SEVENTH EDITION

ADVANCED ENGINEERING MATHEMATICS

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for t real. These parametric equations are

$$x = -1 + 8t,\ y = -1,\ z = 7 - 3t$$

for t real. We obtain P_0 when $t = 0$ and P_1 when $t = 1$. In this example, the y-coordinate of every point on the line is -1, so the line is in the plane $y = -1$.

We may also say that this line consists of all points $(-1 + 8t, -1, 7 - 3t)$ for t real. ♦

SECTION 6.1 PROBLEMS

In each of Problems 1 through 5, compute $\mathbf{F} + \mathbf{G}$, $\mathbf{F} - \mathbf{G}$, $2\mathbf{F}$, $3\mathbf{G}$, and $\| \mathbf{F} \|$.

1. $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{G} = \sqrt{2}\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$
2. $\mathbf{F} = \mathbf{i} - 3\mathbf{k}$, $\mathbf{G} = 4\mathbf{j}$
3. $\mathbf{F} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{G} = \mathbf{i} + 5\mathbf{j} - \mathbf{k}$
4. $\mathbf{F} = \sqrt{2}\mathbf{i} - \mathbf{j} - 6\mathbf{k}$, $\mathbf{G} = 8\mathbf{i} + 2\mathbf{k}$
5. $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{G} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

In each of Problems 6 through 9, find a vector having the given length and in the direction from the first point to the second.

6. 5, (0, 1, 4), (−5, 2, 2)
7. 9, (1, 2, 1), (−4, −2, 3)
8. 12, (−4, 5, 1), (6, 2, −3)
9. 4, (0, 0, 1), (−4, 7, 5)

In each of Problems 10 through 15, find the parametric equations of the line containing the given points.

10. (1, 0, 4), (2, 1, 1)
11. (3, 0, 0), (−3, 1, 0)
12. (2, 1, 1), (2, 1, −2)
13. (0, 1, 3), (0, 0, 1)
14. (1, 0, −4), (−2, −2, 5)
15. (2, −3, 6), (−1, 6, 4)

6.2 The Dot Product

The *dot product* $\mathbf{F} \cdot \mathbf{G}$ of $\mathbf{F}$ and $\mathbf{G}$ is the real number formed by multiplying the two first components, then the two second components, then the two third components, and adding these three numbers. If $\mathbf{F} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{G} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, then

$$\mathbf{F} \cdot \mathbf{G} = a_1 a_2 + b_1 b_2 + c_1 c_2.$$

Again, this dot product is a number, not a vector. For example,

$$(\sqrt{3}\mathbf{i} + 4\mathbf{j} - \pi\mathbf{k}) \cdot (-2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) = -2\sqrt{3} + 24 - 3\pi.$$

The dot product has the following properties.

1. $\mathbf{F} \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{F}$.
2. $(\mathbf{F} + \mathbf{G}) \cdot \mathbf{H} = \mathbf{F} \cdot \mathbf{H} + \mathbf{G} \cdot \mathbf{H}$.
3. $\alpha(\mathbf{F} \cdot \mathbf{G}) = (\alpha\mathbf{F}) \cdot \mathbf{G} = \mathbf{F} \cdot (\alpha\mathbf{G})$.
4. $\mathbf{F} \cdot \mathbf{F} = \| \mathbf{F} \|^2$.
5. $\mathbf{F} \cdot \mathbf{F} = 0$ if and only if $\mathbf{F} = \mathbf{O}$.
6. $\| \alpha\mathbf{F} + \beta\mathbf{G} \|^2 = \alpha^2 \| \mathbf{F} \|^2 + 2\alpha\beta\mathbf{F} \cdot \mathbf{G} + \beta^2 \| \mathbf{G} \|^2$.

Dot products of vectors can be computed using MAPLE and the `DotProduct` command, which is in the `VectorCalculus` package of subroutines. This command also applies to n-dimensional vectors, which are introduced in Section 6.4.

Conclusions (1), (2), and (3) are routine computations. Conclusion (4) is often used in computations. To verify conclusion (4), suppose

$$\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Then

$$\mathbf{F} \cdot \mathbf{F} = a^2 + b^2 + c^2 = \| \mathbf{F} \|^2 .$$

Conclusion (5) follows easily from (4), since $\mathbf{O}$ is the only vector having length 0. For conclusion (6), use conclusions (1) through (4) to write

$$\begin{aligned}\| \alpha\mathbf{F} + \beta\mathbf{G} \|^2 &= (\alpha\mathbf{F} + \beta\mathbf{G}) \cdot (\alpha\mathbf{F} + \beta\mathbf{G}) \\ &= \alpha^2\mathbf{F} \cdot \mathbf{F} + \alpha\beta\mathbf{F} \cdot \mathbf{G} + \alpha\beta\mathbf{G} \cdot \mathbf{F} + \beta^2\mathbf{G} \cdot \mathbf{G} \\ &= \alpha^2 \| \mathbf{F} \|^2 + 2\alpha\beta\mathbf{F} \cdot \mathbf{G} + \beta^2 \| \mathbf{G} \|^2 .\end{aligned}$$

The dot product can be used to find an angle between two vectors. Recall the law of cosines: For the upper triangle of Figure 6.11 with θ being the angle opposite the side of length c, the law of cosines states that

$$a^2 + b^2 - 2ab\cos(\theta) = c^2.$$

Apply this to the vector triangle of Figure 6.11 (lower), which has sides of length $a = \| \mathbf{G} \|$, $b = \| \mathbf{F} \|$, and $c = \| \mathbf{G} - \mathbf{F} \|$. Using property (6) of the dot product, we obtain

$$\begin{aligned}\| \mathbf{G} \|^2 + \| \mathbf{F} \|^2 - 2 \| \mathbf{F} \| \| \mathbf{G} \| \cos(\theta) &= \| \mathbf{G} - \mathbf{F} \|^2 \\ &= \| \mathbf{G} \|^2 + \| \mathbf{F} \|^2 - 2\mathbf{G} \cdot \mathbf{F}.\end{aligned}$$

Assuming that neither $\mathbf{F}$ nor $\mathbf{G}$ is the zero vector, this gives us

$$\cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{G}}{\| \mathbf{F} \| \| \mathbf{G} \|}. \tag{6.1}$$

Since $|\cos(\theta)| \leq 1$ for all θ, equation (6.1) implies the *Cauchy-Schwarz inequality*:

$$|\mathbf{F} \cdot \mathbf{G}| \leq \| \mathbf{F} \| \| \mathbf{G} \| .$$

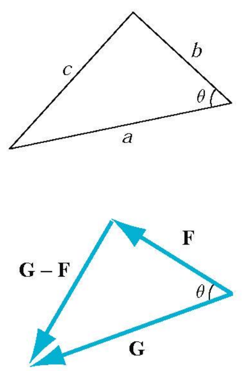

FIGURE 6.11 *The law of cosines and the angle between vectors.*

EXAMPLE 6.2

The angle θ between $\mathbf{F} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{G} = 2\mathbf{j} - 4\mathbf{k}$ is given by

$$\cos(\theta) = \frac{(-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{j} - 4\mathbf{k})}{\| -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \| \| 2\mathbf{j} - 4\mathbf{k} \|}$$

$$= \frac{(-1)(0) + (3)(2) + (1)(-4)}{\sqrt{1^2 + 3^2 + 1^2}\sqrt{2^2 + 4^2}} = \frac{2}{\sqrt{220}}.$$

Then $\theta \approx 1.436$ radians. ♦

EXAMPLE 6.3

Lines L_1 and L_2 have parametric equations

$$L_1 : x = 1 + 6t,\ y = 2 - 4t,\ z = -1 + 3t$$

and

$$L_2 : x = 4 - 3p,\ y = 2p,\ z = -5 + 4p.$$

The parameters t and p can take on any real values. We want an angle θ between these lines.

The strategy is to take a vector $\mathbf{V}_1$ along L_1 and a vector $\mathbf{V}_2$ along L_2 and find the angle between these vectors. For $\mathbf{V}_1$, find two points on L_1, say $(1, 2, -1)$ when $t = 0$ and $(7, -2, 2)$ when $t = 1$, and form

$$\mathbf{V}_1 = (7 - 1)\mathbf{i} + (-2 - 2)\mathbf{j} + (2 - (-1))\mathbf{k} = 6\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}.$$

On L_2, take $(4, 0, -5)$ with $p = 0$ and $(1, 2, -1)$ with $p = 1$, forming

$$\mathbf{V}_2 = 3\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}.$$

Now compute

$$\cos(\theta) = \frac{6(3) - 4(-2) + 3(-4)}{\sqrt{36 + 16 + 9}\sqrt{9 + 4 + 16}} = \frac{14}{\sqrt{1769}}.$$

An angle between L_1 and L_2 is $\arccos(14/\sqrt{1769})$, which is approximately 1.23 radians. ♦

Two nonzero vectors $\mathbf{F}$ and $\mathbf{G}$ are *orthogonal* (perpendicular) when the angle θ between them is $\pi/2$ radians. This happens exactly when

$$\cos(\theta) = 0 = \frac{\mathbf{F} \cdot \mathbf{G}}{\| \mathbf{F} \| \| \mathbf{G} \|}$$

which occurs when $\mathbf{F} \cdot \mathbf{G} = 0$. It is convenient to also agree that $\mathbf{O}$ is orthogonal to every vector. With this convention, two vectors are orthogonal if and only if their dot product is zero.

EXAMPLE 6.4

Let $\mathbf{F} = -4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{G} = 2\mathbf{i} + 4\mathbf{k}$ and $\mathbf{H} = 6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Then $\mathbf{F} \cdot \mathbf{G} = 0$, so $\mathbf{F}$ and $\mathbf{G}$ are orthogonal. But $\mathbf{F} \cdot \mathbf{H}$ and $\mathbf{G} \cdot \mathbf{H}$ are not zero, so $\mathbf{F}$ and $\mathbf{H}$ are not orthogonal and $\mathbf{G}$ and $\mathbf{H}$ are not orthogonal. ♦

Property (6) of the dot product has a particularly simple form when the vectors are orthogonal. In this case, $\mathbf{F} \cdot \mathbf{G} = 0$, and upon setting $\alpha = \beta = 1$, we have

$$\| \mathbf{F} + \mathbf{G} \|^2 = \| \mathbf{F} \|^2 + \| \mathbf{G} \|^2.$$

This is the familiar *Pythagorean theorem*, because the vectors **F** and **G** form the sides of a right triangle with hypotenuse $\mathbf{F}+\mathbf{G}$ (imagine Figure 6.5 with **F** and **G** forming a right angle).

EXAMPLE 6.5

Suppose two lines are defined parametrically by

$$L_1: x=2-4t,\ y=6+t,\ z=3t$$

and

$$L_2: x=-2+p,\ y=7+2p,\ z=3-4p.$$

We want to know if these lines are orthogonal. Note that the question makes sense even if L_1 and L_2 do not intersect.

The idea is to form a vector along each line and test these vectors for orthogonality. For a vector along L_1, take two points on this line, say $(2,6,0)$ when $t=0$ and $(-2,7,3)$ when $t=1$. Then $\mathbf{V}_1=-4\mathbf{i}+\mathbf{j}+3\mathbf{k}$ is parallel to L_1. Similarly, $(-2,7,3)$ is on L_2 when $p=0$, and $(-1,9,-1)$ is on L_2 when $p=1$, so $\mathbf{V}_2=\mathbf{i}+2\mathbf{j}-4\mathbf{k}$ is parallel to L_2. Compute $\mathbf{V}_1\cdot\mathbf{V}_2=-14\neq 0$. Therefore, L_1 and L_2 are not orthogonal. ♦

Orthogonality is also useful for determining the equation of a plane in 3-space. Any plane has an equation of the form

$$ax+by+cz=d.$$

As suggested by Figure 6.12, if we specify a point on the plane and a vector orthogonal to the plane, then the plane is completely determined. Example 6.6 suggests a strategy for finding the equation of this plane.

EXAMPLE 6.6

We will find the equation of the plane Π containing the point $(-6,1,1)$ and orthogonal to the vector $\mathbf{N}=-2\mathbf{i}+4\mathbf{j}+\mathbf{k}$. Such a vector **N** is said to be *normal* to Π and is called a *normal vector* to Π.

Here is a strategy. Because $(-6,1,1)$ is on Π, a point (x,y,z) is on Π exactly when the vector between $(-6,1,1)$ and (x,y,z) lies in Π. But then $(x+6)\mathbf{i}+(y-1)\mathbf{j}+(z-1)\mathbf{k}$ must be orthogonal to **N**, so

$$\mathbf{N}\cdot((x+6)\mathbf{i}+(y-1)\mathbf{j}+(z-1)\mathbf{k})=0.$$

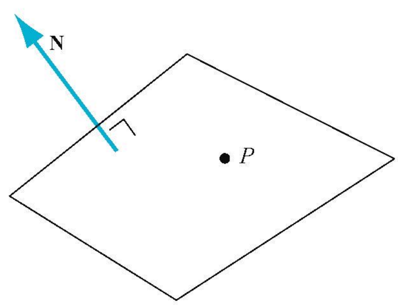

FIGURE 6.12 *A point P and a normal vector* **N** *determine a plane.*

Then

$$-2(x+6)+4(y-1)+(z-1)=0,$$

or

$$-2x+4y+z=17.$$

This is the equation of Π. ♦

Following this reasoning in general, the equation of a plane containing a point $P_0:(x_0, y_0, z_0)$ and having a normal vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is

$$\mathbf{N} \cdot [(x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}] = 0$$

or

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0. \tag{6.2}$$

It is also sometimes convenient to notice that the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is always a normal vector to a plane $ax + by + cz = d$, for any d. Changing the value of d moves the plane in 3-space but does not change its orientation with respect to the axes, so the normal vector remains the same and is determined by the coefficients a, b, and c only.

Another use for the dot product is in forming vector projections.

Let $\mathbf{u}$ and $\mathbf{v}$ be given, nonzero vectors, represented as arrows from a common point (for convenience). The *projection* of $\mathbf{v}$ onto $\mathbf{u}$ is a vector $\text{proj}_{\mathbf{u}}\mathbf{v}$ in the direction of $\mathbf{u}$ having magnitude equal to the length of the perpendicular projection of the arrow representing $\mathbf{v}$ onto the line along the arrow representing $\mathbf{u}$. This projection is done by constructing a perpendicular line from the tip of $\mathbf{v}$ onto the line through $\mathbf{u}$. The base of the right triangle having $\mathbf{v}$ as hypotenuse is the length d of $\text{proj}_{\mathbf{u}}\mathbf{v}$ (Figure 6.13).

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos(\theta) = \frac{d}{\| \mathbf{v} \|}.$$

Then

$$d = \| \mathbf{v} \| \cos(\theta) = \| \mathbf{v} \| \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \|}.$$

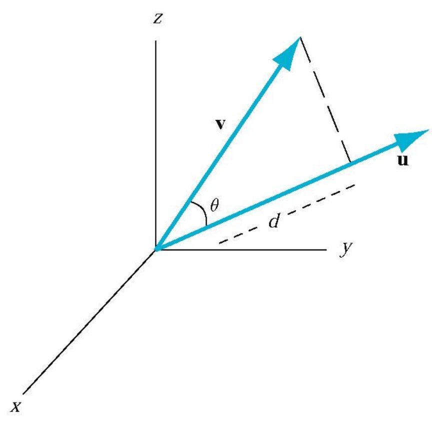

FIGURE 6.13 *Orthogonal projection of* $\mathbf{v}$ *onto* $\mathbf{u}$.

To obtain a vector in the direction of $\mathbf{u}$ and of length d, divide $\mathbf{u}$ by its length to obtain a unit vector, then multiply this vector by d. Therefore,

$$\text{proj}_{\mathbf{u}}\mathbf{v} = d\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

As an example, suppose $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Then

$$\mathbf{u}\cdot\mathbf{v} = 9 \text{ and } \|\mathbf{u}\|^2 = 6,$$

so

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{9}{6}\mathbf{u} = \frac{3}{2}(\mathbf{i} - \mathbf{j} + 2\mathbf{k}).$$

If we think of these vectors as forces, we may interpret $\text{proj}_{\mathbf{u}}\mathbf{v}$ as the effect of $\mathbf{v}$ in the direction of $\mathbf{u}$.

SECTION 6.2 PROBLEMS

In each of Problems 1 through 6, compute the dot product of the vectors and the cosine of the angle between them. Also determine if the vectors are orthogonal.

1. $\mathbf{i}, 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
2. $2\mathbf{i} - 6\mathbf{j} + \mathbf{k}, \mathbf{i} - \mathbf{j}$
3. $-4\mathbf{i} - 2\mathbf{i} + 3\mathbf{k}, 6\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
4. $8\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, -8\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
5. $\mathbf{i} - 3\mathbf{k}, 2\mathbf{j} + 6\mathbf{k}$
6. $\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

In each of Problems 7 through 12, find the equation of the plane containing the given point and orthogonal to the given vector.

7. $(-1, 1, 2), 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
8. $(-1, 0, 0), \mathbf{i} - 2\mathbf{j}$
9. $(2, -3, 4), 8\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$
10. $(-1, -1, -5), -3\mathbf{i} + 2\mathbf{j}$
11. $(0, -1, 4), 7\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$
12. $(-2, 1, -1), 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

In each of Problems 13, 14, and 15, find the projection of $\mathbf{v}$ onto $\mathbf{u}$.

13. $\mathbf{v} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}, \mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
14. $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{u} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
15. $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}, \mathbf{u} = 2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$

6.3 The Cross Product

The dot product produces a scalar from two vectors. The cross product produces a vector from two vectors.

Let $\mathbf{F} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{G} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. The *cross product* of $\mathbf{F}$ with $\mathbf{G}$ is the vector $\mathbf{F} \times \mathbf{G}$ defined by

$$\mathbf{F} \times \mathbf{G} = (b_1c_2 - b_2c_1)\mathbf{i} + (a_2c_1 - a_1c_2)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Here is a simple device for remembering and computing these components. Form the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$