = 0$ if and only if $\lim_{x \to c} |g(x)| = 0$, and (4.7) implies

$$0 \leq |f(x)g(x)| \leq C|g(x)|, \qquad \forall x \in I(c) \setminus \{c\}.$$

The claim follows by applying Theorem 4.5. □

Theorem 4.8 (Second comparison theorem – infinite case) *Let f, g be given functions and*

$$\lim_{x \to c} f(x) = +\infty.$$

If there exists a neighbourhood $I(c)$ of c, where both functions are defined (except possibly at c), such that

$$f(x) \leq g(x), \qquad \forall x \in I(c) \setminus \{c\}, \tag{4.8}$$

then

$$\lim_{x \to c} g(x) = +\infty.$$

A result of the same kind for f holds when the limit of g is $-\infty$.

Proof. The proof is, with the necessary changes, like that of Theorem 4.5, hence left to the reader. □

Example 4.9

Compute the limit of $g(x) = x + \sin x$ when $x \to +\infty$. Using (4.6) we have

$$x - 1 \leq x + \sin x, \qquad \forall x \in \mathbb{R}.$$

Set $f(x) = x - 1$; since $\lim_{x \to +\infty} f(x) = +\infty$, the theorem tells us

$$\lim_{x \to +\infty} (x + \sin x) = +\infty.$$

□

4.1.3 Algebra of limits. Indeterminate forms of algebraic type

This section is devoted to the interaction of limits with the algebraic operations of sum, difference, product and quotient of functions.

First though, we must extend arithmetic operations to treat the symbols $+\infty$ and $-\infty$. Let us set:

$$+\infty + s = +\infty \qquad (\text{if } s \in \mathbb{R} \text{ or } s = +\infty)$$

$$-\infty + s = -\infty \qquad (\text{if } s \in \mathbb{R} \text{ or } s = -\infty)$$

$$\pm\infty \cdot s = \pm\infty \qquad (\text{if } s > 0 \text{ or } s = +\infty)$$

$$\pm\infty \cdot s = \mp\infty \qquad (\text{if } s < 0 \text{ or } s = -\infty)$$

$$\frac{\pm\infty}{s} = \pm\infty \qquad (\text{if } s > 0)$$

$$\frac{\pm\infty}{s} = \mp\infty \qquad (\text{if } s < 0)$$

$$\frac{s}{0} = \infty \qquad (\text{if } s \in \mathbb{R} \setminus \{0\} \text{ or } s = \pm\infty)$$

$$\frac{s}{\pm\infty} = 0 \qquad (\text{if } s \in \mathbb{R})$$

Instead, the following expressions are *not* defined

$$\pm\infty + (\mp\infty), \qquad \pm\infty - (\pm\infty), \qquad \pm\infty \cdot 0, \qquad \frac{\pm\infty}{\pm\infty}, \qquad \frac{0}{0}.$$

A result of the foremost importance comes next.

Theorem 4.10 *Suppose f admits limit ℓ (finite or infinite) and g admits limit m (finite or infinite) for $x \to c$. Then*

$$\lim_{x \to c} \big(f(x) \pm g(x)\big) = \ell \pm m,$$

$$\lim_{x \to c} \big(f(x)\, g(x)\big) = \ell\, m,$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\ell}{m},$$

provided the right-hand-side expressions make sense. (In the last case one assumes $g(x) \neq 0$ on some $I(c)\setminus\{c\}$.)

Proof. We shall prove two relations only, referring the reader to ⇝ **Limits** for the ones left behind. The first we concentrate upon is

$$\lim_{x \to c} \big(f(x) + g(x)\big) = \ell + m$$

when ℓ and m are finite. Fix $\varepsilon > 0$, and consider the neighbourhood of ℓ of radius $\varepsilon/2$. By assumption there is a neighbourhood $I'(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon/2.$$

For the same reason there is also an $I''(c)$ with

$$\forall x \in \operatorname{dom} g, \qquad x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon/2.$$

Put $I(c) = I'(c) \cap I''(c)$. Then if $x \in \operatorname{dom} f \cap \operatorname{dom} g$ belongs to $I(c) \setminus \{c\}$, both inequalities hold; the triangle inequality (1.1) yields

$$\begin{aligned}|(f(x) + g(x)) - (\ell + m)| &= |(f(x) - \ell) + (g(x) - m)| \\ &\le |f(x) - \ell| + |g(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

proving the assertion.

The second relation is

$$\lim_{x \to c} \big(f(x)\, g(x)\big) = +\infty$$

with $\ell = +\infty$ and $m > 0$ finite. For a given real $A > 0$, consider the neighbourhood of $+\infty$ with end-point $B = 2A/m > 0$. We know there is a neighbourhood $I'(c)$ such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) > B.$$

On the other hand, considering the neighbourhood of m of radius $m/2$, there exists an $I''(c)$ such that

$$\forall x \in \operatorname{dom} g, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < m/2,$$

i.e., $m/2 < g(x) < 3m/2$. Set $I(c) = I'(c) \cap I''(c)$. If $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is in $I(c) \setminus \{c\}$, the previous relations will be both fulfilled, whence

$$f(x)\, g(x) > f(x)\,\frac{m}{2} > B\,\frac{m}{2} = A.$$

□

Corollary 4.11 *If f and g are continuous maps at a point $x_0 \in \mathbb{R}$, then also $f(x) \pm g(x)$, $f(x)\, g(x)$ and $\dfrac{f(x)}{g(x)}$ (provided $g(x_0) \neq 0$) are continuous at x_0.*

Proof. The condition that f and g are continuous at x_0 is equivalent to $\lim\limits_{x \to x_0} f(x) = f(x_0)$ and $\lim\limits_{x \to x_0} g(x) = g(x_0)$ (recall (3.9)). The previous theorem allows to conclude. □

Corollary 4.12 *Rational functions are continuous on their domain. In particular, polynomials are continuous on* $\mathbb{R}$.

Proof. We verified in Example 3.17, part i), that the constants $y = a$ and the linear function $y = x$ are continuous on $\mathbb{R}$. Consequently, maps like $y = ax^n$ ($n \in \mathbb{N}$) are continuous. But then so are polynomials, being sums of the latter. Rational functions, as quotients of polynomials, inherit the property wherever the denominator does not vanish. □

Examples 4.13

i) Calculate

$$\lim_{x \to 0} \frac{2x - 3\cos x}{5 + x \sin x} = \ell.$$

The continuity of numerator and denominator descends from algebraic operations on continuous maps, and the denominator is not zero at $x = 0$. The substitution of 0 to x produces $\ell = -3/5$.

ii) Discuss the limit behaviour of $y = \tan x$ when $x \to \frac{\pi}{2}$. Since

$$\lim_{x \to \frac{\pi}{2}} \sin x = \sin \frac{\pi}{2} = 1 \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}} \cos x = \cos \frac{\pi}{2} = 0,$$

the above theorem tells

$$\lim_{x \to \frac{\pi}{2}} \tan x = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} = \frac{1}{0} = \infty.$$

But one can be more precise by looking at the sign of the tangent around $\frac{\pi}{2}$. Since $\sin x > 0$ in a neighbourhood of $\frac{\pi}{2}$, while $\cos x > 0$ (< 0) in a left (resp. right) neighbourhood of $\frac{\pi}{2}$, it follows

$$\lim_{x \to \frac{\pi}{2}^{\pm}} \tan x = \mp\infty.$$

iii) Let $R(x) = \dfrac{P(x)}{Q(x)}$ be rational and reduced, meaning the polynomials P, Q have no common factor. Call $x_0 \in \mathbb{R}$ a zero of Q, i.e., a point such that $Q(x_0) = 0$. Clearly $P(x_0) \neq 0$, otherwise P and Q would be both divisible by $(x - x_0)$. Then

$$\lim_{x \to x_0} R(x) = \infty$$

follows. In this case too, the sign of $R(x)$ around of x_0 retains some information. For instance, $y = \dfrac{x^2 - 3x + 1}{x^2 - x}$ is positive on a left neighbourhood of $x_0 = 1$ and negative on a right neighbourhood, so

$$\lim_{x\to 1^{\pm}} \frac{x^2-3x+1}{x^2-x} = \mp\infty.$$

In contrast, the function $y = \dfrac{x-2}{x^2-2x+1}$ is negative in a whole neighbourhood of $x_0 = 1$, hence

$$\lim_{x\to 1} \frac{x-2}{x^2-2x+1} = -\infty.$$ □

Theorem 4.10 gives no indication about the limit behaviour of an algebraic expression in three cases, listed below. The expressions in question are called **indeterminate forms** of algebraic type.

i) Consider $f(x)+g(x)$ (resp. $f(x)-g(x)$) when both f, g tend to ∞ with different (resp. same) signs. This gives rise to the indeterminate form denoted by the symbol

$$\infty - \infty.$$

ii) The product $f(x)\,g(x)$, when one function tends to ∞ and the other to 0, is the indeterminate form with symbol

$$\infty \cdot 0.$$

iii) Relatively to $\dfrac{f(x)}{g(x)}$, in case both functions tend to ∞ or 0, the indeterminate forms are denoted with

$$\frac{\infty}{\infty} \qquad \text{or} \qquad \frac{0}{0}.$$

In presence of an indeterminate form, the limit behaviour cannot be told a priori, and there are examples for each possible limit: infinite, finite non-zero, zero, even non-existing limit. Every indeterminate form should be treated singularly and requires often a lot of attention.

Later we shall find the actual limit behaviour of many important indeterminate forms. With those and this section's theorems we will discuss more complicated indeterminate forms. Additional tools to analyse this behaviour will be provided further on: they are the local comparison of functions by means of the Landau symbols (Sect. 5.1), de l'Hôpital's Theorem (Sect. 6.11), the Taylor expansion (Sect. 7.1).

Examples 4.14

i) Let x tend to $+\infty$ and define functions $f_1(x) = x + x^2$, $f_2(x) = x+1$, $f_3(x) = x + \frac{1}{x}$, $f_4(x) = x + \sin x$. Set $g(x) = x$. Using Theorem 4.10, or Example 4.9, one verifies easily that all maps tend to $+\infty$. One has

$$\lim_{x\to+\infty} [f_1(x) - g(x)] = \lim_{x\to+\infty} x^2 = +\infty,$$

$$\lim_{x\to+\infty} [f_2(x) - g(x)] = \lim_{x\to+\infty} 1 = 1,$$

$$\lim_{x\to+\infty} [f_3(x) - g(x)] = \lim_{x\to+\infty} \frac{1}{x} = 0,$$

whereas the limit of $f_4(x) - g(x) = \sin x$ does not exist: the function $\sin x$ is periodic and assumes each value between -1 and 1 infinitely many times as $x \to +\infty$.

ii) Consider now $x \to 0$. Let $f_1(x) = x^3$, $f_2(x) = x^2$, $f_3(x) = x$, $f_4(x) = x^2 \sin \frac{1}{x}$, and $g(x) = x^2$. All functions converge to 0 (for f_4 apply Corollary 4.7). Now

$$\lim_{x\to 0} \frac{f_1(x)}{g(x)} = \lim_{x\to 0} x = 0,$$

$$\lim_{x\to 0} \frac{f_2(x)}{g(x)} = \lim_{x\to 0} 1 = 1,$$

$$\lim_{x\to 0} \frac{f_3(x)}{g(x)} = \lim_{x\to 0} \frac{1}{x}, = \infty,$$

but $\dfrac{f_4(x)}{g(x)} = \sin \dfrac{1}{x}$ does not admit limit for $x \to 0$ (Remark 4.19 furnishes a proof of this).

iii) Let us consider a polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0 \qquad (a_n \neq 0)$$

for $x \to \pm\infty$. A function of this sort can give rise to an indeterminate form $\infty - \infty$ according to the coefficients' signs and the degree of the monomials involved. The problem is sorted by factoring out the leading term (monomial of maximal degree) x^n

$$P(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

The part in brackets converges to a_n when $x \to \pm\infty$, so

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n = \infty$$

The sign of the limit is easily found. For instance,

$$\lim_{x\to-\infty} (-5x^3 + 2x^2 + 7) = \lim_{x\to-\infty} (-5x^3) = +\infty.$$

Take now a reduced rational function

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} \qquad (a_n, b_m \neq 0,\ m > 0).$$

When $x \to \pm\infty$, an indeterminate form $\frac{\infty}{\infty}$ arises. With the same technique as before,

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m} = \begin{cases} \infty & \text{if } n > m\,, \\ \dfrac{a_n}{b_m} & \text{if } n = m\,, \\ 0 & \text{if } n < m\,. \end{cases}$$

For example:

$$\lim_{x\to+\infty} \frac{3x^3-2x+1}{x-x^2} = \lim_{x\to+\infty} \frac{3x^3}{-x^2} = -\infty\,,$$

$$\lim_{x\to-\infty} \frac{-4x^5+2x^3-7}{8x^5-x^4+5x} = \lim_{x\to-\infty} \frac{-4x^5}{8x^5} = -\frac{1}{2}\,,$$

$$\lim_{x\to-\infty} \frac{6x^2-x+5}{-x^3+9} = \lim_{x\to-\infty} \frac{6x^2}{-x^3} = 0.$$

iv) The function $y = \dfrac{\sin x}{x}$ becomes indeterminate $\frac{0}{0}$ for $x \to 0$; we proved in part i), Examples 4.6 that y converges to 1. From this, we can deduce the behaviour of $y = \dfrac{1-\cos x}{x^2}$ as $x \to 0$, another indeterminate form of the type $\frac{0}{0}$. In fact,

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2} \cdot \lim_{x\to 0} \frac{1}{1+\cos x}.$$

The fundamental trigonometric equation $\cos^2 x + \sin^2 x = 1$ together with Theorem 4.10 gives

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 1.$$

The same theorem tells also that the second limit is $\frac{1}{2}$, so we conclude

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.}$$

□

With these examples we have taken the chance to look at the behaviour of elementary functions at the boundary points of their domains. For completeness we gather the most significant limits relative to the elementary functions of Sect. 2.6. For explanations ⇝ `Elementary functions.`

$$\lim_{x\to+\infty} x^\alpha = +\infty\,, \qquad \lim_{x\to 0^+} x^\alpha = 0 \qquad \alpha > 0$$

$$\lim_{x\to+\infty} x^\alpha = 0\,, \qquad \lim_{x\to 0^+} x^\alpha = +\infty \qquad \alpha < 0$$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

$$\lim_{x\to+\infty} a^x = +\infty\,, \qquad \lim_{x\to-\infty} a^x = 0 \qquad a > 1$$

$$\lim_{x\to+\infty} a^x = 0\,, \qquad \lim_{x\to-\infty} a^x = +\infty \qquad a < 1$$

$$\lim_{x\to+\infty} \log_a x = +\infty\,, \qquad \lim_{x\to 0^+} \log_a x = -\infty \quad a > 1$$

$$\lim_{x\to+\infty} \log_a x = -\infty\,, \qquad \lim_{x\to 0^+} \log_a x = +\infty \quad a < 1$$

$$\lim_{x\to\pm\infty} \sin x\,, \qquad \lim_{x\to\pm\infty} \cos x\,, \qquad \lim_{x\to\pm\infty} \tan x \qquad \text{do not exist}$$

$$\lim_{x\to(\frac{\pi}{2}+k\pi)^{\pm}} \tan x = \mp\infty\,, \quad \forall k \in \mathbb{Z}$$

$$\lim_{x\to\pm 1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm 1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

4.1.4 Substitution theorem

The so-called Substitution theorem is important in itself for theoretical reasons, besides providing a very useful method to compute limits.

Theorem 4.15 *Suppose a map f admits limit*

$$\lim_{x\to c} f(x) = \ell, \tag{4.9}$$

finite or not. Let g be defined on a neighbourhood of ℓ (excluding possibly the point ℓ) and such that

i) if $\ell \in \mathbb{R}$, g is continuous at ℓ;

ii) if $\ell = +\infty$ or $\ell = -\infty$, the limit $\lim\limits_{y\to\ell} g(y)$ exists, finite or not.

Then the composition $g \circ f$ admits limit for $x \to c$ and

$$\lim_{x\to c} g(f(x)) = \lim_{y\to\ell} g(y). \tag{4.10}$$

Proof. Set $m = \lim\limits_{y\to\ell} g(y)$ (noting that under *i)*, $m = g(\ell)$). Given any neighbourhood $I(m)$ of m, by *i)* or *ii)* there will be a neighbourhood $I(\ell)$ of ℓ such that

$$\forall y \in \operatorname{dom} g, \qquad y \in I(\ell) \quad \Rightarrow \quad g(y) \in I(m).$$

Note that in case *i)* we can use $I(\ell)$ instead of $I(\ell) \setminus \{\ell\}$ because g is continuous at ℓ (recall (3.7)), while ℓ does not belong to $I(\ell)$ for case *ii)*. With such $I(\ell)$, assumption (4.9) implies the existence of a neighbourhood $I(c)$ of c with

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell).$$

Since $x \in \operatorname{dom} g \circ f$ means $x \in \operatorname{dom} f$ plus $y = f(x) \in \operatorname{dom} g$, the previous two implications now give

$$\forall x \in \operatorname{dom} g \circ f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad g(f(x)) \in I(m).$$

But $I(m)$ was arbitrary, so

$$\lim_{x \to c} g(f(x)) = m. \qquad \square$$

Remark 4.16 An alternative condition that yields the same conclusion is the following:

i') if $\ell \in \mathbb{R}$, there is a neighbourhood $I(c)$ of c where $f(x) \neq \ell$ for all $x \neq c$, and the limit $\lim_{y \to \ell} g(y)$ exists, finite or infinite.

The proof is analogous. $\square$

In case $\ell \in \mathbb{R}$ and g is continuous at ℓ (case *i)*), then $\lim_{y \to \ell} g(y) = g(\ell)$, so (4.10) reads

$$\boxed{\lim_{x \to c} g(f(x)) = g(\lim_{x \to c} f(x)).} \tag{4.11}$$

An imprecise but effective way to put (4.11) into words is to say that a continuous function *commutes* (exchanges places) with the symbol of limit.

Theorem 4.15 implies that continuity is inherited by composite functions, as we discuss hereby.

Corollary 4.17 *Let f be continuous at x_0, and define $y_0 = f(x_0)$. Let furthermore g be defined around y_0 and continuous at y_0. Then the composite $g \circ f$ is continuous at x_0.*

Proof. From (4.11)

$$\lim_{x \to x_0} (g \circ f)(x) = g(\lim_{x \to x_0} f(x)) = g(f(x_0)) = (g \circ f)(x_0),$$

which is equivalent to the claim. $\square$

A few practical examples will help us understand how the Substitution theorem and its corollary are employed.

Examples 4.18

i) The map $h(x) = \sin(x^2)$ is continuous on $\mathbb{R}$, being the composition of the continuous functions $f(x) = x^2$ and $g(y) = \sin y$.