SEVENTH EDITION

ADVANCED ENGINEERING MATHEMATICS

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To obtain a vector in the direction of **u** and of length d, divide **u** by its length to obtain a unit vector, then multiply this vector by d. Therefore,

$$\text{proj}_{\mathbf{u}}\mathbf{v} = d\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

As an example, suppose $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Then

$$\mathbf{u}\cdot\mathbf{v} = 9 \text{ and } \|\mathbf{u}\|^2 = 6,$$

so

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{9}{6}\mathbf{u} = \frac{3}{2}(\mathbf{i} - \mathbf{j} + 2\mathbf{k}).$$

If we think of these vectors as forces, we may interpret $\text{proj}_{\mathbf{u}}\mathbf{v}$ as the effect of **v** in the direction of **u**.

SECTION 6.2 PROBLEMS

In each of Problems 1 through 6, compute the dot product of the vectors and the cosine of the angle between them. Also determine if the vectors are orthogonal.

1. $\mathbf{i}, 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
2. $2\mathbf{i} - 6\mathbf{j} + \mathbf{k}, \mathbf{i} - \mathbf{j}$
3. $-4\mathbf{i} - 2\mathbf{i} + 3\mathbf{k}, 6\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
4. $8\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, -8\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
5. $\mathbf{i} - 3\mathbf{k}, 2\mathbf{j} + 6\mathbf{k}$
6. $\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

In each of Problems 7 through 12, find the equation of the plane containing the given point and orthogonal to the given vector.

7. $(-1, 1, 2), 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
8. $(-1, 0, 0), \mathbf{i} - 2\mathbf{j}$
9. $(2, -3, 4), 8\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$
10. $(-1, -1, -5), -3\mathbf{i} + 2\mathbf{j}$
11. $(0, -1, 4), 7\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$
12. $(-2, 1, -1), 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

In each of Problems 13, 14, and 15, find the projection of **v** onto **u**.

13. $\mathbf{v} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}, \mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
14. $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{u} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
15. $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}, \mathbf{u} = 2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$

6.3 The Cross Product

The dot product produces a scalar from two vectors. The cross product produces a vector from two vectors.

Let $\mathbf{F} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{G} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. The *cross product* of **F** with **G** is the vector $\mathbf{F} \times \mathbf{G}$ defined by

$$\mathbf{F} \times \mathbf{G} = (b_1c_2 - b_2c_1)\mathbf{i} + (a_2c_1 - a_1c_2)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Here is a simple device for remembering and computing these components. Form the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

having the standard unit vectors in the first row, the components of **F** in the second row, and the components of **G** in the third row. If this determinant is expanded by the first row, we get exactly $\mathbf{F} \times \mathbf{G}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \mathbf{k}$$

$$= (b_1 c_2 - b_2 c_1)\mathbf{i} + (a_2 c_1 - a_1 c_2)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$
$$= \mathbf{F} \times \mathbf{G}.$$

The cross product of two 3-vectors can be computed in MAPLE using the `CrossProduct` command, which is part of the `VectorCalculus` package.

We will summarize some properties of the cross product.

1. $\mathbf{F} \times \mathbf{G} = -\mathbf{G} \times \mathbf{F}$.
2. $\mathbf{F} \times \mathbf{G}$ is orthogonal to both **F** and **G**. This is shown in Figure 6.14.
3. $\| \mathbf{F} \times \mathbf{G} \| = \| \mathbf{F} \| \| \mathbf{G} \| \sin(\theta)$ in which θ is the angle between **F** and **G**.
4. If **F** and **G** are nonzero vectors, then $\mathbf{F} \times \mathbf{G} = \mathbf{O}$ if and only if **F** and **G** are parallel.
5. $\mathbf{F} \times (\mathbf{G} + \mathbf{H}) = \mathbf{F} \times \mathbf{G} + \mathbf{F} \times \mathbf{H}$.
6. $\alpha(\mathbf{F} \times \mathbf{G}) = (\alpha\mathbf{F}) \times \mathbf{G} = \mathbf{F} \times (\alpha\mathbf{G})$.

Property (1) of the cross product follows from the fact that interchanging two rows of a determinant changes its sign. In computing $\mathbf{F} \times \mathbf{G}$, the components of **F** are in the second row of the determinant, and those of **G** in the third row. These rows are interchanged in computing $\mathbf{G} \times \mathbf{F}$.

For property (2), compute the dot product

$$\mathbf{F} \cdot (\mathbf{F} \times \mathbf{G})$$
$$= a_1[b_1 c_2 - b_2 c_1] + b_1[a_2 c_1 - a_1 c_2] + c_1[a_1 b_2 - a_2 b_1] = 0.$$

Therefore, **F** is orthogonal to $\mathbf{F} \times \mathbf{G}$. Similarly, **G** is orthogonal to $\mathbf{F} \times \mathbf{G}$.

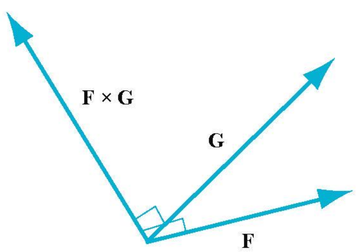

FIGURE 6.14 $\mathbf{F} \times \mathbf{G}$ *is orthogonal to* **F** *and to* **G**.

To derive property (3), suppose both vectors are nonzero and recall that $\cos(\theta) = (\mathbf{F}\cdot\mathbf{G})/\|\mathbf{F}\|\|\mathbf{G}\|$, where θ is the angle between $\mathbf{F}$ and $\mathbf{G}$. Now write

$$\begin{aligned}&\|\mathbf{F}\|^2\|\mathbf{G}\|^2 - (\mathbf{F}\cdot\mathbf{G})^2\\ &= \|\mathbf{F}\|^2\|\mathbf{G}\|^2 - \|\mathbf{F}\|^2\|\mathbf{G}\|^2\cos^2(\theta)\\ &= \|\mathbf{F}\|^2\|\mathbf{G}\|^2\sin^2(\theta).\end{aligned}$$

It is therefore enough to show that

$$\|\mathbf{F}\times\mathbf{G}\|^2 = \|\mathbf{F}\|^2\|\mathbf{G}\|^2 - (\mathbf{F}\cdot\mathbf{G})^2,$$

and this is a tedious but routine calculation.

Property (4) follows from (3), since two nonzero vectors are parallel exactly when the angle θ between them is zero, and in this case, $\sin(\theta) = 0$. Properties (5) and (6) are routine computations.

Property (4) provides a test for three points to be *collinear*, that is, to lie on a single line. Let P, Q, and R be the points. These points will be collinear exactly when the vector $\mathbf{F}$ from P to Q is parallel to the vector $\mathbf{G}$ from P to R. By property (4), this occurs when $\mathbf{F}\times\mathbf{G} = \mathbf{O}$.

One of the primary uses of the cross product is to produce a vector orthogonal to two given vectors. This can be used to find the equation of a plane containing three given points. The strategy is to pick one of the points and write the vectors from this point to the other two. The cross product of these two vectors is normal to the plane containing the points. Now we know a normal vector and a point (in fact three points) on the plane, so we can use equation (6.2) to write the equation of the plane.

This strategy fails if the cross product is zero. But by property (4), this only occurs if the given points are collinear, hence do not determine a unique plane (there are infinitely many planes through any line in 3-space).

EXAMPLE 6.7

Find the equation of a plane containing the points $P : (-1, 4, 2)$, $Q : (6, -2, 8)$, and $R : (5, -1, -1)$.

Use the three given points to form two vectors in the plane:

$$\mathbf{PQ} = 7\mathbf{i} - 6\mathbf{j} + 6\mathbf{k} \text{ and } \mathbf{PR} = 6\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}.$$

The cross product of these vectors is orthogonal to the plane of these vectors, so

$$\mathbf{N} = \mathbf{PQ}\times\mathbf{PR} = 48\mathbf{i} + 57\mathbf{j} + \mathbf{k}$$

is a normal vector. By equation (6.2), the equation of the plane is

$$48(x+1) + 57(y-4) + (z-2) = 0,$$

or

$$48x + 57y + z = 182. \; \blacklozenge$$

SECTION 6.3 PROBLEMS

In each of Problems 1 through 4, compute $\mathbf{F}\times\mathbf{G}$ and $\mathbf{G}\times\mathbf{F}$ and verify the anticommutativity of the cross product.

1. $\mathbf{F} = -3\mathbf{i} + 6\mathbf{j} + \mathbf{k}$, $\mathbf{G} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
2. $\mathbf{F} = 6\mathbf{i} - \mathbf{k}$, $\mathbf{G} = \mathbf{j} + 2\mathbf{k}$
3. $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{G} = -3\mathbf{i} + 2\mathbf{j}$
4. $\mathbf{F} = 8\mathbf{i} + 6\mathbf{j}$, $\mathbf{G} = 14\mathbf{j}$

In each of Problems 5 through 9, determine whether the points are collinear. If they are not, determine an equation for the plane containing these points.

5. $(-1, 1, 6), (2, 0, 1), (3, 0, 0)$
6. $(4, 1, 1), (-2, -2, 3), (6, 0, 1)$
7. $(1, 0, -2), (0, 0, 0), (5, 1, 1)$
8. $(0, 0, 2), (-4, 1, 0), (2, -1, -1)$
9. $(-4, 2, -6), (1, 1, 3), (-2, 4, 5)$

In each of Problems 10, 11, and 12, find a vector normal to the given plane. There are infinitely many such vectors.

10. $8x - y + z = 12$
11. $x - y + 2z = 0$
12. $x - 3y + 2z = 9$
13. Let **F** and **G** be nonparallel vectors and let R be the parallelogram formed by representing these vectors as arrows from a common point. Show that the area of this parallelogram is $\| \mathbf{F} \times \mathbf{G} \|$.
14. Form a parallelepiped (skewed rectangular box) having as incident sides the vectors **F**, **G**, and **H** drawn as arrows from a common point. Show that the volume of this parallelepiped is

$$|\mathbf{F} \cdot (\mathbf{G} \times \mathbf{H})|.$$

This quantity is called the *scalar triple product* of **F**, **G**, and **H**.

6.4 The Vector Space R^n

For systems involving n variables we may consider n-vectors

$$< x_1, x_2, \cdots, x_n >$$

having n components. The jth component of this n-vector is x_j and this is a real number. The totality of such n-vectors is denoted R^n and is called "n-space". R^1 is the real line, consisting of all real numbers. We can think of numbers as 1-vectors, although we do not usually do this. R^2 is the familiar plane, consisting of vectors with two components. And R^3 is in 3-space. R^n has an algebraic structure which will prove useful when we consider matrices, systems of linear algebraic equations, and systems of linear differential equations.

Two n-vectors are equal exactly when their respective components are equal:

$$< x_1, x_2, \cdots, x_n > = < y_1, y_2, \cdots, y_n >$$

if and only if

$$x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n.$$

Add n-vectors, and multiply them by scalars, in the natural ways:

$$< x_1, x_2, \cdots, x_n > + < y_1, y_2, \cdots, y_n > = < x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n >$$

and

$$\alpha < x_1, x_2, \cdots, x_n > = < \alpha x_1, \alpha x_2, \cdots, \alpha x_n > .$$

These operations have the properties we expect of vector addition and multiplication by scalars. If **F**, **G**, and **H** are in R^n and α an β are real numbers, then

1. $\mathbf{F} + \mathbf{G} = \mathbf{G} + \mathbf{F}$.
2. $\mathbf{F} + (\mathbf{G} + \mathbf{H}) = (\mathbf{F} + \mathbf{G}) + \mathbf{H}$.
3. $\mathbf{F} + \mathbf{O} = \mathbf{F}$,