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Mathematical Analysis I

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6

Differential calculus

The precise definition of the notion of derivative, studying a function's differentiability and computing its successive derivatives, the use of derivatives to analyse the local and global behaviours of functions are all constituents of Differential Calculus.

6.1 The derivative

We start by defining the derivative of a function.

Let $f : \text{dom} f \subseteq \mathbb{R} \to \mathbb{R}$ be a real function of one real variable, take $x_0 \in \text{dom} f$ and suppose f is defined in a neighbourhood $I_r(x_0)$ of x_0. With $x \in I_r(x_0)$, $x \neq x_0$ fixed, denote by

$$\Delta x = x - x_0$$

the (positive or negative) **increment of the independent variable between** x_0 **and** x, and by

$$\Delta f = f(x) - f(x_0)$$

the corresponding **increment of the dependent variable**. Note that $x = x_0 + \Delta x$, $f(x) = f(x_0) + \Delta f$.

The ratio

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called **difference quotient of** f **between** x_0 **and** x.

In this manner Δf represents the *absolute increment* of the dependent variable f when passing from x_0 to $x_0 + \Delta x$, whereas the difference quotient detects the *rate of increment* (while $\Delta f / f$ is the *relative increment*). Multiplying the difference quotient by 100 we obtain the so-called *percentage increment*. Suppose a rise by $\Delta x = 0.2$ of the variable x prompts an increment $\Delta f = 0.06$ of f; the difference quotient $\frac{\Delta f}{\Delta x}$ equals $0.3 = \frac{30}{100}$, corresponding to a 30% increase.

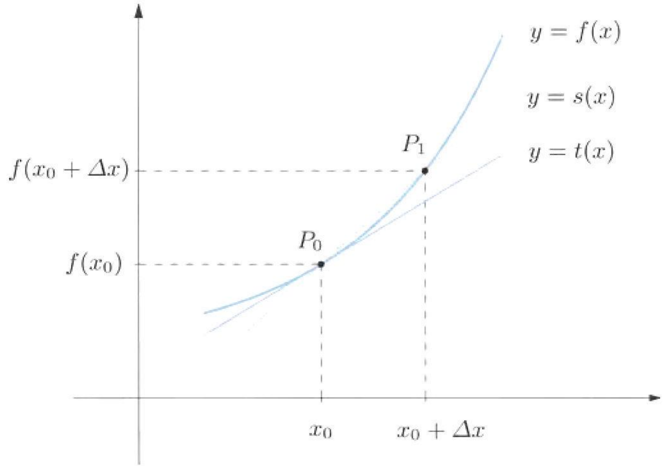

Figure 6.1. Secant and tangent lines to the graph of f at P_0

Graphically, the difference quotient between x_0 and a point x_1 around x_0 is the slope of the straight line s passing through $P_0 = \big(x_0, f(x_0)\big)$ and $P_1 = \big(x_1, f(x_1)\big)$, points that belong to the graph of the function; this line is called **secant** of the graph of f at P_0 and P_1 (Fig. 6.1). Putting $\Delta x = x_1 - x_0$ and $\Delta f = f(x_1) - f(x_0)$, the equation of the secant line reads

$$y = s(x) = f(x_0) + \frac{\Delta f}{\Delta x}(x - x_0), \qquad x \in \mathbb{R}. \tag{6.1}$$

A typical application of the difference quotient comes from physics. Let M be a point-particle moving along a straight line; call $s = s(t)$ the x-coordinate of the position of M at time t, with respect to a reference point O. Between the instants t_0 and $t_1 = t_0 + \Delta t$, the particle changes position by $\Delta s = s(t_1) - s(t_0)$. The difference quotient $\frac{\Delta s}{\Delta t}$ represents the *average velocity* of the particle in the given interval of time.

How does the difference quotient change, as Δx approaches 0? This is answered by the following notion.

Definition 6.1 *A map f defined on a neighbourhood of $x_0 \in \mathbb{R}$ is called* **differentiable at** x_0 *if the limit of the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x exists and is finite, as x approaches x_0. The real number*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called (**first**) **derivative of** f **at** x_0.

The derivative at x_0 is variously denoted, for instance also by

$$y'(x_0), \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(x_0), \qquad \mathrm{D}f(x_0).$$

The first symbol goes back to Newton, the second is associated to Leibniz.

From the geometric point of view $f'(x_0)$ is the slope of the **tangent line** at $P_0 = \big(x_0, f(x_0)\big)$ to the graph of f: such line t is obtained as the limiting position of the secant s at P_0 and $P = \big(x, f(x)\big)$, when P approaches P_0. From (6.1) and the previous definition we have

$$y = t(x) = f(x_0) + f'(x_0)(x - x_0), \qquad x \in \mathbb{R}.$$

In the physical example given above, the derivative $v(t_0) = s'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ is the instantaneous *velocity* of the particle M at time t_0.

Let

$$\mathrm{dom}\, f' = \{x \in \mathrm{dom}\, f \ : \ f \text{ is differentiable at } x\}$$

and define the function $f' : \mathrm{dom}\, f' \subseteq \mathbb{R} \to \mathbb{R},\ f' : x \mapsto f'(x)$ mapping $x \in \mathrm{dom}\, f'$ to the value of the derivative of f at x. This map is called **(first) derivative of** f.

Definition 6.2 *Let I be a subset of* $\mathrm{dom}\, f$. *We say that f is* **differentiable on** I *(or* **in** I*) if f is differentiable at each point of I.*

A first yet significant property of differentiable maps is the following.

Proposition 6.3 *If f is differentiable at x_0, it is also continuous at x_0.*

Proof. Continuity at x_0 prescribes

$$\lim_{x \to x_0} f(x) = f(x_0), \qquad \text{that is} \qquad \lim_{x \to x_0} \big(f(x) - f(x_0)\big) = 0.$$

If f is differentiable at x_0, then

$$\begin{aligned}\lim_{x \to x_0} \big(f(x) - f(x_0)\big) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0}(x - x_0) \\ &= f'(x_0) \cdot 0 = 0.\end{aligned}$$

□

Not all continuous maps at a point are differentiable though. Consider the map $f(x) = |x|$: it is *continuous* at the origin, yet the difference quotient between the origin and a point $x \neq 0$ is

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases} \tag{6.2}$$

so the limit for $x \to 0$ does not exist. Otherwise said, *f is not differentiable* at the origin. This particular example shows that the implication of Proposition 6.3 can not be reversed: differentiability is thus a *stronger* property than continuity, an aspect to which Sect. 6.3 is entirely devoted.

6.2 Derivatives of the elementary functions. Rules of differentiation

We begin by tackling the issue of differentiability for elementary functions using Definition 6.1.

i) Consider the affine map $f(x) = ax + b$, and let $x_0 \in \mathbb{R}$ be arbitrary. Then

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\big(a(x_0 + \Delta x) + b\big) - (ax_0 + b)}{\Delta x} = \lim_{\Delta x \to 0} a = a,$$

in agreement with the fact that the graph of f is a straight line of slope a. The derivative of $f(x) = ax + b$ is then the constant map $f'(x) = a$.

In particular if f is constant ($a = 0$), its derivative is identically zero.

ii) Take $f(x) = x^2$ and $x_0 \in \mathbb{R}$. Since

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = \lim_{\Delta x \to 0} (2x_0 + \Delta x) = 2x_0,$$

the derivative of $f(x) = x^2$ is the function $f'(x) = 2x$.

iii) Now let $f(x) = x^n$ with $n \in \mathbb{N}$. The binomial formula (1.13) yields

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x_0^n + nx_0^{n-1}\Delta x + \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^k - x_0^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(nx_0^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^{k-1} \right) = nx_0^{n-1}. \end{aligned}$$

for all $x_0 \in \mathbb{R}$. Therefore, $f'(x) = nx^{n-1}$ is the derivative of $f(x) = x^n$.

iv) Even more generally, consider $f(x) = x^\alpha$ where $\alpha \in \mathbb{R}$, and let $x_0 \neq 0$ be a point of the domain. Then

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^\alpha - x_0^\alpha}{\Delta x} = \lim_{\Delta x \to 0} \frac{x_0^\alpha \left[\left(1 + \frac{\Delta x}{x_0}\right)^\alpha - 1\right]}{\Delta x}$$
$$= x_0^{\alpha - 1} \lim_{\Delta x \to 0} \frac{\left(1 + \frac{\Delta x}{x_0}\right)^\alpha - 1}{\frac{\Delta x}{x_0}}.$$

Substituting $y = \frac{\Delta x}{x_0}$ brings the latter into the form of the fundamental limit (4.13), so

$$f'(x_0) = \alpha x_0^{\alpha-1}.$$

When $\alpha > 1$, f is differentiable at $x_0 = 0$ as well, and $f'(0) = 0$. The function $f(x) = x^\alpha$ is thus differentiable at all points where the expression $x^{\alpha-1}$ is well defined; its derivative is $f'(x) = \alpha x^{\alpha-1}$.

For example $f(x) = \sqrt{x} = x^{1/2}$, defined on $[0, +\infty)$, is differentiable on $(0, +\infty)$ with derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$. The function $f(x) = \sqrt[3]{x^5} = x^{5/3}$ is defined on $\mathbb{R}$, where it is also differentiable, and $f'(x) = \frac{5}{3}x^{2/3} = \frac{5}{3}\sqrt[3]{x^2}$.

v) Now consider the trigonometric functions. Take $f(x) = \sin x$ and $x_0 \in \mathbb{R}$. Formula (2.14) gives

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x} = \lim_{\Delta x \to 0} \frac{2 \sin \frac{\Delta x}{2} \cos(x_0 + \frac{\Delta x}{2})}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \lim_{\Delta x \to 0} \cos\left(x_0 + \frac{\Delta x}{2}\right).$$

The limit (4.5) and the cosine's continuity tell

$$f'(x_0) = \cos x_0.$$

Hence the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$.

Using in a similar way formula (2.15), we can see that the derivative of $f(x) = \cos x$ is the function $f'(x) = -\sin x$.

vi) Eventually, consider the exponential function $f(x) = a^x$. By (4.12) we have

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{a^{x_0 + \Delta x} - a^{x_0}}{\Delta x} = a^{x_0} \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = a^{x_0} \log a,$$

showing that the derivative of $f(x) = a^x$ is $f'(x) = (\log a)a^x$.

As $\log \mathrm{e} = 1$, the derivative of $f(x) = \mathrm{e}^x$ is $f'(x) = \mathrm{e}^x = f(x)$, whence the derivative f' coincides at each point with the function f itself. This is a crucial fact, and a reason for choosing e as privileged base for the exponential map.

We next discuss differentiability in terms of operations (algebraic operations, composition, inversion) on functions. We shall establish certain *differentiation rules* to compute derivatives of functions that are built from the elementary ones, without resorting to the definition each time. The proofs may be found at ⇝ `Derivatives`.

Theorem 6.4 (Algebraic operations) *Let $f(x), g(x)$ be differentiable maps at $x_0 \in \mathbb{R}$. Then the maps $f(x) \pm g(x)$, $f(x)g(x)$ and, if $g(x_0) \neq 0$, $\frac{f(x)}{g(x)}$ are differentiable at x_0. To be precise,*

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0), \tag{6.3}$$

$$(f\,g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0), \tag{6.4}$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}. \tag{6.5}$$

Corollary 6.5 ('Linearity' of the derivative) *If $f(x)$ and $g(x)$ are differentiable at $x_0 \in \mathbb{R}$, the map $\alpha f(x) + \beta g(x)$ is differentiable at x_0 for any $\alpha, \beta \in \mathbb{R}$ and*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0). \tag{6.6}$$

Proof. Consider (6.4) and recall that differentiating a constant gives zero; then $(\alpha f)'(x_0) = \alpha f'(x_0)$ and $(\beta g)'(x_0) = \beta g'(x_0)$ follow. The rest is a consequence of (6.3). □

Examples 6.6

i) To differentiate a polynomial, we use the fact that $D\,x^n = nx^{n-1}$ and apply the corollary repeatedly. So, $f(x) = 3x^5 - 2x^4 - x^3 + 3x^2 - 5x + 2$ differentiates to

$$f'(x) = 3 \cdot 5x^4 - 2 \cdot 4x^3 - 3x^2 + 3 \cdot 2x - 5 = 15x^4 - 8x^3 - 3x^2 + 6x - 5.$$

ii) For rational functions, we compute the numerator and denominator's derivatives and then employ rule (6.5), to the effect that

$$f(x) = \frac{x^2 - 3x + 1}{2x - 1}$$

has derivative

$$f'(x) = \frac{(2x-3)(2x-1) - (x^2 - 3x + 1)2}{(2x-1)^2} = \frac{2x^2 - 2x + 1}{4x^2 - 4x + 1}.$$

iii) Consider $f(x) = x^3 \sin x$. The product rule (6.4) together with $(\sin x)' = \cos x$ yield

$$f'(x) = 3x^2 \sin x + x^3 \cos x.$$

iv) The function

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

can be differentiated with (6.5)

$$f'(x) = \frac{\cos x\, \cos x - \sin x\,(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x.$$

Another possibility is to use $\cos^2 x + \sin^2 x = 1$ to obtain

$$f'(x) = \frac{1}{\cos^2 x}.$$

□

Theorem 6.7 (**"Chain rule"**) *Let $f(x)$ be differentiable at $x_0 \in \mathbb{R}$ and $g(y)$ a differentiable map at $y_0 = f(x_0)$. Then the composition $g \circ f(x) = g\big(f(x)\big)$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0) = g'\big(f(x_0)\big) f'(x_0). \tag{6.7}$$

Examples 6.8

i) The map $h(x) = \sqrt{1 - x^2}$ is the composite of $f(x) = 1 - x^2$, whose derivative is $f'(x) = -2x$, and $g(y) = \sqrt{y}$, for which $g'(y) = \dfrac{1}{2\sqrt{y}}$. Then (6.7) directly gives

$$h'(x) = \frac{1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

ii) The function $h(x) = \mathrm{e}^{\cos 3x}$ is composed by $f(x) = \cos 3x$, $g(y) = \mathrm{e}^y$. But $f(x)$ is in turn the composite of $\varphi(x) = 3x$ and $\psi(y) = \cos y$; thus (6.7) tells $f'(x) = -3 \sin 3x$. On the other hand $g'(y) = \mathrm{e}^y$. Using (6.7) once again we conclude

$$h'(x) = -3\mathrm{e}^{\cos 3x} \sin 3x.$$

□

Theorem 6.9 (**Derivative of the inverse function**) *Suppose $f(x)$ is a continuous, invertible map on a neighbourhood of $x_0 \in \mathbb{R}$, and differentiable at x_0, with $f'(x_0) \neq 0$. Then the inverse map $f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}. \tag{6.8}$$

Examples 6.10

i) The function $y = f(x) = \tan x$ has derivative $f'(x) = 1 + \tan^2 x$ and inverse $x = f^{-1}(y) = \arctan y$. By (6.8)

$$(f^{-1})'(y) = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}.$$

Setting for simplicity $f^{-1} = g$ and denoting the independent variable with x, the derivative of $g(x) = \arctan x$ is the function $g'(x) = \dfrac{1}{1 + x^2}$.

ii) We are by now acquainted with the function $y = f(x) = \sin x$: it is invertible on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, namely $x = f^{-1}(y) = \arcsin y$. Moreover, f differentiates to $f'(x) = \cos x$. Using $\cos^2 x + \sin^2 x = 1$, and taking into account that on that interval $\cos x \geq 0$, one can write the derivative of f in the equivalent form $f'(x) = \sqrt{1 - \sin^2 x}$. Now (6.8) yields

$$(f^{-1})'(y) = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}.$$

Put once again $f^{-1} = g$ and change names to the variables: the derivative of $g(x) = \arcsin x$ is $g'(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

In similar fashion $g(x) = \arccos x$ differentiates to $g'(x) = -\dfrac{1}{\sqrt{1 - x^2}}$.

iii) Consider $y = f(x) = a^x$. It has derivative $f'(x) = (\log a)a^x$ and inverse $x = f^{-1}(y) = \log_a y$. The usual (6.8) gives

$$(f^{-1})'(y) = \frac{1}{(\log a)a^x} = \frac{1}{(\log a)y}.$$

Defining $f^{-1} = g$ and renaming x the independent variable gives $g'(x) = \dfrac{1}{(\log a)x}$ as derivative of $g(x) = \log_a x$ $(x > 0)$.

Take now $h(x) = \log_a(-x)$ (with $x < 0$), composition of $x \mapsto -x$ and $g(y)$: then $h'(x) = \dfrac{1}{(\log a)(-x)}(-1) = \dfrac{1}{(\log a)x}$. Putting all together shows that $g(x) = \log_a |x|$ $(x \neq 0)$ has derivative $g'(x) = \dfrac{1}{(\log a)x}$.

With the choice of base $a = \mathrm{e}$ the derivative of $g(x) = \log |x|$ is $g'(x) = \dfrac{1}{x}$. □

Remark 6.11 Let $f(x)$ be differentiable and strictly positive on an interval I. Due to the previous result and the Chain rule, the derivative of the composite map $g(x) = \log f(x)$ is

$$g'(x) = \frac{f'(x)}{f(x)}.$$

The expression $\dfrac{f'}{f}$ is said **logarithmic derivative** of the map f. □

The section ends with a useful corollary to the Chain rule 6.7.

Property 6.12 *If f is an even (or odd) differentiable function on all its domain, the derivative f' is odd (resp. even).*

Proof. Since f is even, $f(-x) = f(x)$ for any $x \in \operatorname{dom} f$. Let us differentiate both sides. As $f(-x)$ is the composition of $x \mapsto -x$ and $y \mapsto f(y)$, its derivative reads $-f'(-x)$. Then $f'(-x) = -f'(x)$ for all $x \in \operatorname{dom} f$, so f' is odd. Similarly if f is odd. □

We reckon it could be useful to collect the derivatives of the main elementary functions in one table, for reference.

$$\mathrm{D}\, x^\alpha = \alpha x^{\alpha-1} \qquad (\forall \alpha \in \mathbb{R})$$

$$\mathrm{D}\, \sin x = \cos x$$

$$\mathrm{D}\, \cos x = -\sin x$$

$$\mathrm{D}\, \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\mathrm{D}\, \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arctan x = \frac{1}{1+x^2}$$

$$\mathrm{D}\, a^x = (\log a)\, a^x \qquad \text{in particular,} \quad \mathrm{D}\, \mathrm{e}^x = \mathrm{e}^x$$

$$\mathrm{D}\, \log_a |x| = \frac{1}{(\log a)\, x} \qquad \text{in particular,} \quad \mathrm{D}\, \log |x| = \frac{1}{x}$$

6.3 Where differentiability fails

It was noted earlier that the function $f(x) = |x|$ is continuous but not differentiable at the origin. At each other point of the real line f is differentiable, for it coincides with the line $y = x$ when $x > 0$, and with $y = -x$ for $x < 0$. Therefore $f'(x) = +1$