Mavzu: 1-ajoyib limit.
2-ajoyib limit.
Aniqmasliklarni ochish.

Reja

1. 2-ajoyib.
2. 1-ajoyib limit.
3. limitAniqmasliklarni ochish

1-ajoyib limit

Ko’pchilik hollarda limitlarni hisoblash masalasi

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

formula yordamida hal etilishi mumkin. Bu formula 1-ajoyib limitdir.

Birinchi ajoyib limit tushunchasini kiritishdan oldin quyidagi ma`lumotlarni eslash o`rinlidir.

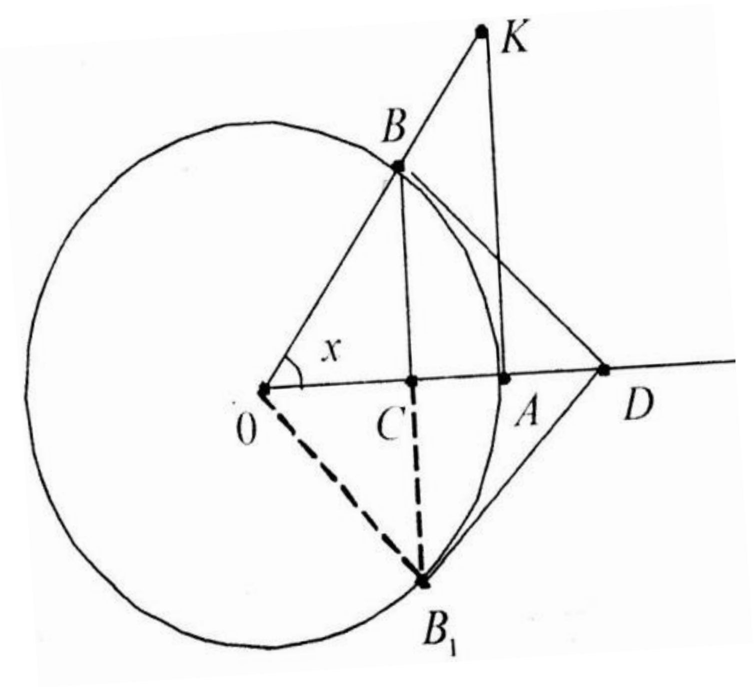
1)Berilgan butun songa teskari son birning shu songa nisbatiga teng. Masalan, a ga teskari son $\frac{1}{a}$ dir. $\frac{a}{b}$ kasrga teskari son $\frac{b}{a}$ ga teng.

2) Agar a va b sonlar $0 < a < b$ tengsizlikni qanoatlantirsa, bu sonlarning teskarisi quyidagi tengsizlikni qanoatlantiradi:

$$\frac{1}{a} > \frac{1}{b}$$

3) Kamayuvchi o`zgarmasdan, ayiruvchi kamaya borsa, ayirma orta boradi. Endi $\frac{\sin x}{x}$ funktsiyani tekshiramiz. Radiusi birga teng bo`lgan birlik aylana olamiz va unda AB yoy ajratamiz. AB yoy tortib turuvchi x burchakni belgilaymiz. B uchidan radiusga perpendikulyar tushirib, kesishish nuqtasini C deb olamiz hamda uni davom ettirib, yoy bilan kesishtiramiz. Kesishish nuqtasini B_1 bilan belgilaymiz.

Ma`lumki, $BC$ - sinus chizig`idir. Shuningdek, AK - tangens chiziqni va BD urinmani ham o`tkazamiz. U holda, $\angle OAK = \angle OBD = 90^0$, $\angle AOB$ - umumiy va $OA = OB = 1$ bo`lganligi uchun $\Delta OAK = \Delta OBD$.



Uchburchaklar tengligidan $BD = AK$, ya`ni $BD$ ning tangens chizig`iga tengligi kelib chiqadi.

Chizmada $B_1B = B_1C + CB = 2CB, \quad B_1B + BD = 2AK$ hamda

$$B_1B = 2\sin x, \quad B_1D + BD = 2tgx \qquad (1)$$

Har qanday vatar o`zini tortib turuvchi yoydan kichik bo`lganligi uchun

$$2\sin x < B_1\breve{A}B = 2x \qquad (2)$$

ekanligi kelib chiqadi. Aylana tashqarisiga chizilgan siniq chiziq uzunligi unga tegishli bo`lgan yoy uzunligidan kattaligi hisobga olinsa, quyidagi o`rinli bo`ladi:

$$B_1D + BD > B_1\breve{A}B \quad \text{yoki} \quad 2tgx > 2x. \tag{3}$$

(2) tengsizlikdan

$$0 < \sin x < x, \tag{4}$$

(3) tengsizlikdan esa

$$tgx > x. \tag{5}$$

(4) va (5) ni birlashtirib, quyidagini hosil qilamiz:

$$0 < \sin x < x < tgx.$$

Bu tengsizlikni $\sin x$ ga bo`lsak, quyidagi hosil bo`ladi:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}. \tag{6}$$

Agar 3) ma`lumotdan foydalansak:

$$1 > \frac{\sin x}{x} > \cos x. \tag{7}$$

Tengsizlikning har bir hadidan 1ni ayramiz. U holda,

$$0 < 1 - \frac{\sin x}{x} < 2 \cdot \sin^2 \frac{x}{2}. \tag{8}$$

(4)dan foydalanib, quyidagini hosil qilamiz:

$$\sin\frac{x}{2} < \frac{x}{2} \quad \text{yoki} \quad \sin^2\frac{x}{2} < \left(\frac{x}{2}\right)^2. \tag{9}$$

Shuning uchun ham (8)dan:

$$0 < 1 - \frac{\sin x}{x} < \frac{x^2}{2}. \qquad (10)$$

x cheksiz kichik son bo`lganligi uchun $\frac{x^2}{2}$ ham cheksiz kichikdir. Bundan $1 - \frac{\sin x}{x}$ ning ham cheksiz kichikligi kelib chiqadi. Demak, x ning nolga yaqinlashishidan $1 - \frac{\sin x}{x}$ ham nolga yaqinlashadi. Buni quyidagicha yozish mumkin: $1 - \frac{\sin x}{x} \to 0$ yoki $\frac{\sin x}{x} \to 1$.

Bundan esa

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \qquad (11)$$

(11)ni quyidagi ko`rinishda ham yozish mumkin:

$$\lim_{x \to 0} \frac{x}{\sin x} = 1. \qquad (12)$$

(11) va (12) tengliklarga ***birinchi ajoyib limit*** deyiladi.

Misol

$$\lim_{x\to 0}\frac{1-cosx}{x^2}=\lim_{x\to 0}\frac{(1-cosx)(1+cosx)}{x(1+cosx)}=$$

$$\lim_{x\to 0}\frac{1-cos^2x}{x^2}\cdot\lim_{x\to 0}\frac{1}{1+cosx}$$

$$\lim_{x\to 0}\frac{sin^2x}{x^2}=\left(\frac{sinx}{x}\right)^2=(\lim_{x\to 0}\frac{sinx}{x})^2=1$$

$$\lim_{x\to 0}\frac{1}{1+cosx}=\frac{1}{2}$$

Demak natija $\frac{1}{2}$ teng

. *Ikkinchi ajoyib limit va «e» soni*

Quyidagi $\{x_n\}$ ketma –ketlikni qaraylik, ya`ni:

$$x_n = \left(1+\frac{1}{n}\right)^n, \quad \left(n=\overline{1,n}\right) \qquad (1)$$

Agar $n=1,\ n=2,\ n=3,\ldots$ bo`lsa,

$$(1+1)^1, \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{3}\right)^3, \ldots, \left(1+\frac{1}{n}\right)^n, \ldots. \qquad (2)$$

(2) ketma –ketlikning yaqinlashishini ko`rsatamiz. Buning uchun $\{x_n\}$ ketma –ketlikning o`suvchi va yuqoridan chegaralanganligini ko`rsatish yetarlidir.

(1) ketma –ketlik uchun Nyuton binomi formulasini qo`llaymiz. U holda:

$$x_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots + \frac{n(n-1)(n-2)\ldots[n-(n-1)]}{n!}\cdot\frac{1}{n^n}.$$

Bundan

$$x_n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right). \qquad (3)$$

n ni $n+1$ bilan almashtirsak (4) hosil bo`ladi:

$$x_{n+1} = 2 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \ldots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{n}{n+1}\right). \qquad (4)$$

(4)dan ko`rinib turibdiki, $0<k<n$ da $\left(1-\frac{k}{n}\right)<\left(1-\frac{k}{n+1}\right)$dir. Shuning uchun $x_n < x_{n+1}$, ya`ni $\{x_n\}$ ketma –ketlik o`suvchi va quyidan chegaralangan. Yuqoridan chegaralanganligini ko`rsatishda (3) ketma –ketlikka murojaat qilamiz. (3)dan ko`rinadiki, har bir qavsning ichi 1 dan kichik. Bundan tashqari, $n>2$ bo`lganda $\frac{1}{n!}<\frac{1}{2^{n-1}}$ ni hisobga olsak, quyidagini hosil qilamiz:

$$x_n < 2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}. \qquad (5)$$

Oxirgi ifoda uchun geometrik progressiya hadlarining yig`indisi formulasini qo`llasak:

$$x_n < 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 3 - \frac{1}{2^{n-1}} < 3 \qquad (6)$$

hosil bo`ladi. Bu esa yuqoridan chegaralanganligidan dalolat beradi.

Demak, $\{x_n\}$ ketma –ketlik o`suvchi va yuqoridan chegaralanganligi uchun u chekli limitga ega bo`ladi. Bunday limitni «e» soni deb qabul qilingan. Uning algebraik ifodasi quyidagicha:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad (7) \text{ yoki } \quad \lim_{n\to 0}(1+n)^{\frac{1}{n}} = e. \qquad (7^1)$$

(7) va (7^1) tengliklarga «e» ***soni*** yoki ***ikkinchi ajoyib limit*** deyiladi. Shuni hisobga olish lozimki, (3) va (6) lardan

$2 \le e \le 3$ (8) ekanligi kelib chiqadi.

«e» soni $2{,}71828\ldots$ga teng bo`lib, u irrasional sondir. (7) ni quyidagi ko`rinishda ham yozish mumkin: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2{,}71828\ldots$

- $\lim_{x\to+\infty}(\frac{x-1}{x+3})^{x-2} = \lim_{x\to+\infty}(\frac{x+3-4}{x+3})^{x-2} =$

- $\lim_{x\to+\infty}(1-\frac{1}{x+3})^{x-2} = \left| \begin{array}{l} \frac{-1}{x+3} = t\,, x+3 = \frac{-1}{t} \\ x = \frac{-1}{t} - 3\,, x \to +\infty \\ \quad t \to 0 \end{array} \right.$

$= \lim_{t\to 0}(1+t)^{\frac{-1}{t}-5} = \varepsilon^{-1} = \frac{1}{\varepsilon}$

Aniqmasliklarni ochish.

limit turli qiymatlarga ega bo`lishi yoki mutlaqo mavjud bo`lmasligi mumkin.

Faraz qilaylik, $x \to 0$ da $\frac{f(x)}{g(x)}$ dagi $f(x)$ va $g(x)$ larning ikkalasi ham bir vaqtning o`zida nolga intilsin. U holda,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad (1)$$

hosil bo`ladi, ammo $\frac{0}{0}$ shakldagi natijani javob sifatida qabul qilib bo`lmaydi.

$x \to \infty$ da ham $\frac{\infty}{\infty}$ nisbat haqida shunday fikrni aytish mumkin:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} \qquad (2)$$

(1) va (2) hollarda $\frac{f(x)}{g(x)}$ nisbatga $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko`rinishlardagi ***aniqmasliklar*** deyiladi. $\frac{0}{0}$ - shaklidagi aniqmasliklarni ochish uchun berilgan kasrning surat va maxrajini ko`paytuvchilarga ajratish va o`xshash hadlarini qisqartirish lozim. Hosil bo`lgan kasrning limiti aniq ifodaga aylanadi.

$\frac{\infty}{\infty}$ - shaklidagi aniqmasliklarni ochish uchun berilgan kasrning surat va maxrajini x –ning eng kata darajasiga bo'linadi, natijada kasrning limiti aniq ifodaga aylanadi.

Bulardan tashqari $0 \cdot \infty$, $\infty - \infty$, 1^{∞} kabi aniqmasliklar ham uchraydi. Bunday aniqmasliklarni ochish uchun yuqoridagi aniqmasliklarga keltiriladi

$$\lim_{x\to-\infty}\frac{x^3+x^2+x}{2x^2-x+3}=\lim_{x\to-\infty}\frac{x^3\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\left(2-\frac{1}{x}+\frac{3}{x^2}\right)}=\lim_{x\to-\infty}\frac{x}{2}=-\infty$$

$$\lim_{x\to 0} x\cdot ctgx=\lim_{x\to\to 0}\frac{xcosx}{sinx}=\lim_{x\to 0} cosx\cdot\lim_{x\to 0}\frac{x}{sinx}=1$$

$$\lim_{x\to 0}\frac{x}{sinx}=1,\quad \lim_{x\to 0} cosx=1$$