Федеральное государственное образовательное бюджетное учреждение высшего образования

«ФИНАНСОВЫЙ УНИВЕРСИТЕТ ПРИ ПРАВИТЕЛЬСТВЕ РОССИЙСКОЙ ФЕДЕРАЦИИ»

(Финансовый университет)

Кафедра «Математика-1»

ЛИНЕЙНАЯ АЛГЕБРА: ТЕКСТЫ ЛЕКЦИЙ НА АНГЛИЙСКОМ ЯЗЫКЕ

О.Е. Орел

Для студентов,
обучающихся по направлению
38.03.01 «Экономика»
(программа подготовки бакалавра)

Москва 2015

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Пособие содержит курс лекций, читаемый студентам Международного финансового факультета Финуниверситета на английском языке. Пособие охватывает следующие вопросы дисциплины: системы линейных уравнений, матрицы и определители, векторная алгебра, линейные преобразования, комплексные числа, квадратичные формы, элементы аналитической геометрии, выпуклые множества. Пособие снабжено списком рекомендуемой литературы на русском и английском языках.

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Содержание

Введение

Пособие содержит курс лекций, читаемый студентам Международного финансового факультета Финуниверситета на английском языке. Пособие охватывает следующие вопросы дисциплины: системы линейных уравнений, матрицы и определители, векторная алгебра, линейные преобразования, комплексные числа, квадратичные формы, элементы аналитической геометрии, выпуклые множества. Весь теоретический материал сопровождается примерами, иллюстрирующими основные положения теории. Пособие снабжено списком рекомендуемой литературы на русском и английском языках.

Перед студентами, изучающими курс линейной алгебры на Международном финансовом факультете на английском языке, ставится нелегкая задача: освоить данный, очень непростой предмет, сразу на двух языках. А это означает не только научиться понимать и использовать соответствующую специальную терминологию на языке, но и приспособиться к западному стилю подачи материала. Последнее особенно актуально для студентов, желающих продолжать обучение за рубежом. Поэтому все основные термины снабжены переводами на русский язык, приводятся русскоязычные названия теорем, а также при подаче материала порой отдается предпочтение подходу, принятому в зарубежном курсе “Linear Algebra”.

FINANCIAL UNIVERSITY
UNDER THE GOVERNMENT OF THE RUSSIAN FEDERATION

Linear Algebra

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Contents of the Course

- Systems of Linear Equations
- Matrices and Determinants
- Vector Spaces
- Polynomials
- Complex Numbers
- Linear Transformations
- Bilinear and Quadratic Forms
- Analytic Geometry (Lines, Planes, and Conics)
- Convex Sets

Lecture 1

Solving Systems of Linear Equations

Basic Terminology

- ***System of Linear Equations* (*система линейных уравнений*)** or ***Linear System:*** a finite set of linear equations.
- ***Solution:*** numbers $s_1, s_2, \dots, s_n$ such that every equation of the system is satisfied when we substitute $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$.
- ***Solution Set* (*множество решений*)** or ***General Solution* (*общее решение*):** the set of all solutions of the system.

- An ***inconsistent* (*несовместная*)** system has no solutions
- A ***consistent* (*совместная*)** system has at least one solution

EXAMPLE of an inconsistent system:

$$\begin{cases} x + y = 4 \\ x + y = 3 \end{cases}$$

Every system of linear equations

- *has no solutions, or*
- *has exactly one solution, or*
- *has infinitely many solutions.*

An arbitrary system of *m* linear equations in *n* unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

constant terms

where

- $x_1, x_2, \ldots, x_n$ are the *unknowns*
- *a*'s and *b*'s are *constants*

EXAMPLE of a general system of 3 equations in 4 unknowns (общая система из 3-х уравнений с 4-мя неизвестными):

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \end{cases}$$

$$a_{ij}$$

number of the equation

number of the unknown

Terminology

- **Augmented Matrix (расширенная матрица):**

$$(A|\mathbf{b}) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

- **Coefficient Matrix (матрица коэффициентов):**

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

entry (элемент матрицы) ← a_{1n}

List of operations to eliminate unknowns:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another.

The corresponding operations on the rows of the augmented matrix (*elementary row operations*):

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

EXAMPLE

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Add
- -2 x the 1st row to the 2nd
- -3 x the 1st row to the 3rd

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

Multiply the 2nd row by 1/2

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

Add
- -1 x the 2nd row to the 1st
- -3 x the 2nd row to the 3rd

$$\left(\begin{array}{ccc|c} 1 & 0 & 11/2 & 35/2 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{array}\right)$$

Multiply the 2nd row by -2

$$\left(\begin{array}{ccc|c} 1 & 0 & 11/2 & 35/2 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Add
- -11/2 x the 3RD row to the 1st
- 7/2 x the 3RD row to the 2nd

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Solution: $x = 1$, $y = 2$, $z = 3$.

Row-Echelon Form of a Matrix

DEFINITION A row of a matrix has k ***leading zeros*** if the first k elements of the row are zeros and the $(k + 1)$th element of the row is not zero.

EXAMPLE

$$(\underbrace{0 \quad 0 \quad 0 \quad 0}_{\text{four leading zeros}} \quad 4 \quad 3 \quad 0 \quad -1 \quad 2)$$

DEFINITION A matrix is in ***row-echelon form*** (ступенчатый вид) if each row has more leading zeros than the row preceding it.

DEFINITION The first nonzero entry in each row of a matrix in row-echelon form is called a ***pivot*** (разрешающий элемент).

EXAMPLE

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 0.8 & -0.2 & 100 \\ 0 & 0 & 0.7 & 210 \end{array}\right)$$

pivots

Reduced Row-Echelon Form

DEFINITION A row-echelon matrix in which

- each pivot is a 1 and
- each column containing a pivot contains no other nonzero entries

is said to be in ***reduced row-echelon form*** (приведенный ступенчатый вид).

EXAMPLES

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solving Linear Systems

EXAMPLE 1

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right) \rightarrow \begin{cases} x = 5 \\ y = -2 \\ z = 4 \end{cases}$$

EXAMPLE 2

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right)$$

$$\begin{cases} x_1 \qquad\qquad + 4x_4 = -1 \\ \quad x_2 \qquad + 2x_4 = 6 \\ \qquad\quad x_3 + 3x_4 = 2 \end{cases} \rightarrow \begin{cases} x_1 = -1 - 4x_4 \\ x_2 = 6 - 2x_4 \\ x_3 = 2 - 3x_4 \end{cases}$$

- x_1, x_2, and x_3 are ***pivots*** (базисные переменные)
- x_4 is a ***free variable*** (свободная переменная)

General solution:

$x_1 = -1 - 4t,\ x_2 = 6 - 2t,\ x_3 = 2 - 3t,\ x_4 = t,$

or $\mathbf{x}_G = (-1 - 4t, 6 - 2t, 2 - 3t, t),\ t \in R.$

Basis solution:

$$t = 0 \ \rightarrow \ \mathbf{x}_B = (-1, 6, 2, 0).$$

EXAMPLE 3

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

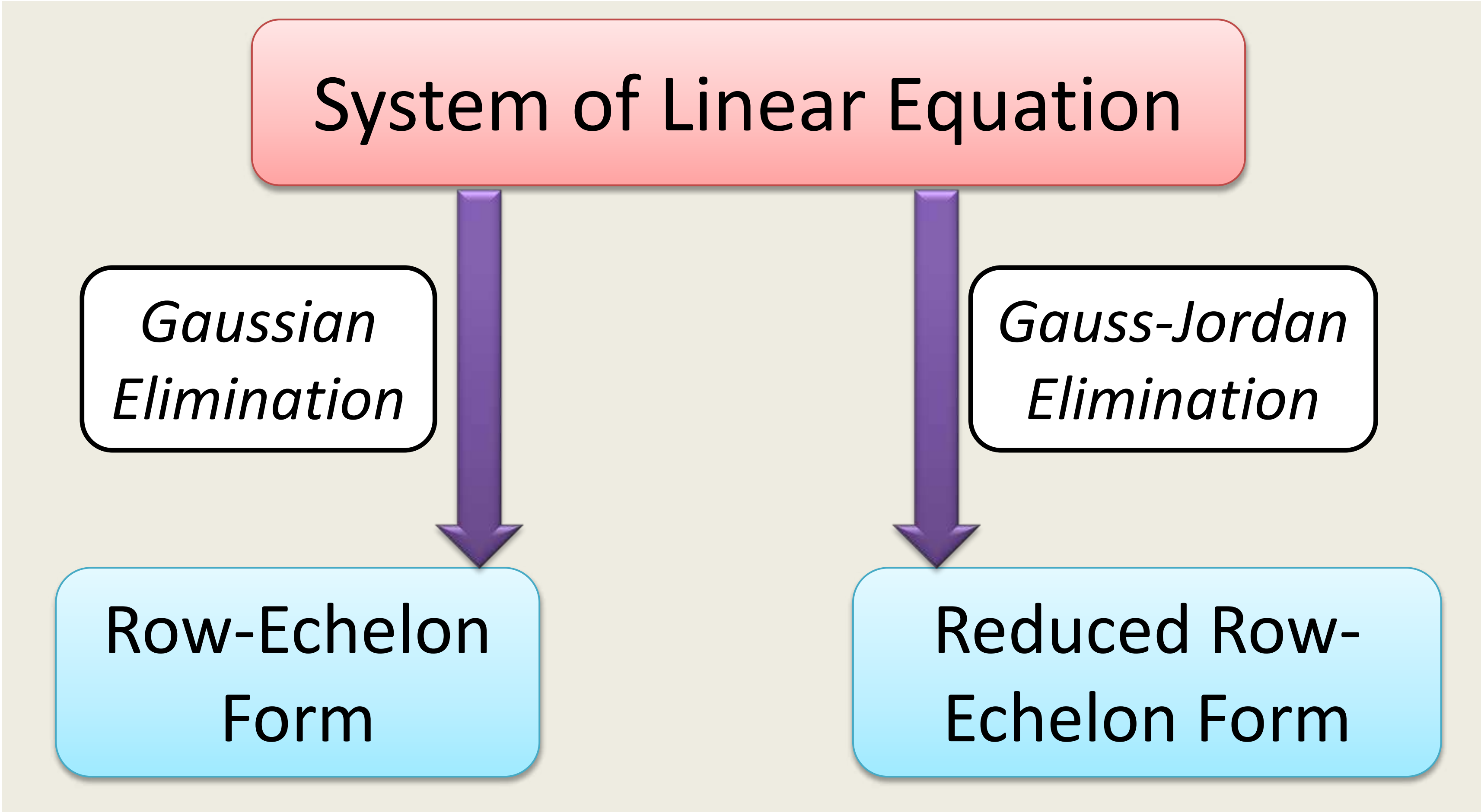
The last equation is

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1.$$

This equation cannot be satisfied $\Rightarrow$ there is no solution to the system.

The system is **inconsistent**!

Elimination Methods



Step-by-step Elimination: Example and Algorithm

Step 1. Locate the leftmost column that does not consist entirely of zeros:

$$\left(\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right)$$

↑ *Leftmost nonzero column*

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column:

$$\left(\begin{array}{ccccc|c} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right) \longleftarrow$$

The first and the second rows in the preceding matrix were interchanged

Step 3. If the entry that is now at the top of the column is *a*, multiply the first row by 1/*a* in order to get 1:

$$\left(\begin{array}{ccccc|c} \textcircled{1} & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right) \longleftarrow$$

The first row of the preceding matrix was multiplied by 1/2

Step 4. Add suitable multiples of the top row to the rows below so that all entries below and above the pivot becomes zero:

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right)$$

← -2 times the first row of the preceding matrix was added to the third row

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix. Continue in this way until the *entire* matrix is in row-echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3.5 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

↑ *Leftmost nonzero column in the submatrix*

Reduced row-echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Corresponding system:

$$\begin{cases} x_1 + 2x_2 \quad + 3x_4 & = & 7 \\ \qquad\qquad x_3 & = & 1 \\ \qquad\qquad\qquad x_5 & = & 2 \end{cases} \rightarrow \begin{cases} x_1 & = & 7 - 2x_2 - 3x_4 \\ x_3 & = & 1 \\ x_5 & = & 2 \end{cases}$$

General solution:

$\mathbf{x}_G = (7 - 2t - 3s, t, 1, s, 2),\ t, s \in R.$

Lecture 2

Matrices and Matrix Operations

Homogeneous Systems

DEFINITION A system of linear equations is said to be ***homogeneous (однородная)*** if the constant terms are all zero:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \qquad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Properties of Homogeneous Systems

- Every homogeneous system of linear equations is consistent.
- There are only two possibilities for its solutions:
  - The system has only ***trivial solution***: $x_1 = 0, x_2 = 0, \ldots, x_n = 0.$
  - The system has ***nontrivial solutions*** in addition to the trivial solution.

THEOREM 1 *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.*

Matrix: Rectangular Array Of Numbers

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{is} \quad m \times n \quad \text{matrix}$$

m is the number of rows

n is the number of columns

EXAMPLE

$$\begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & -1 \end{pmatrix} \quad \text{is a } 2 \times 3 \text{ matrix}$$

Notation: $a_{ij} = (A)_{ij}$

Special Matrices

- a ***column matrix*** (or a ***column vector***):

$$\mathbf{b} = \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- a ***row matrix*** (or a ***row vector***):

$$\mathbf{a} = \vec{a} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

- a ***square matrix of order n*** (порядка *n*):

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

the main diagonal

Arithmetic of Matrices

DEFINITION Two matrices are ***equal*** if they have the same size and their corresponding entries are equal.

DEFINITION If A and B are matrices of the same size, then

- the ***sum*** $A + B$:

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

- the ***difference*** $A - B$:

$$(A-B)_{ij} = (A)_{ij} - (B)_{ij}$$

DEFINITION If A is any matrix and c is any scalar, then the matrix cA is a ***scalar multiple*** of A:

$$(cA)_{ij} = c(A)_{ij}$$

If $A_1, A_2, \ldots, A_n$ are matrices of the same size and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with ***coefficients*** $c_1, c_2, \ldots, c_n$.

EXAMPLE

$$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 7 & 4 \end{pmatrix}$$

is the linear combination of A and B with scalar coefficients 2 and −1.

Multiplying Matrices

DEFINITION If *A* is an *m* x *r* matrix and *B* is an *r* x *n* matrix:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1r} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{ir} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mr} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ b_{r1} & \dots & b_{rj} & \dots & b_{rn} \end{pmatrix}$$

then the ***product*** *AB* is the *m* x *n* matrix with entries $\left(AB\right)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ir}b_{rj}$

EXAMPLE $(2\times 3)(3\times 4) \rightarrow (2\times 4)$

$$AB = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{pmatrix} = \begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix}$$

$$(2\cdot 4)+(6\cdot 3)+(0\cdot 5)=26$$

$$AB = \begin{pmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{pmatrix}$$

Transpose of a Matrix

DEFINITION If A is any m x n matrix, then the ***transpose of A (транспонированная)***, denoted by A^T, is the n x m matrix that results from interchanging the rows and columns of A; that is $(A^T)_{ij} = (A)_{ji}$

EXAMPLE

$$A = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{pmatrix}$$

Trace of a Matrix

DEFINITION If A is a square matrix, then the ***trace of A (след)***, denoted by Tr*(A),* is the sum of the entries on the main diagonal of A:

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

EXAMPLE

$$A = \begin{pmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{pmatrix}$$

$$\operatorname{tr}(A) = -1 + 5 + 7 + 0 = 11$$

THEOREM 2 Properties of Matrix Arithmetic

The following rules of matrix arithmetic are valid:

- $A + B = B + A$ (Commutative law for addition)
- $A + (B + C) = (A + B) + C$ (Associative law for addition)
- $A(BC) = (AB)C$ (Associative law for multiplication)
- $A(B + C) = AB + AC$ (Left distributive law)
- $(B + C) A = BA + CA$ (Right distributive law)
- $a(B + C) = aB + aC$
- $(a + b)C = aC + bC$
- $a(bC) = (ab)C$
- $a(BC) = (aB)C = B(aC)$

Zero and Identity Matrices

DEFINITION A ***zero*** matrix is a matrix, all of whose entries are zero:

$$0_{m\times n} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

DEFINITION An ***identity matrix (единичная матрица)*** is a square matrix with 1's on the main diagonal and 0's off the main diagonal:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

THEOREM 3

The following rules of matrix arithmetic are valid:

- $A + 0 = 0 + A = A$
- $A - A = 0$
- $0 - A = -A$
- $A0 = 0; \; 0A = 0$
- If A is an m x n matrix, then

$$AI_n = A, \qquad I_m A = A.$$

THEOREM 4 *If R is the reduced row-echelon form of an* n x n *matrix A, then either R has a row of zeros or R is the identity matrix* I_n.

Inverse of a Matrix

DEFINITION If A is a square matrix, and if B is a matrix of the same size and $AB = BA = I$, then A is said to be ***invertible*** (***обратимая матрица***) and B is called an ***inverse*** (***обратная матрица***) of A. If there is no such matrix B, then A is said to be ***singular*** (***вырожденная матрица***).

The inverse of A is uniquely defined and is denoted by A^{-1}

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I$$

THEOREM 6 The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof You should verify that $AA^{-1} = I$ and $A^{-1}A = I$. □

THEOREM 7 *If A and B are invertible matrices of the same size, then AB is invertible and*

$$\left(AB\right)^{-1} = B^{-1}A^{-1}$$

THEOREM 8 *If A is an invertible matrix, then:*

- A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$
- *For any nonzero scalar k, the matrix kA is invertible and* $\left(kA\right)^{-1} = \frac{1}{k}A^{-1}$

THEOREM 9 *If the sizes of the matrices are such that the stated operations can be performed, then*

- $\left(A^T\right)^T = A$
- $(A+B)^T = A^T + B^T$ *and* $(A-B)^T = A^T - B^T$
- $(kA)^T = kA^T$, *where k is any scalar*
- $(AB)^T = B^T A^T$

THEOREM 10 *If A is an invertible matrix, then* A^T *is also invertible and* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Lecture 3

Inverse Matrices and Determinants

A Method for Inverting Matrices

$$A \xrightarrow{\textit{a sequence of elementary row operations}} I_n$$

$$I_n \xrightarrow{\textit{the same sequence of elementary row operations}} A^{-1}$$

EXAMPLE Find the inverse of $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

Answer: $A^{-1} = \begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}$

Matrix Form of a Linear System

Consider any system of *m* linear equations in *n* unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Replace this system by the single matrix equation:

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Write it as a product:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or $A\mathbf{x} = \mathbf{b}$

THEOREM 2 *Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.*

THEOREM 3 *If A is an invertible n x n matrix, then for each n x 1 matrix* **b**, *the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

EXAMPLE Solve the linear system

$$\begin{cases} x_1 + 4x_2 = 7 \\ 2x_1 + 7x_2 = 12 \end{cases}$$

Solution. Write system in matrix form: $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$$

A is invertible and

$$A^{-1} = \begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}$$

By Theorem 3, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 7 \\ 12 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Answer: $x_1 = -1$, $x_2 = 2$

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is a ***diagonal matrix*** **(диагональная матрица)**:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero and

$$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{pmatrix}$$

Triangular Matrices

- ***Lower triangular*** matrix:

  (***нижнетреугольная*** матрица)

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

- ***Upper triangular*** matrix:

  (***верхнетреугольная*** матрица)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

- ***Triangular*** matrix: either upper triangular or lower triangular.

THEOREM 5 *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

Symmetric Matrices

A square matrix A is called ***symmetric*** (***симметрическая***) if $A = A^T$.

EXAMPLES

$$\begin{pmatrix} 7 & -3 \\ -3 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{pmatrix}$$

THEOREM 6 *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

- *A^T and kA are symmetric.*
- *$A + B$ and $A - B$ are symmetric.*
- *If A is an invertible symmetric matrix, then A^{-1} is symmetric.*

Determinants: Introduction

Recall that the 2 x 2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if $ad - bc \neq 0$. The expression $ad - bc$ is called the ***determinant*** **(*определитель*)** of A and is denoted by det(A) or $|A|$. With this notation

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Minors and Cofactors

DEFINITION If A is a square matrix, then the ***minor of entry*** (***минор элемента***) a_{ij} is denoted by M_{ij} and is the determinant of the submatrix that remains after the ith row and jth column are deleted from A.

$$M_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{in} & \dots & a_{nn} \end{vmatrix}$$

$C_{ij} = (-1)^{i+j} M_{ij}$ is called the ***cofactor of entry*** (***алгебраическое дополнение***) a_{ij}.

EXAMPLE Let

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Then

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5 \cdot 8 - 6 \cdot 4 = 16$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

and

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 3 \cdot 6 + 2 \cdot 4 = 26$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

Cofactor Expansions

DEFINITION The determinant of an *n* x *n* matrix is

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

This method of evaluating det (*A*) is called ***cofactor expansion*** along the first row of *A* (***разложение определителя*** по первой строке).

EXAMPLE Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Then

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} =$$

$$= 3 \cdot \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \cdot \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} =$$

$$= 3(-4) - 1(-11) + 0 = -1$$

THEOREM 7 Expansions by Cofactors

The determinant of an n x n matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \le i \le n$ *and* $1 \le j \le n$:

- Cofactor expansion along the *j*th column:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

- Cofactor expansion along the *i*th row:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

EXAMPLE Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Then

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} =$$

$$= 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} =$$

$$= 3(-4) - (-2)(-2) + 5 \cdot 3 = -1$$

Lecture 4

Determinants and Vectors

Adjoint of a Matrix

DEFINITION If A is any n x n matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

is called the ***matrix of cofactors from A***.

The transpose of this matrix is called the ***adjoint of A*** (***присоединенная матрица***) and is denoted by **adj (*A*)**.

EXAMPLE Suppose

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

The cofactors of A:

C_{11} = 12, C_{12} = 6, C_{13} = – 16, C_{21} = 4, C_{22} = 2,
C_{23} = 16, C_{31} = 12, C_{32} = – 10, C_{33} = 16

The matrix of cofactors:

$$\begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$$

The adjoint of A:

$$\text{adj}\,(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

THEOREM 1

- *The square matrix is invertible if and only if* $\det(A)$ *is not zero.*
- *If* A *is an invertible matrix, then*

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

EXAMPLE Consider

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

The determinant of *A*:

$$\det(A) = 64$$

The adjoint of *A*:

$$\text{adj}\,(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

The inverse of *A*:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}\,(A) = \frac{1}{64} \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

$$= \begin{pmatrix} 3/16 & 1/16 & 3/16 \\ 3/32 & 1/32 & -5/32 \\ -1/4 & 1/4 & 1/4 \end{pmatrix}$$

THEOREM 2 *If A is an n x n triangular matrix (upper triangular, lower triangular, or diagonal), then* det (A) *is the product of the entries on the main diagonal of the matrix; that is,* $\det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$

THEOREM 3 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, *then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}, \qquad \ldots, \qquad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

EXAMPLE Use Cramer's rule to solve

$$\begin{cases} 7x_1 - 5x_2 = 3 \\ 2x_1 + 3x_2 = 4 \end{cases}$$

Solution.

$$A = \begin{pmatrix} 7 & -5 \\ 2 & 3 \end{pmatrix} \quad A_1 = \begin{pmatrix} 3 & -5 \\ 4 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\det(A) = 21 + 10 = 31$$

$$\det(A_1) = 9 + 20 = 29 \qquad \det(A_2) = 28 - 6 = 22$$

$$x_1 = \frac{29}{31} \qquad x_2 = \frac{22}{31}$$

THEOREM 4 *Let A be a square matrix. If A has a row of zeros or a column of zeros, then* $\det(A) = 0$.

THEOREM 5 *Let A be a square matrix. Then* $\det(A) = \det(A^T)$.

THEOREM 9 *If A is a square matrix with two proportional rows or two proportional columns, then* $\det(A) = 0$.

THEOREM 6 *Let A be a square matrix. If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then* det $(B) = k \cdot$det (A).

EXAMPLE

(for the first row of 3 x 3 matrix)

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Proof (for this case)

$$\det(B) = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) = k \cdot \det(A) \qquad \square$$

THEOREM 7 *Let A be a square matrix. If B is the matrix that results when two rows or two columns of A are interchanged, then* det $(B) = -$ det (A).

EXAMPLE

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\square$

THEOREM 8 *Let A be a square matrix. If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then* det (B) = det (A).

EXAMPLE

$$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\square$

EXAMPLE

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

The 1st and the 2nd rows of *A* were interchanged

$$= -3 \cdot \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the 1st row was taken through the determinant sign

$$= -3 \cdot \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

–2 times the 1st row was added to the 3rd row

$$= -3 \cdot \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

–10 times the 2nd row was added to the 3rd row

$$= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} = (-3)(-55)(1) = 165$$

A common factor of -55 from the last row was taken through the determinant sign

Basic Properties of Determinants

1

$$\det(kA) = k^n \det(A)$$

EXAMPLE

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

EXAMPLE det(A + B) ≠ det(A) + det(B)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \qquad A + B = \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}$$

$$\det(A) = 1$$

$$\det(B) = 8$$

$$\det(A + B) = 23$$

2

Let A, B, and C be n x n matrices that differ only in a single row, say the rth, and

the rth row of C = *the rth row of A* + *the rth row of B*

Then det(*C*) = det(*A*) + det(*B*).

The same result holds for columns.

EXAMPLE

$$\begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{vmatrix} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

3

A square matrix A is invertible if and only if det (*A*) ≠ 0.

EXAMPLE Test the matrix for invertibility

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}$$

$$\det(A) = 0$$ → *A* is not invertible

THEOREM 10 *If A and B are square matrices of the same size, then*

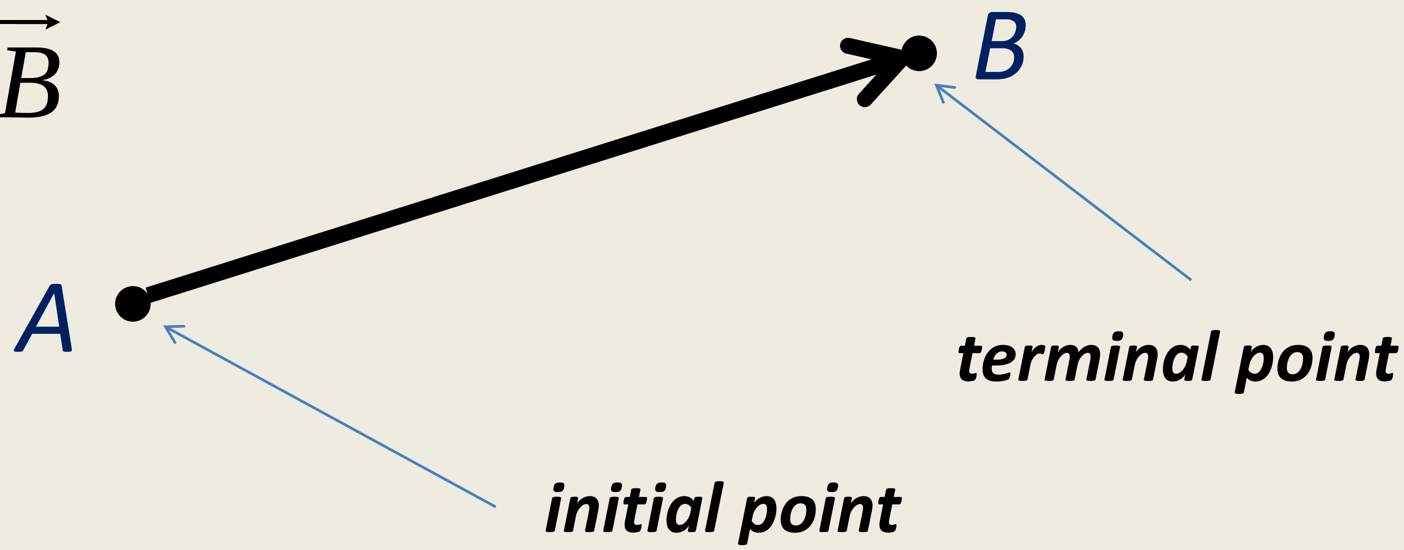
$$\det(AB) = \det(A) \cdot \det(B).$$

COROLLARY *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Geometric Vectors

$$\mathbf{v} = \overrightarrow{AB}$$



Notation:

- vectors: **v**, **w**, **a**, **b**,...
- scalars: $k, x, a, t, \ldots$

Two vectors **v** = (v_1, v_2) and **w** = (w_1, w_2) are equal if and only if $v_1 = w_1$ and $v_2 = w_2$

If **v** = (v_1, v_2) and **w** = (w_1, w_2), then
v + **w** = $(v_1 + w_1, v_2 + w_2)$ and
v − **w** = $(v_1 - w_1, v_2 - w_2)$

If k is any scalar, then $k \cdot$**v** = (kv_1, kv_2)

EXAMPLE If **v =** (1, – 2) and **w** = (7, 6), then
v + w = (8, 4) and 4**v** = (4, – 8).

Vectors in 3-Space

If **v** = (v_1, v_2, v_3) and **w** = (w_1, w_2, w_3), then

- ***v*** and ***w*** are equal if and only if

  $v_1 = w_1$, $v_2 = w_2$, and $v_3 = w_3$
- $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
- $k \cdot \mathbf{v} = (kv_1, kv_2, kv_3)$, where k is any scalar

If the vector $\overrightarrow{P_1P_2}$ has initial point $P_1 = (x_1, y_1, z_1)$ and terminal point $P_2 = (x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Lecture 5

Vector Spaces

Vectors in *n*-Space

DEFINITION If n is a positive integer, then an ***ordered n-tuple*** (***упорядоченный набор из n чисел***) is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by $\mathbb{R}^n$. The elements of this set are called ***generalized vector***.

Notation: $\mathbf{v} = (a_1, a_2, \ldots, a_n)$.

ordered 2-tuple ↔ ordered pair

ordered 3-tuple ↔ ordered triple

1-tuple ↔ a real number; $\mathbb{R}^1 = \mathbb{R}$

Standard Operations on $\mathbb{R}^n$

DEFINITION Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ are called ***equal*** if

$$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n.$$

- The ***sum*** $\mathbf{u} + \mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

- If k is any scalar, the ***scalar multiple*** $k\cdot\mathbf{u}$ is defined by

$$k\cdot\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

- The ***zero vector*** in $\mathbb{R}^n$: $\mathbf{0} = (0, 0, ..., 0)$.
- If $\mathbf{u} = (u_1, u_2, ..., u_n)$ is any vector in $\mathbb{R}^n$, then the ***negative*** (or ***additive inverse***) of $\mathbf{u}$ is defined by $-\mathbf{u} = (-u_1, -u_2, ..., -u_n)$.

- The ***difference*** of vectors in $\mathbb{R}^n$ is defined by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

$$= (u_1 - v_1, u_2 - v_2, ..., u_n - v_n).$$

Properties of Vector Operations in *n*-Space

THEOREM 1 *If* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, *and* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ *are vectors in* $\mathbb{R}^n$ *and k and m are scalars, then:*

(1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(2) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(3) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(4) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$; *that is,* $\mathbf{u} - \mathbf{u} = \mathbf{0}$

(5) $k(m\mathbf{u}) = (mk)\mathbf{u}$

(6) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(7) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(8) $1\mathbf{u} = \mathbf{u}$

DEFINITION Let *V* be an arbitrary nonempty set of objects with two operations : ***addition***, and ***multiplication*** by scalars.

Addition	each pair of objects **u** and **v**	→	an object **u** + **v**, the ***sum*** of **u** and **v**
Scalar multiplication	each scalar *k* and each object **u** in *V*	→	an object *k***u**, the ***scalar multiple*** of **u** by *k*

Suppose that

- If **u** and **v** are objects in *V*, then **u** + **v** is in *V.*
- If *k* is any scalar and **u** is any object in *V*, then *k***u** is in *V*.
- There is an object **0** in *V*, called a ***zero vector*** for *V*.
- For each **u** in *V*, there is an object – **u** in *V*, a ***negative*** of **u**.
- Eight axioms from Theorem 1 are satisfied

Then *V* is a ***vector space*** and the objects in *V* are **vectors**.

EXAMPLE 1 **$\mathbb{R}^n$ is a vector space**

EXAMPLE 2 **The set *V* of all 2 x 2 matrices is a vector space** if

- addition = matrix addition,
- scalar multiplication = matrix scalar multiplication.

Solution

1) Consider $\mathbf{u} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$

Then **u** + **v** is an object in *V* and *k***u** is an object in *V*.

2) If $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then **0** + **u** = **u** + **0** = **u** for all **u** in *V*.

3) For any **u** in *V* put $-\mathbf{u} = \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$

Then **u** + (– **u**) = **0** and (– **u**) + **u** = **0**.

Axioms 1–8 follow from properties of matrix operation.

EXAMPLE 3 The set V of m x n matrices is a vector space M_{mn}.

EXAMPLE 4 A vector space of real-valued functions

Let V be the set of real-valued functions defined on $(-\infty, \infty)$. If **f** = $f(x)$ and **g** = $g(x)$ are two functions and k is any real number, define

- the sum function **f + g** by $(f + g)(x) = f(x) + g(x)$
- the scalar multiple k**f** by $(kf)(x) = k \cdot f(x)$

The vector **0** is identically zero for all values of x.

The negative of a vector **f** is the function –**f** = $-f(x)$.

EXAMPLE 5 The zero vector space

Let V consist of a single object, which we denote by **0**, and define **0 + 0 = 0** and k**0 = 0** for all scalars k.

Euclidean *n*-Space

DEFINITION

If **u** = (u_1, u_2, …, u_n) and **v** = (v_1, v_2, …, v_n) are any vectors in $\mathbb{R}^n$, then the ***Euclidean inner product*** (***скалярное произведение***) is

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 + u_nv_n$$

Sometimes it is also called ***dot product***.

EXAMPLE Find the Euclidean inner product of the vectors **u** = (–1, 3, 5, 7) and **v** = (5, –4, 7, 0) in $\mathbb{R}^4$.

Answer: **u·v =** 18

THEOREM 2 (Properties of Euclidean Inner Product) *If* **u, v,** *and* **w** *are vectors in* $\mathbb{R}^n$ *and k is any scalar, then:*

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(c) $(k\mathbf{u})\cdot\mathbf{v} = k\,(\mathbf{u}\cdot\mathbf{v})$

(d) $\mathbf{v}\cdot\mathbf{v} \geq 0$. *Further,* $\mathbf{v}\cdot\mathbf{v} = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$.

***Proof* (c)** Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$.

Then $(k\mathbf{u})\cdot\mathbf{v} = (ku_1, ku_2, \ldots, ku_n)\cdot(v_1, v_2, \ldots, v_n)$

$$= ku_1v_1 + ku_2v_2 + \ldots + ku_nv_n =$$

$$= k(u_1v_1 + u_2v_2 + \ldots + u_nv_n) = k(\mathbf{u}\cdot\mathbf{v})$$

***Proof* (d)** We have $\mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 + \ldots + v_n^2 \geq 0$. Equality holds if and only if $v_1 = v_2 = \ldots = v_n = 0$— that is, if and only if $\mathbf{v} = \mathbf{0}$. □

EXAMPLE Given $\mathbf{u}\cdot\mathbf{u} = 2$, $\mathbf{u}\cdot\mathbf{v} = -3$, $\mathbf{v}\cdot\mathbf{v} = 1$, evaluate $(3\mathbf{u} + 2\mathbf{v})\cdot(4\mathbf{u} + \mathbf{v})$.

Solution

$$(3\mathbf{u} + 2\mathbf{v})\cdot(4\mathbf{u} + \mathbf{v}) =$$

$$= 3\mathbf{u}\cdot(4\mathbf{u} + \mathbf{v}) + 2\mathbf{v}\cdot(4\mathbf{u} + \mathbf{v})$$

$$= 3\mathbf{u}\cdot4\mathbf{u} + 3\mathbf{u}\cdot\mathbf{v} + 2\mathbf{v}\cdot4\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v}$$

$$= 12\mathbf{u}\cdot\mathbf{u} + 11\mathbf{u}\cdot\mathbf{v} + 2\mathbf{v}\cdot\mathbf{v}$$

$$= 12\cdot2 + 11\cdot(-3) + 2\cdot1 = -7$$

Norm and Distance in Euclidean *n*-Space

The ***Euclidean norm*** (or ***Euclidean length***) (***норма, длина***) of a vector **u** = (u_1, u_2, …, u_n) in $\mathbb{R}^n$ is

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

The ***Euclidean distance*** between the points

u = (u_1, u_2, …, u_n) and **v** = (v_1, v_2, …, v_n) in $\mathbb{R}^n$ is

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2}$$

EXAMPLE 3 Finding Norm and Distance

Suppose **u** = (1, 3, –2, 7) and **v** = (0, 7, 2, 2).

Then

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (-2)^2 + (7)^2} = \sqrt{63} = 3\sqrt{7}$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

THEOREM 3 Cauchy–Bunyakovsky–Schwarz Inequality (неравенство Коши-Буняковского-Шварца) in $\mathbb{R}^n$

If **u** = $(u_1, u_2, \ldots, u_n)$ *and* **v** = $(v_1, v_2, \ldots, v_n)$ *are vectors in* $\mathbb{R}^n$, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$$

In terms of components, this inequality is the same as

$$\left|u_1v_1 + u_2v_2 + \ldots + u_nv_n\right| \leq$$

$$\leq \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \cdot \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

Angle between Two Vectors in *n*-Space

- If **u** and **v** are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and θ is the angle between them, then

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \quad \text{or} \quad \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

- If **u** = (u_1, u_2, …, u_n) and **v** = (v_1, v_2, …, v_n) are two nonzero vectors in $\mathbb{R}^n$, then the ***cosine*** of the angle θ between **u** and **v** is defined by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

The Cauchy–Bunyakovsky–Schwarz inequality $\Longrightarrow$ $|\cos\theta| \le 1$

EXAMPLE For the given vectors **u** = (1, −2, 3, −1) and **v** = (3, 0, 2, −5), find the cosine of the angle θ between **u** and **v**.

Solution

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 3 + 2 \cdot 3 + (-1) \cdot (-5) = 14$$

$$||\mathbf{u}|| = \sqrt{1^2 + (-2)^2 + 3^2 + (-1)^2} = \sqrt{15}$$

$$||\mathbf{v}|| = \sqrt{3^2 + 0^2 + 2^2 + (-5)^2} = \sqrt{38}$$

$$\cos\theta = \frac{14}{\sqrt{15} \cdot \sqrt{38}}$$

THEOREM 4 Properties of Length in $\mathbb{R}^n$

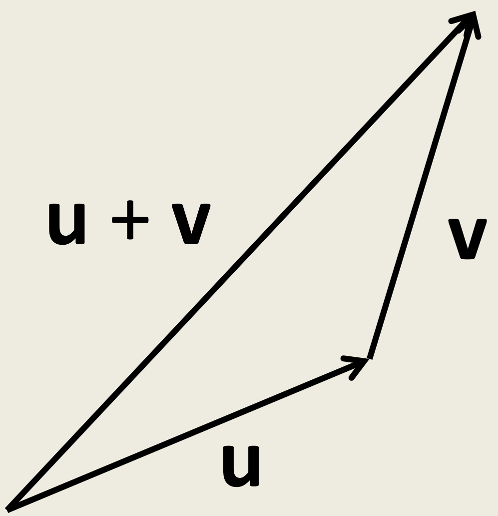
If **u** *and* **v** *are vectors in* $\mathbb{R}^n$ *and k is any scalar, then:*

(a) $\|\mathbf{u}\| \geq 0$

(b) $\|\mathbf{u}\| = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$

(c) $\|k\mathbf{u}\| = |k|\,\|\mathbf{u}\|$

(d) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ *(Triangle inequality)*



THEOREM 5 Properties of Distance in $\mathbb{R}^n$

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* $\mathbb{R}^n$, *then*:

(a) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(b) $d(\mathbf{u}, \mathbf{v}) = 0$ *if and only if* $\mathbf{u} = \mathbf{v}$.

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

(d) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ *(Triangle inequality)*

Orthogonality

DEFINITION Two vectors **u** and **v** in $\mathbb{R}^n$ are called *orthogonal* if $\mathbf{u}\cdot\mathbf{v} = 0$.

EXAMPLE Are the vectors

$$\mathbf{u} = (-2, 3, 1, 4) \quad \text{and} \quad \mathbf{v} = (1, 2, 0, -1)$$

in $\mathbb{R}^4$ orthogonal?

$$\mathbf{u}\cdot\mathbf{v} = -2\cdot 1 + 3\cdot 2 + 1\cdot 0 + 4\cdot(-1) = 0$$

Lecture 6

Subspaces and Linear Independence

DEFINITION Let *V* be an arbitrary nonempty set of objects with two operations : ***addition***, and ***multiplication*** by scalars.

Addition	each pair of objects **u** and **v** →	an object **u** + **v** (the ***sum*** of **u** and **v**)
Scalar multiplication	each scalar *k* and each object **u** in *V* →	an object *k***u** (the ***scalar multiple*** of **u** by *k*)

Suppose that

- If **u** and **v** are objects in *V*, then **u** + **v** is in *V*.
- If *k* is any scalar and **u** is any object in *V*, then *k***u** is in *V*.
- There is an object **0** in *V*, called a ***zero vector*** for *V*.
- For each **u** in *V*, there is an object – **u** in *V*, a ***negative*** of **u**.
- Eight axioms (see the next slide) are satisfied.

Then *V* is a ***vector space*** and the objects in *V* are **vectors**.

Eight Axioms of Vector Space

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in V, and k and m are scalars, then:*

(1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(2) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(3) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(4) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$; *that is,* $\mathbf{u} - \mathbf{u} = \mathbf{0}$

(5) $k(m\mathbf{u}) = (mk)\mathbf{u}$

(6) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(7) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(8) $1\mathbf{u} = \mathbf{u}$

Concept of a Subspace

DEFINITION A subset W of a vector space V is called a ***subspace*** (***подпространство***) of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Question Is it necessary to verify 8 vector space axioms to show that a set W forms a vector space?

No, because these axioms are “inherited” from V.

THEOREM 1 *If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.*

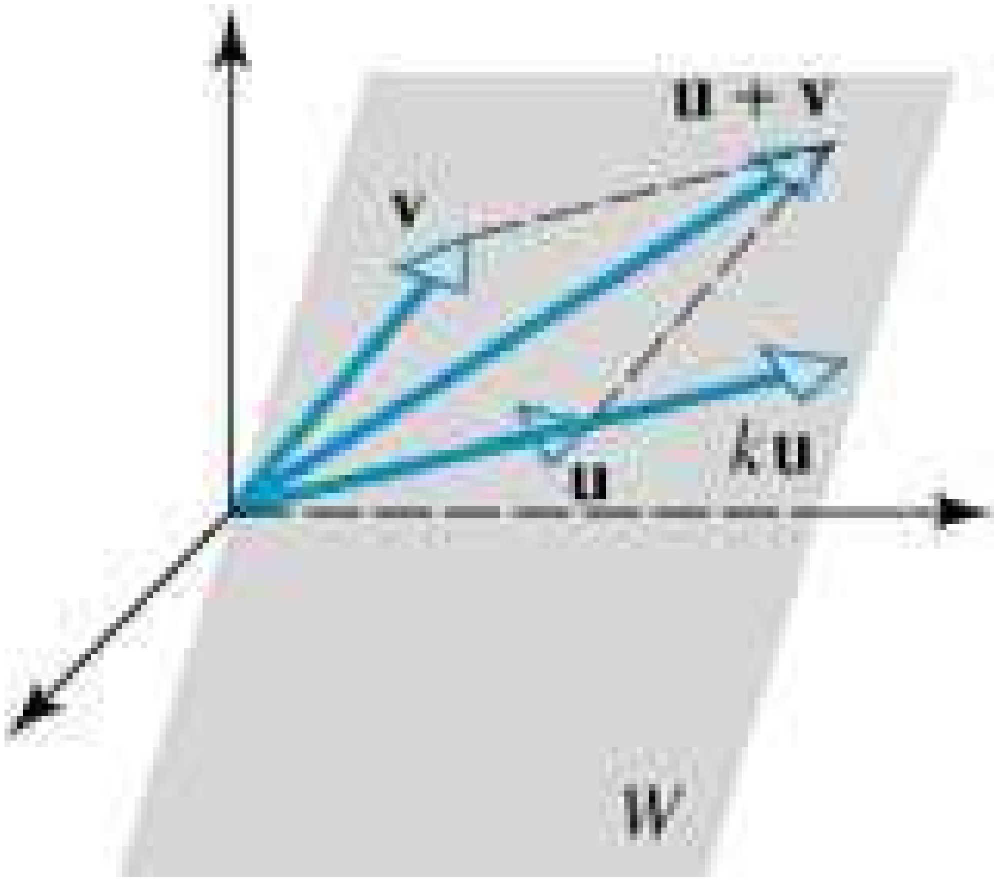
(a) *If* **u** *and* **v** *are vectors in W, then* **u** + **v** *is in W.*

(b) *If k is any scalar and* **u** *is any vector in W, then k***u** *is in W.*

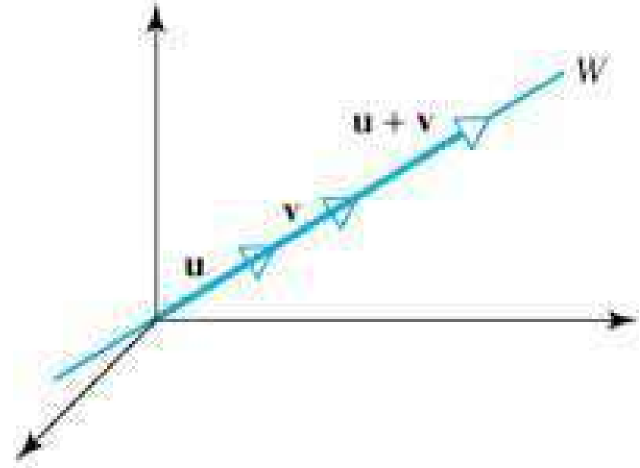
Remark Theorem 1 states that *W is a subspace of V if and only if W is* ***closed*** *under* (***замкнуто*** *относительно*) *addition and* ***closed*** *under scalar multiplication.*

Testing for a Subspace

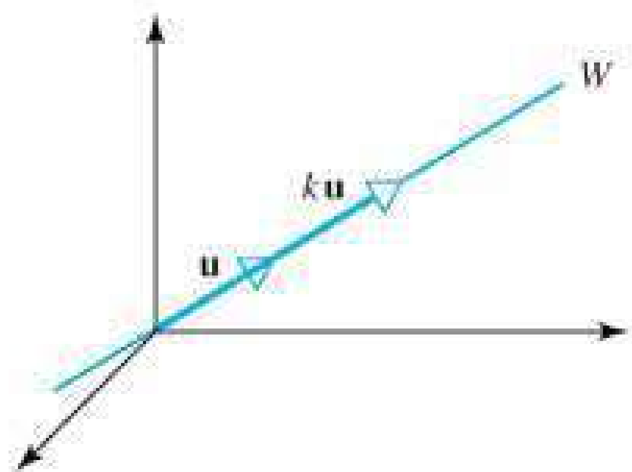
EXAMPLE 1 Every plane through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.



EXAMPLE 2 Every line through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.



(*a*) *W* is closed under addition.



(*b*) *W* is closed under scalar multiplication.

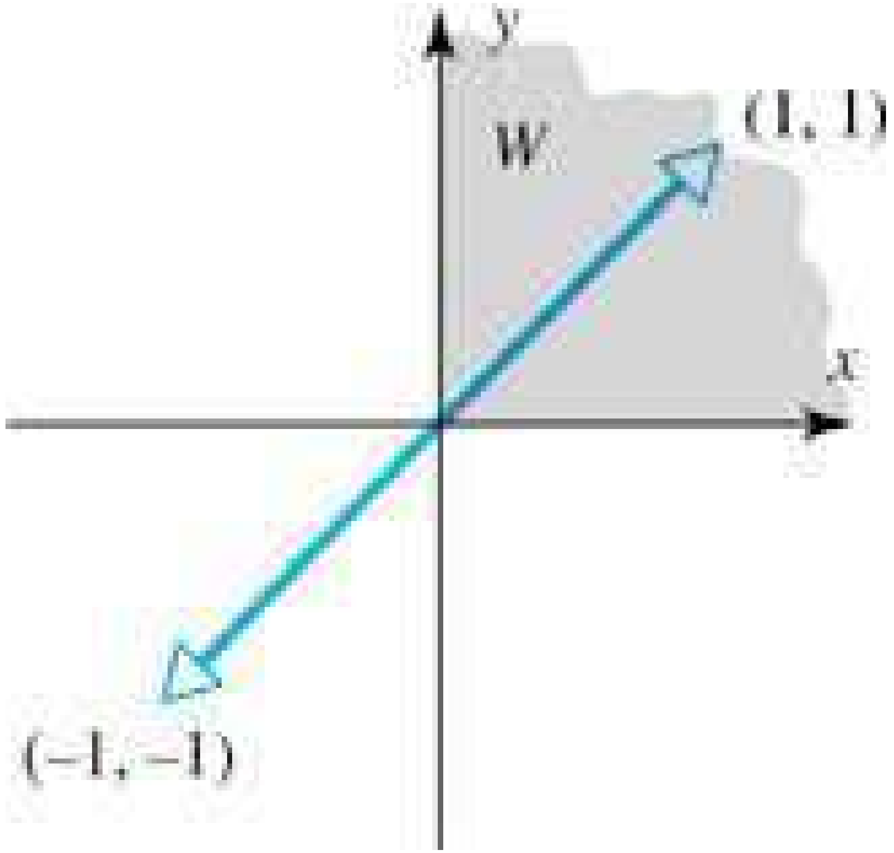
EXAMPLE 3 Let W be the set of all points (x, y) in $\mathbb{R}^2$ such that $x \geq 0$ and $y \geq 0$.

Question Is the set W a subspace of $\mathbb{R}^2$?

Solution $\mathbf{v} = (1, 1)$ lies in W, but its negative $-\mathbf{v} = (-1, -1)$ does not $\Longrightarrow$?

W is not closed under scalar multiplication $\Longrightarrow$?

***W* is not a subspace!!!**



List of Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

Subspaces of $\mathbb{R}^2$	Subspaces of $\mathbb{R}^3$
$\{\mathbf{0}\}$	$\{\mathbf{0}\}$
Lines through the origin	Lines through the origin
	Planes through the origin
$\mathbb{R}^2$	$\mathbb{R}^3$

Solution Spaces of Homogeneous Systems

Let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations.

Solution vector: a vector $\mathbf{x}$ that satisfies this system.

THEOREM 2 *If* $A\mathbf{x} = \mathbf{0}$ *is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of* $\mathbb{R}^n$.

Linear Combinations of Vectors

DEFINITION A vector **w** is called a ***linear combination*** (***линейной комбинацией***) of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

EXAMPLE Every vector $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$ is expressible as a linear combination of the standard basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ since

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

THEOREM 3

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *are vectors in a vector space V, then*

(a) *The set W of all linear combinations of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *is a subspace of V.*

(b) *W is the smallest subspace of V that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *in the sense that every other subspace of V that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *must contain W.*

Spanning

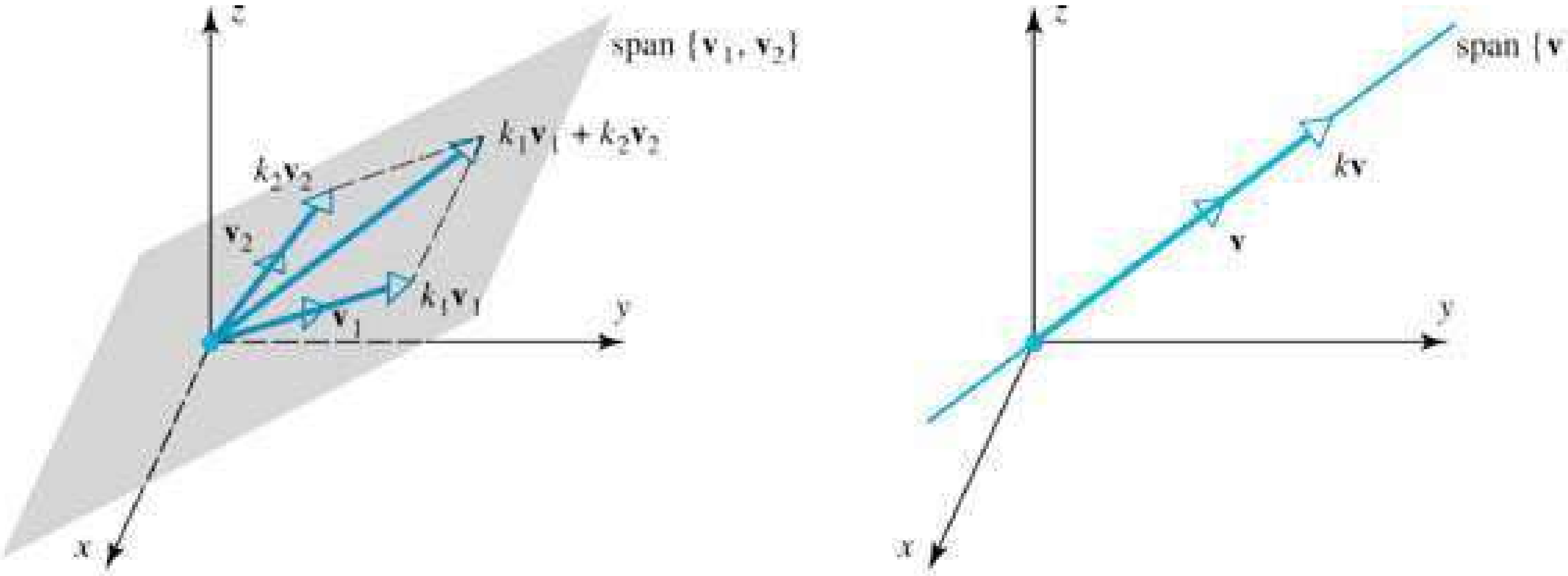
DEFINITION If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called the ***space spanned*** by (***линейной оболочкой векторов***) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, and we say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ ***span*** (***стягивают, порождают***) W.

Write:

$$W = \text{span}(S) \quad \text{or} \quad W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

EXAMPLE If $\mathbf{v}_1$ and $\mathbf{v}_2$ are noncollinear vectors in $\mathbb{R}^3$ with their initial points at the origin, then span{$\mathbf{v}_1$, $\mathbf{v}_2$} is the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Similarly, if $\mathbf{v}$ is a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, then span{$\mathbf{v}$} is the line determined by $\mathbf{v}$.



(a) Span {$\mathbf{v}_1$, $\mathbf{v}_2$} is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Span {$\mathbf{v}$} is the line through the origin determined by $\mathbf{v}$.

Linear Independence

DEFINITION If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vectors, then the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$$

has at least one solution, namely

$$k_1 = 0,\ k_2 = 0,\ \ldots,\ k_r = 0$$

If this is the only solution, then S is called a ***linearly independent*** (***линейно независимое***) set. If there are other solutions, then S is called a ***linearly dependent*** (***линейно зависимое***) set.

EXAMPLE 1 If $\mathbf{v}_1 = (2, -1, 0, 3)$, $\mathbf{v}_2 = (1, 2, 5, -1)$, and $\mathbf{v}_3 = (7, -1, 5, 8)$, then

$3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 =$ **0** → the set of vectors is linearly dependent

EXAMPLE 2

Consider $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ in $\mathbb{R}^3$.

In terms of components, the vector equation

$$k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$$

becomes $k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)$

$k_1 = 0, k_2 = 0, k_3 = 0$ → the set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent

Similarly, the vectors

$\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, …, $\mathbf{e}_n = (0, 0, \ldots, 1)$

form a linearly independent set in $\mathbb{R}^n$.

THEOREM 4 *A set S with two or more vectors is*

(a) *Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.*

(b) *Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.*

Proof **(a)**

1) $\Longrightarrow$

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, $r \geq 2$.

S is linearly dependent $\Longrightarrow$ there are scalars k_1, k_2, …, k_r, not all zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$$

Suppose that $k_1 \neq 0$. Then this formula $\Longrightarrow$

$$\mathbf{v}_1 = \left(-\frac{k_2}{k_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{k_r}{k_1}\right)\mathbf{v}_r$$

which expresses $\mathbf{v}_1$ as a linear combination of the other vectors in S.

2) $\Longleftarrow$

Assume that at least one of the vectors in *S* is expressible as a linear combination of the others. Suppose that

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \ldots + c_r\mathbf{v}_r$$

so

$$\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \ldots - c_r\mathbf{v}_r = \mathbf{0}$$

The equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$$

is satisfied by

$$k_1 = 1, \quad k_2 = -c_2, \quad \ldots, \quad k_r = -c_r$$

which are not all zero. Therefore, *S* is linearly dependent. $\square$

EXAMPLE The vectors

$$\mathbf{v}_1 = (2, -1, 0, 3),\ \mathbf{v}_2 = (1, 2, 5, -1),\ \mathbf{v}_3 = (7, -1, 5, 8)$$

form a linearly dependent set, since (see Example 1)

$$3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$$

Task. Express each vector as a linear combination of the other two.

Answer.

$$\mathbf{v}_1 = -\frac{1}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3$$

$$\mathbf{v}_2 = -3\mathbf{v}_1 + \mathbf{v}_3 \qquad \mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$$

THEOREM 5

(a) *A finite set of vectors that contains the zero vector is linearly dependent.*

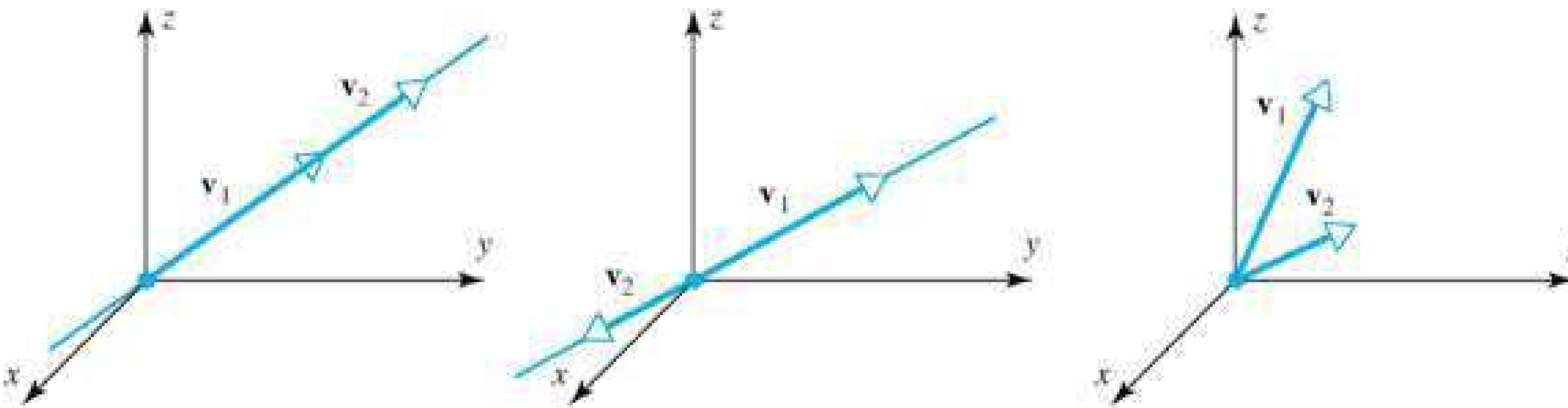
(b) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

Proof **(a)** For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{0}\}$ is linearly dependent since

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r + 1 \cdot \mathbf{0} \qquad \square$$

Geometric Interpretation of Linear Independence in $\mathbb{R}^2$ and $\mathbb{R}^3$

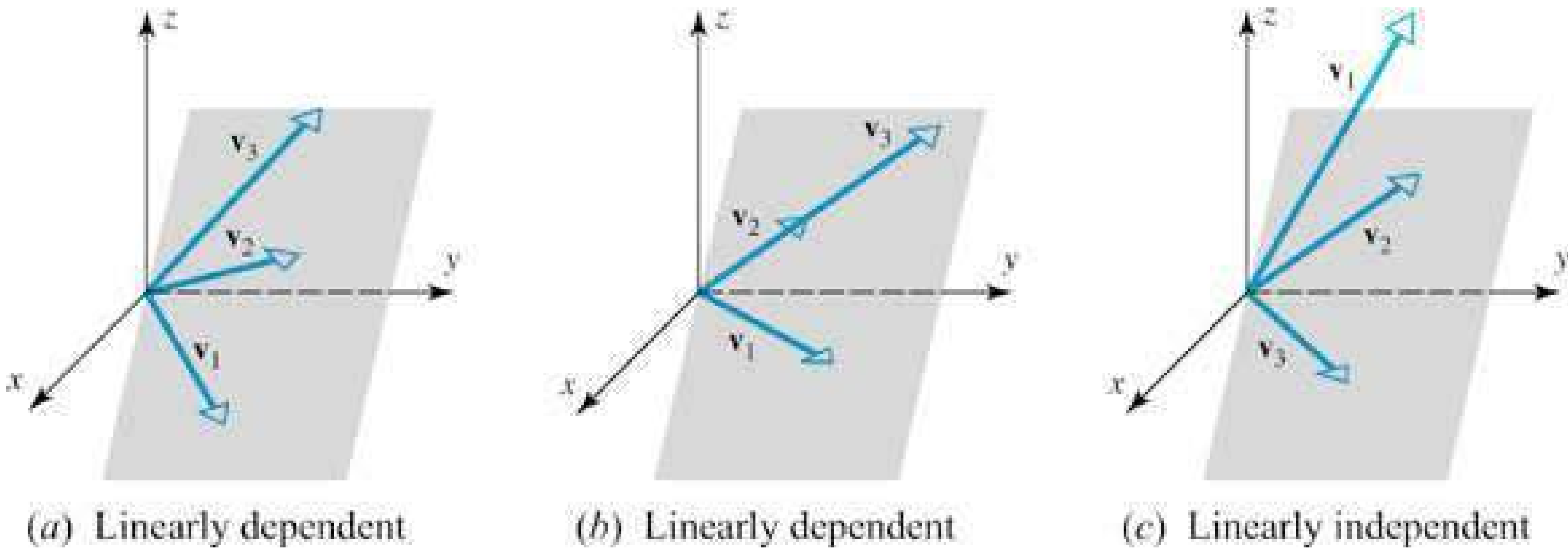
A set of two vectors is linearly independent ⇔ the vectors do not lie on the same line when they are placed with their initial points at the origin.



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

Geometric Interpretation of Linear Independence in $\mathbb{R}^3$

A set of three vectors is linearly independent $\Leftrightarrow$ the vectors do not lie in the same plane when they are placed with their initial points at the origin.



THEOREM 6

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

Lecture 7

Basis and Dimension

Concept of a Basis

DEFINITION If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a set of vectors in V, then S is called a ***basis*** (***базис***) for V if the following two conditions hold:

(a) S is linearly independent.

(b) S spans V, that is, any vector $\mathbf{v} \in V$ can be represented as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$.

THEOREM 1 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a basis for a vector space* V*, then every vector* $\mathbf{v}$ *in* V *can be expressed in the form* $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ *in exactly one way.*

Proof Suppose that some vector **v** can be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

and also as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$$

Therefore,

$$\mathbf{0} = (c_1 - k_1)\,\mathbf{v}_1 + (c_2 - k_2)\,\mathbf{v}_2 + \ldots + (c_n - k_n)\,\mathbf{v}_n$$

The linear independence of $S \Longrightarrow$

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \;\ldots, \quad c_n - k_n = 0$$

that is, $c_1 = k_1, \quad c_2 = k_2, \;\ldots, \quad c_n = k_n.$ □

Coordinates Relative to a Basis

DEFINITION If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

is the expression for a vector **v** in terms of the basis S, then the scalars $c_1, c_2, \dots, c_n$ are called the ***coordinates*** of **v** relative to the basis S (***координаты вектора в базисе***).

The vector $(c_1, c_2, \dots, c_n)$ in $\mathbb{R}^n$ is called the ***coordinate vector of* v *relative to S***.

Notation:

$$(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$$

EXAMPLE 1 Standard Basis for $\mathbb{R}^n$

Consider $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, …, $\mathbf{e}_n = (0, 0, 0, \ldots, 1)$. Then:

- $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is linearly independent
- S spans $\mathbb{R}^n$ since for any vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Thus, S is the ***standard basis for*** $\mathbb{R}^n$. We have

$$\mathbf{v} = (\mathbf{v})_S = (v_1, v_2, \ldots, v_n)$$

Remark In $\mathbb{R}^2$ and $\mathbb{R}^3$, the standard basis vectors are commonly denoted by **i**, **j**, and **k**, rather than by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

EXAMPLE 2 Suppose $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (1, 1)$.

(a) Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$.

(b) Find the coordinate vector of $\mathbf{v} = (5, -1)$ with respect to S.

Solution (a) Show that S spans $\mathbb{R}^2$. Let $\mathbf{b} = (b_1, b_2)$ be an arbitrary vector and try to express it as $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_2 \end{pmatrix} \Rightarrow c_1 = b_1 - b_2,\ c_2 = b_2$$

Show that S is linearly independent. Find the solution of

$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ $\Rightarrow$ $c_1 = 0$, $c_2 = 0$

Solution (b) We must find the scalars c_1 and c_2 such that

$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. $\Rightarrow$ $c_1 = 6$, $c_2 = -1$

$\mathbf{v} = 6\mathbf{v}_1 - \mathbf{v}_2$ $\Rightarrow$ $(\mathbf{v})_S = (6, -1)$

EXAMPLE 5 Basis for the Subspace span(*S*)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a *linearly independent* set in a vector space *V*, then *S* is a basis for the subspace span(*S*).

THEOREM 2

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *be any basis for a vector space V.*

(a) *If a set has more than n vectors, then it is linearly dependent.*

(b) *If a set has fewer than n vectors, then it does not span V.*

THEOREM 3 *All bases for a vector space have the same number of vectors.*

DEFINITION The ***dimension* (*размерность*)** of a vector space *V*, is the number of vectors in a basis for *V*. The zero vector space has dimension zero.

Notation: dim(V).

Dimensions of Some Vector Spaces

$\dim(\mathbb{R}^n) = n$	The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ has n vectors
$\dim(P_n) = n + 1$	The standard basis $\{1, x, x^2, \ldots, x^n\}$ has $n + 1$ vectors
$\dim(M_{mn}) = mn$	The standard basis has mn vectors

THEOREM 4 *If V is an n-dimensional vector space, and if S is a set in V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.*

EXAMPLE Show that $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for $\mathbb{R}^2$.

Solution

neither vector is a scalar multiple of the other → the two vectors form a linearly independent set in $\mathbb{R}^2$ → they form a basis by Theorem 4

THEOREM 5 *Let S be a set of vectors in a vector space V.*

(a) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(b) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

THEOREM 6 *If W is a subspace of a vector space V, then* $\dim(W) \le \dim(V)$*; moreover, if* $\dim(W) = \dim(V)$*, then* $W = V$.

DEFINITION For an m x n matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

we consider the row vectors

$$\mathbf{r}_1 = (a_{11} \quad a_{12} \quad \cdots \quad a_{1n})$$
$$\mathbf{r}_2 = (a_{21} \quad a_{22} \quad \cdots \quad a_{2n})$$
$$\cdots$$
$$\mathbf{r}_m = (a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn})$$

and the column vectors

$$\mathbf{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ \mathbf{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ \ldots, \mathbf{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

DEFINITION If A is an m x n matrix, then

- the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the ***row space*** (***пространство строк***) of A
- the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the ***column space*** (***пространство столбцов***) of A

General and Particular Solutions

- The vector $\mathbf{x}_0$ is called a ***particular solution*** of $A\mathbf{x} = \mathbf{b}$.
- The expression $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ is called the ***general solution*** of $A\mathbf{x} = \mathbf{0}$.
- The expression $\mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ is called the ***general solution*** of $A\mathbf{x} = \mathbf{b}$.

Theorem 8 *The general solution of* $A\mathbf{x} = \mathbf{b}$ *is the sum of any particular solution of* $A\mathbf{x} = \mathbf{b}$ *and the general solution of* $A\mathbf{x} = \mathbf{0}$.

EXAMPLE Consider the system

$$\begin{cases} x_1 - 2x_2 \qquad\qquad + x_4 = 5 \\ \qquad 3x_2 + x_3 - 7x_4 = -1 \end{cases} \qquad (1)$$

Then

$x_1 = 5 + 2t - s, \quad x_2 = t, \quad x_3 = -1 - 3t + 7s, \quad x_4 = s.$

This result can be written in vector form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 + 2t - s \\ t \\ -1 - 3t + 7s \\ s \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 7 \\ 1 \end{pmatrix}$$

the general solution of (1)

$\mathbf{x}_0$, a particular solution of (1)

the general solution of $\begin{cases} x_1 - 2x_2 \qquad + x_4 = 0 \\ \quad 3x_2 + x_3 - 7x_4 = 0 \end{cases}$

THEOREM 9 *Elementary row operations do not change the row space of a matrix.*

THEOREM 10
(*a*) *Elementary row operations do not alter relationships of linear independence among the column vectors.*
(*b*) *Elementary row operations do not alter the formulas (linear combinations) that relate linearly dependent column vectors.*

COROLLARY *If A and B are row equivalent matrices, then a given set of column vectors of A forms a basis for the column space of A*

if and only if

the corresponding column vectors of B form a basis for the column space of B.

EXAMPLE (a) Find a subset of the vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-2, 0, 1)$, $\mathbf{v}_3 = (-3, 2, 5)$ that forms a basis for the space spanned by these vectors.

(b) Express each vector not in the basis as a linear combination of the basis vectors.

Solution

Step 1. Form the matrix A having $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ as its column vectors:

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & 2 \\ 3 & 1 & 5 \end{pmatrix}$$

Step 2. Reduce the matrix A to its reduced row-echelon form R:

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & 2 \\ 3 & \boxed{1} & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 7 & 0 & 7 \\ 2 & 0 & 2 \\ 3 & 1 & 5 \end{pmatrix} \xrightarrow{:2} \begin{pmatrix} 7 & 0 & 7 \\ \boxed{1} & 0 & 1 \\ 3 & 1 & 5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ be the column vectors of R.

Step 3. Identify the columns that contain the pivots in R $\rightarrow$ $\mathbf{w}_1$ and $\mathbf{w}_2$.

The corresponding column vectors of A are the basis vectors for span(S) $\rightarrow$ $\mathbf{v}_1$ and $\mathbf{v}_2$.

Step 4. Express each column vector of R that does *not* contain a pivot as a linear combination of column vectors that *do* contain pivots. This yields a set of ***dependency equations***:

$$\mathbf{w}_3 = \mathbf{w}_1 + 2\mathbf{w}_2$$

The corresponding equations for the column vectors of A express the vectors that are not in the basis as linear combinations of the basis vectors:

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2 \qquad \square$$

Lecture 8

Rank and Orthogonality

Row Space, Column Space, and Nullspace

DEFINITION If A is an m x n matrix, then

- the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the ***row space*** of A,
- the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the ***column space*** of A,
- the solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $\mathbb{R}^n$, is called the ***nullspace*** of A.

THEOREM 1 *If A is any matrix, then the row space and column space of A have the same dimension.*

DEFINITION The common dimension of the row space and column space of a matrix *A* is called the ***rank* (*ранг*)** of *A* and is denoted by rank(*A*).

$$\text{rank}(A) = \text{rank}(A^T)$$

The dimension of the nullspace of *A* is denoted by $\dim(V_A)$.

EXAMPLE Find the rank and the dimension of the nullspace V_A of the matrix

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution Since the row space and column space are both two-dimensional, so rank(*A*) = 2.

To find dim(V_A), solve the linear system $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} x_1 + 4x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \quad \Longrightarrow \quad x_1 = -4t, \quad x_2 = 2t, \quad x_3 = t$$

The general solution of the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$$

Therefore, $\dim(V_A) = 1$

THEOREM 2 (Dimension Theorem for Matrices)

If A is a matrix with n columns, then

$$\text{rank}(A) + \dim(V_A) = n.$$

Proof Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n variables, which are the basic variables (pivots) and the free variables. Thus

number of basic variables + number of free variables = n

- The number of basic variables = rank(A)
- The number of free variables = the number of parameters in the general solution = $\dim(V_A)$

$$\text{rank}(A) + \dim(V_A) = n$$

□

THEOREM 3 *The rank of a matrix is the number of nonzero rows in its row-echelon form.*

EXAMPLE Find the rank of the matrix given in the row-echelon form:

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 0.8 & -0.2 & 100 \\ 0 & 0 & 0.7 & 210 \end{array}\right)$$

Answer: rank(A) = 3.

THEOREM 4 The Consistency Theorem (теорема Кронекера—Капелли)

If $A\mathbf{x} = \mathbf{b}$ *is a linear system of m equations in n unknowns, then the following are equivalent*:

- $A\mathbf{x} = \mathbf{b}$ *is consistent.*
- $\mathbf{b}$ *is in the column space of A.*
- *The coefficient matrix A and the augmented matrix* $[A \mid \mathbf{b}]$ *have the same rank*:

$$\text{rank}(A) = \text{rank}\,[A \mid \mathbf{b}]$$

EXAMPLE If the system of linear equations has the following augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & -23 & 16 & 2 \\ 0 & 1 & -10 & 7 & -8 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

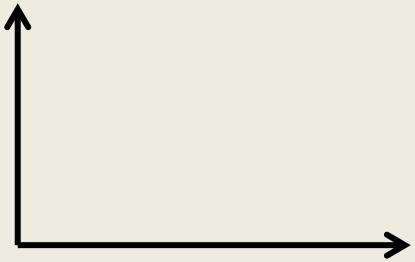
$$\operatorname{rank}(A) = 2$$

$$\operatorname{rank}[A \,|\, \mathbf{b}] = 3$$

→ the system is inconsistent

Orthogonality

Since for nonzero vectors **u** and **v**



$$\cos\theta = \frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$$

then

$$\cos\theta = 0 \quad \Leftrightarrow \quad \theta = \pi/2 \quad \Leftrightarrow \quad (\mathbf{u}, \mathbf{v}) = 0$$

Agreement: consider the angle between **u** and **v** to be $\pi/2$ when either or both of these vectors is **0**.

DEFINITION Two vectors **u** and **v** are called ***orthogonal (ортогональными)*** if (**u**, **v**) = 0.

Notation: **u** ⊥ **v**

DEFINITION A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in $\mathbb{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal:

$$\mathbf{v}_i \perp \mathbf{v}_j \text{ for } i \neq j.$$

An orthogonal set in which each vector has norm 1 is called ***orthonormal***:

$$\forall i \quad \|\mathbf{v}_i\| = 1$$

EXAMPLE

Let $\mathbf{u}_1 = (0, 1, 0)$, $\mathbf{u}_2 = (1, 0, 1)$, $\mathbf{u}_3 = (1, 0, -1)$.

Is this set orthogonal? yes

Is this set orthonormal? no

EXAMPLE Let $\mathbf{u}_1 = (0, 1, 0)$, $\mathbf{u}_2 = (1, 0, 1)$, $\mathbf{u}_3 = (1, 0, -1)$

The Euclidean norms of the vectors are

$$\|\mathbf{u}_1\| = 1, \quad \|\mathbf{u}_2\| = \sqrt{2}, \quad \|\mathbf{u}_3\| = \sqrt{2}$$

Normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = (0, 1, 0) \qquad \mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{v}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

Is the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ orthonormal? yes

$$(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1, \mathbf{v}_3) = (\mathbf{v}_2, \mathbf{v}_3) = 0 \quad \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = 1$$

Orthonormal and Orthogonal Bases

- ***Orthonormal basis***: a basis consisting of orthonormal vectors
- ***Orthogonal basis***: a basis consisting of orthogonal vectors

EXAMPLE the standard basis for $\mathbb{R}^n$

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots,$$

$$\mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

is orthonormal.

Coordinates Relative to Orthogonal Bases

THEOREM 5 *If S = {$\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$} is an orthogonal basis for $\mathbb{R}^n$, and* **u** *is any vector in $\mathbb{R}^n$, then*

$$\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 + \frac{(\mathbf{u}, \mathbf{v}_2)}{(\mathbf{v}_2, \mathbf{v}_2)}\mathbf{v}_2 + \cdots + \frac{(\mathbf{u}, \mathbf{v}_n)}{(\mathbf{v}_n, \mathbf{v}_n)}\mathbf{v}_n$$

or

$$\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v}_1)}{\| \mathbf{v}_1 \|^2}\mathbf{v}_1 + \frac{(\mathbf{u}, \mathbf{v}_2)}{\| \mathbf{v}_2 \|^2}\mathbf{v}_2 + \cdots + \frac{(\mathbf{u}, \mathbf{v}_n)}{\| \mathbf{v}_n \|^2}\mathbf{v}_n$$

Proof Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis, a vector $\mathbf{u}$ can be expressed in the form

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$$

For each vector in S, we have

$$\begin{aligned}(\mathbf{u}, \mathbf{v}_i) &= (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i) \\ &= k_1(\mathbf{v}_1, \mathbf{v}_i) + k_2(\mathbf{v}_2, \mathbf{v}_i) + \ldots + k_n(\mathbf{v}_n, \mathbf{v}_i)\end{aligned}$$

Since S is an orthogonal set, we have

$$(\mathbf{v}_j, \mathbf{v}_i) = 0 \quad \text{if} \quad j \neq i.$$

Therefore, $(\mathbf{u}, \mathbf{v}_i) = k_i(\mathbf{v}_i, \mathbf{v}_i)$ and

$$k_i = \frac{(\mathbf{u}, \mathbf{v}_i)}{(\mathbf{v}_i, \mathbf{v}_i)} \qquad \square$$

EXAMPLE Suppose $\mathbf{v}_1 = (2, 3)$ and $\mathbf{v}_2 = (3, -2)$.

1) Check that $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for $\mathbb{R}^2$.
2) Express the vector $\mathbf{u} = (7, 5)$ as a linear combination of the vectors in S, and find the coordinate vector $(\mathbf{u})_S$.

Solution

$$(\mathbf{v}_1, \mathbf{v}_2) = 2\cdot 3 - 3\cdot 2 = 0$$

$$(\mathbf{u}, \mathbf{v}_1) = 29 \quad \|\mathbf{v}_1\|^2 = 13 \quad \Rightarrow \quad k_1 = \frac{29}{13}$$

$$(\mathbf{u}, \mathbf{v}_2) = 11 \quad \|\mathbf{v}_2\|^2 = 13 \quad \Rightarrow \quad k_2 = \frac{11}{13}$$

$$(\mathbf{u})_S = (29/13,\ 11/13)$$

THEOREM 7 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent.*

Proof Assume that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n = \mathbf{0}$.

For each $\mathbf{v}_i$ in S we have

$$(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i) = (\mathbf{0}, \mathbf{v}_i) = 0$$

or, equivalently,

$$k_1(\mathbf{v}_1, \mathbf{v}_i) + k_2(\mathbf{v}_2, \mathbf{v}_i) + \ldots + k_n(\mathbf{v}_n, \mathbf{v}_i) = 0$$

From the orthogonality of S it follows that $(\mathbf{v}_j, \mathbf{v}_i) = 0$ when $i \neq j$, so this equation reduces to

$$k_i(\mathbf{v}_i, \mathbf{v}_i) = 0$$

Since the vectors in S are nonzero, $(\mathbf{v}_i, \mathbf{v}_i) \neq 0 \Longrightarrow k_i = 0$. So

$$k_1 = k_2 = \ldots = k_n = 0. \qquad \square$$

Notion of a Polynomial

DEFINITION A ***polynomial*** in a single variable can be written in the form

$$f = f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where n is a nonnegative integer. The numbers a_i are ***coefficients***, with a_n the ***leading coefficient*** and a_0 the ***constant term*** of the polynomial. If $a_n \neq 0$, then n is the ***degree*** of the polynomial.

Notation: $\deg(f) = n$.

The ***zero polynomial*** $f(x) = 0$ is not assigned a degree.

Some terminology

A nonzero polynomial = the sum of a finite number of non-zero ***terms***.

A ***monomial***: a one-term polynomial

EXAMPLE $f(x) = 3x^2 - 5x + 4$

term 1: coefficient 3, variable x, and exponent 2

term 2

term 3

$\deg(f) = 2$

Polynomials of small degree

Zeroth degree:	$f(x) = a,\ a \neq 0$	*Constant polynomial*
First degree:	$f(x) = ax + b$	*Linear polynomial*
Second degree:	$f(x) = ax^2 + bx + c$	*Quadratic polynomial*
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	*Cubic polynomial*

Arithmetic of polynomials

addition | multiplication

EXAMPLE Let

$$f(x) = x^3 + 2x + 1 \text{ and } g(x) = x^2 - x + 4.$$

- $f(x) + g(x) = x^3 + x^2 + x + 5$
- $f(x) \cdot g(x) = x^5 - x^4 + 6x^3 - x^2 + 7x + 4$

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$

$$\deg(f \cdot g) = \deg(f) + \deg(g)$$

Polynomial Long Division

two polynomials:

- f (the ***dividend***)
- $g \neq 0$ (the ***divisor***)

→

two polynomials:

- q (a ***quotient***)
- r (a ***remainder***)

$$f(x) = g(x) \cdot q(x) + r(x),$$

where either $r = 0$ or $\deg(r) < \deg(g)$.

These conditions define uniquely $q(x)$ and $r(x)$.

EXAMPLE Divide $x^3 - 2x^2 - 4$ by $x - 3$.

Solution

$$
\begin{array}{rl|l}
- & x^3 - 2x^2 + 0 \cdot x - 4 & x - 3 \\
\hline
 & x^3 - 3x^2 & x^2 + x + 3 \\
\hline
- & x^2 + 0 \cdot x & \\
 & x^2 - 3x & \\
\hline
- & 3x - 4 & \\
 & 3x - 9 & \\
\hline
 & 5 &
\end{array}
$$

Answer:

$$x^3 - 2x^2 - 4 = (x^2 + x + 3)(x - 3) + 5$$

THEOREM 10 (Polynomial Remainder Theorem or **Little Bézout's Theorem)**
The remainder of the division of a polynomial $f(x)$ by a linear polynomial $x - a$ is equal to $f(a)$.

EXAMPLE Suppose $f(x) = x^3 - 2x + 3$.

Polynomial division of $f(x)$ by $x - 2$ gives

- the quotient $x^2 + 2x + 2$
- and the remainder 7

Therefore, $f(2) = 7$

Proof

Polynomial long division

$$f(x) = q(x) \cdot g(x) + r(x)$$

where deg (r) < deg (g)

Take $g(x) = x - a$ as the divisor $\Longrightarrow$ deg $(g) = 1 \Longrightarrow$ deg $(r) = 0$, i.e. $r(x) = r$. Therefore,

$$f(x) = q(x) \cdot (x - a) + r$$

Setting $x = a$ we obtain:

$$f(a) = q(a) \cdot (a - a) + r = r \quad \square$$

COROLLARY (The Factor Theorem)

A polynomial $f(x)$ has a factor $x - a$ if and only if $f(a) = 0$. (i.e. a is a root of the polynomial).

Proof We have

$$f(x) = q(x) \cdot (x - a) + r$$

1) If $f(a) = 0$, then $r = 0$, and $f(x)$ has a factor $x - a$
2) If $f(x)$ has a factor $x - a$, i.e. $f(x) = q(x) \cdot (x - a)$, then $f(a) = 0$. □

Lecture 9

Polynomials and Complex Numbers

DEFINITION If $f(a) = 0$ then a is called the ***root*** (***корень***) of the polynomial $f(x)$.

THEOREM 1 (The Factor Theorem) *A polynomial $f(x)$ has a factor $x - a$ if and only if $f(a) = 0$.*

THEOREM 2
(The integer root theorem)

*If an integer is a root of a polynomial whose coefficients are integers and whose leading coefficient is ±1, then that integer is a **factor** of the constant term.*

EXAMPLE Find the factors at

$$f(x) = x^3 + 7x^2 + 8x + 2.$$

Solution The roots should be among the divisors of 2:   1, –1, 2, –2.

1) Substitute $x = 1$ into f, using the Horner's scheme:

x_0	x^3	x^2	x^1	x^0
	1	7	8	2
1	1	8	16	18

$$f(x) = (x^2 + 8x + 16)(x - 1) + 18$$

2) Next try $x + 1$ (substituting $x = -1$ into the f):

x_0	x^3	x^2	x^1	x^0
	1	7	8	2
-1	1	6	2	0

$$f(x) = (x^2 + 6x + 2)(x + 1)$$

3) $x = -3 \pm \sqrt{7}$

Answer: $$f(x) = (x+1)\left(x+3+\sqrt{7}\right)\left(x+3-\sqrt{7}\right)$$

Divisibility of Polynomials

DEFINITION g ***divides*** f or g is a divisor of f if there exists a polynomial q such that $g \cdot q = f$.

Notation: $g \mid f$

a is a root of f $\Longleftrightarrow$ $(x - a) \mid f$

If $(x - a)^2$ divides f then a is called a ***multiple root*** (***кратный корень***) of f, and otherwise a is called a ***simple root*** of f.

DEFINITION The ***multiplicity*** (***кратность***) of the root a in f is a highest power m such that $(x - a)^m$ divides f.

DEFINITION A polynomial which cannot be factorized into the product of two non constant polynomials is called *an* ***irreducible polynomial*** (***неприводимый многочлен***).

EXAMPLE Polynomial $f(x) = x^2 + 1$ is irreducible.

THEOREM 3 *Any polynomial can be decomposed into the product of a constant by a product of irreducible polynomials. This decomposition is unique up to the order of the factors and the multiplication of any constant factors by a constant.*

Greatest Common Divisor

DEFINITION Let *f* and *g* be polynomials. A ***greatest common divisor*** (***наибольший общий делитель***) of *f* and *g* is a polynomial *d* that divides *f* and *g* and such that every common divisor of *f* and *g* also divides *d*.

Notation: GCD(*f*, *g*).

- If *f* = *g* = 0, the GCD is 0.
- The GCD is unique up to the multiplication by a nonzero constant.

EXAMPLE: Find the GCD of

$$x^2 + 7x + 6 \text{ and } x^2 - 5x - 6.$$

Solution

1) Factor $x^2 + 7x + 6$:

$$x^2 + 7x + 6 = (x + 1)(x + 6)$$

2) Factor $x^2 - 5x - 6$:

$$x^2 - 5x - 6 = (x + 1)(x - 6)$$

Thus, their GCD is

$$x + 1$$

Properties of GCD

- If c is any common divisor of f and g, then c divides their GCD.
- $\mathrm{GCD}(f, g) = \mathrm{GCD}(g, f)$
- $\mathrm{GCD}(f, g) = \mathrm{GCD}(f, g + rf)$ for any polynomial r.
- For any $k \neq 0$, $\mathrm{GCD}(f, g) = \mathrm{GCD}(f, kg)$
- For two polynomials f and g, there exist polynomials a and b, such that

  $$\mathrm{GCD}(f, g) = af + bg$$

  and $\mathrm{GCD}(f, g)$ divides every such linear combination of f and g.

Finding the Greatest Common Divisor

Two ways to find GCD

Factorization

- find the factors of each polynomial
- select the set of common factors

The Euclidean algorithm

Euclidean Algorithm

to find the GCD of two polynomials $a(x)$ and $b(x)$, where $\deg(b(x)) \leq \deg(a(x))$

1) By polynomial long division:

$$a(x) = q_0(x)b(x) + r_0(x),\ \deg(r_0(x)) < \deg(b(x))$$

2) By propetry, $\mathrm{GCD}(a, b) = \mathrm{GCD}(b, r_0)$.

Then set $a_1(x) = b(x)$, $b_1(x) = r_0(x)$.

3) Repeat the process:

$$a_1(x) = q_1(x)b_1(x) + r_1(x)$$

Then set $a_2(x) = b_1(x)$, $b_2(x) = r_1(x)$ and so on.

4) When we reach a point at which $b_N(x) = 0$,

$$\mathrm{GCD}(a, b) = \mathrm{GCD}(a_1, b_1) = \ldots = \mathrm{GCD}(a_N, 0) = \boxed{a_N}$$

EXAMPLE: Find the GCD of $x^3 + 9x^2 + 20x + 12$ and $x^2 - 5x - 6$.

1) $x^3 + 9x^2 + 20x + 12 =$

$$(x^2 - 5x - 6)(x + 14) + 96(x + 1)$$

2) $x^2 - 5x - 6 =$

$$(x + 1)(x - 6) + 0$$

Since $x+1$ is the last nonzero remainder, the GCD of these polynomials is $x+1$

Partial Fraction Decomposition

Partial fraction decomposition of a rational function is the operation that consists in expressing the fraction as a sum of a polynomial (possibly zero) and one or several fractions with a simpler denominator.

Decomposition of $N(x)/D(x)$ into Partial Fractions

- **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if $\deg(N) \geq \deg(D)$), divide $N(x)$ into $D(x)$ to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

- **Factor denominator** into factors of the form

$$(px + q)^m \text{ and } (ax^2 + bx + c)^n,$$

where $ax^2 + bx + c$ is irreducible.

- **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \ldots + \frac{A_m}{(px+q)^m}$$

- **Quadratic factors:** For each factor of the form

$(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \ldots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

EXAMPLE 1

$$\frac{5x^2+20x+6}{x(x+1)^2}=\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}$$

EXAMPLE 2

$$\frac{2x^3-4x-8}{x(x-1)(x^2+4)}=\frac{2}{x}-\frac{2}{x-1}+\frac{2x+4}{x^2+4}$$

Definition of a Complex Number

$x^2 \geq 0$, for every real number x → the equation $x^2 = -1$ has no real solutions

Introduce the "imaginary"

number $i = \sqrt{-1}$

with the property

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

Gerolamo Cardano (1501 –1576)

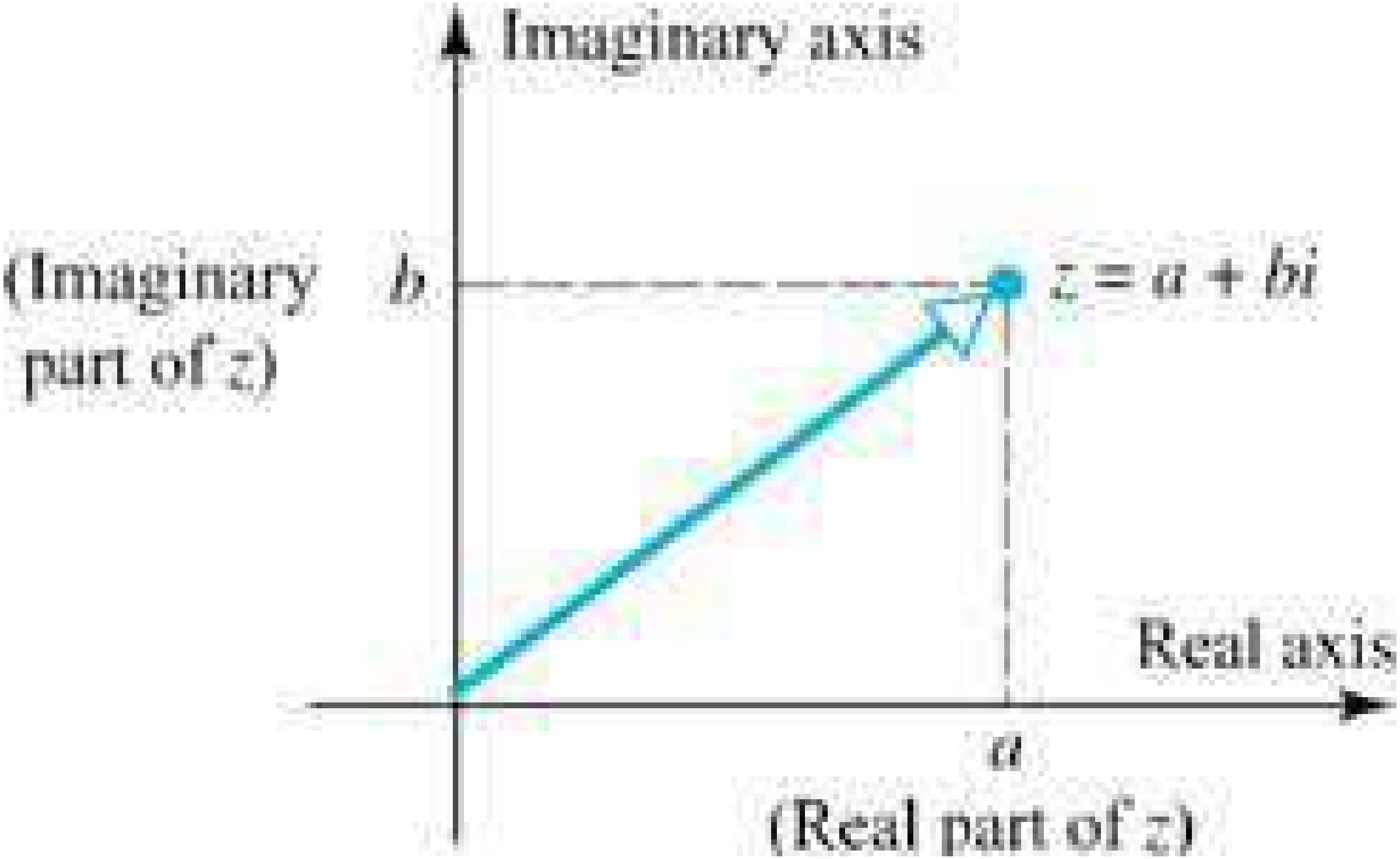
DEFINITION A ***complex number*** (***комплексное число***) is an ordered pair of real numbers, denoted by $a + bi$, where $i^2 = -1$.

EXAMPLES $4 + 3i$, $2 - i$, $-1 + i$.

- the real number a is called the ***real part of z*** (***действительная часть***), $\mathrm{Re}(z) = a$
- the real number b is called the ***imaginary part of z*** (***мнимая часть***), $\mathrm{Im}(z) = b$

EXAMPLE $\mathrm{Re}(4 - 3i) = 4$, $\mathrm{Im}(4 - 3i) = -3$.

The Complex Plane



DEFINITION Two complex numbers, $a + bi$ and $c + di$ are ***equal*** if $a = c$ and $b = d$.

- If $b = 0$, then $a + bi = a$

real numbers ⇔ points on the real axis

- If $a = 0$, then $a + bi = bi$ ⇒

pure imaginary numbers (***чисто мнимые***) ⇔ points on the imaginary axis

Operations on Complex Numbers

- Addition of complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- Subtraction of complex numbers:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

- Multiplication by a *real* number k:

$$k(a + bi) = (ka) + (kb)i$$

Since $(-1)z + z = 0$, we denote $(-1)z$ as $-z$ and call it the ***negative of z***.

EXAMPLE Let $z_1 = 4 - 5i$, $z_2 = -1 + 6i$. Then

$$z_1 + z_2 = (4-5i)+(-1+6i)$$

$$= (4-1)+(-5+6)i = 3+i$$

$$z_1 - z_2 = (4-5i)-(-1+6i)$$

$$= (4+1)+(-5-6)i = 5-11i$$

$$3z_1 = 3(4-5i) = 12-15i$$

$$-z_2 = -(-1+6i) = 1-6i$$

Multiplication of Complex Numbers

Expand the product $(a + bi) \cdot (c + di)$, following the usual rules of algebra but treating i^2 as -1. This yields

$$\begin{aligned}(a + bi) \cdot (c + di) &= ac + bdi^2 + bci + adi \\ &= (ac - bd) + (bc + ad)i\end{aligned}$$

which suggests the following definition:

$$(a + bi) \cdot (c + di) = (ac - bd) + (bc + ad)i$$

EXAMPLE

$$(3+2i)(4+i) = 12+3i+8i+2i^2$$

$$=12+11i-2=10+11i$$

$$\left(5-\frac{1}{2}i\right)(2+3i) = 10+15i-i-\frac{3}{2}i^2$$

$$=10+14i+\frac{3}{2}=\frac{23}{2}+14i$$

$$i(1+i)(1-2i) = 1+3i$$

Lecture 10

Complex Numbers

Complex Conjugates

DEFINITION If $z = a + bi$ is any complex number, then the ***complex conjugate*** (***комплексно-сопряженное***) of z (also called the ***conjugate*** of z) is defined by $\bar{z} = a - bi$.

EXAMPLES

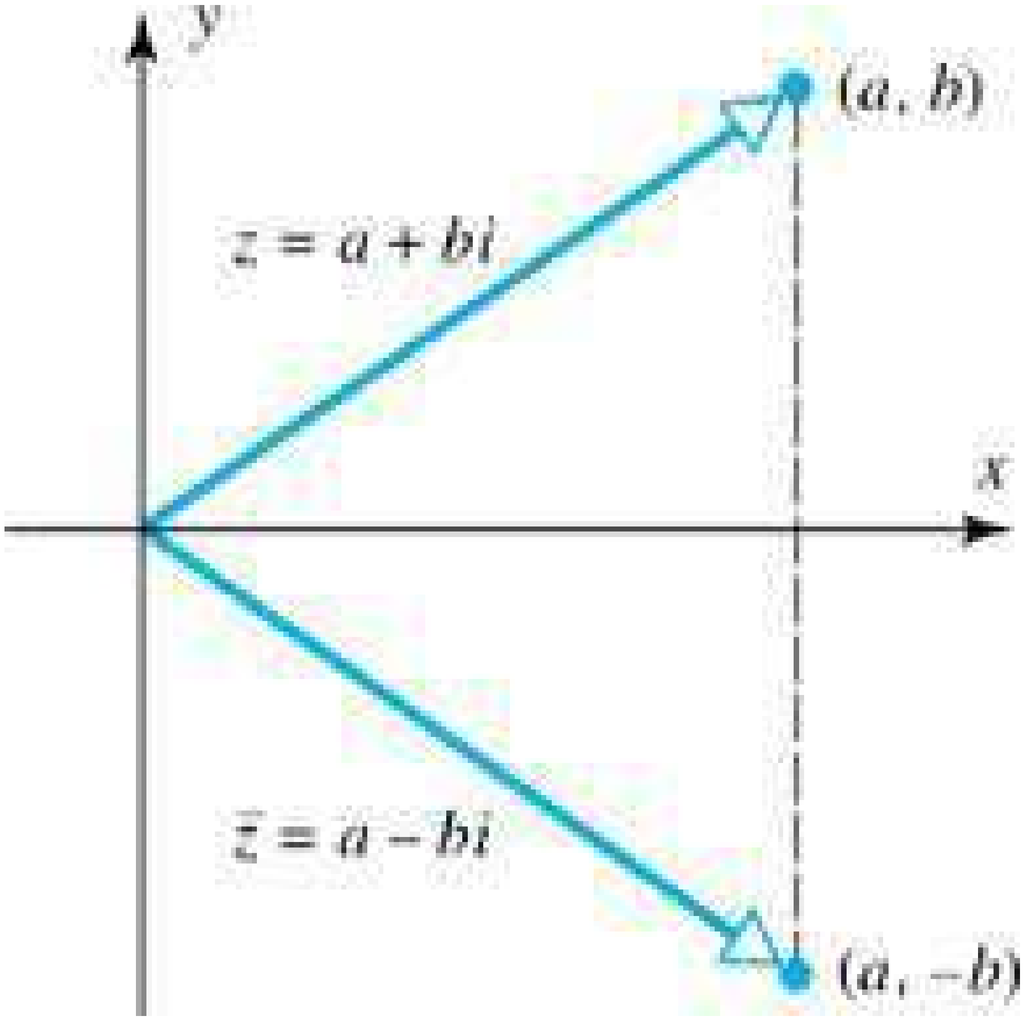
$$z = 3 + 2i \qquad \bar{z} = 3 - 2i$$

$$z = -4 - 7i \qquad \bar{z} = -4 + 7i$$

$$z = i \qquad \bar{z} = -i$$

$$z = 4 \qquad \bar{z} = 4$$

The Conjugate of a Complex Number



Modulus of a Complex Number

DEFINITION The ***modulus*** (***модуль***) of a complex number $z = a + bi$, denoted by $|z|$, is defined by

$$|z| = \sqrt{a^2 + b^2}$$

If $b = 0$, then $z = a$ is a real number, and

$$|z| = \sqrt{a^2 + 0^2} = \sqrt{a^2} = |a|$$

so the modulus of a real number is its absolute value. Thus the modulus of z is also called the ***absolute value*** of z.

EXAMPLE If $z = 3 - 4i$, then $|z| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

THEOREM 1 *For any complex number z,*

$$z\bar{z} = |z|^2$$

Proof If $z = a + bi$, then

$$z\bar{z} = (a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2 = |z|^2$$

$\square$

EXAMPLE $(2 + 5i)(2 - 5i) = 2^2 + 5^2 = 29$

Division of Complex Numbers

If $z_2 \neq 0$, then

$$z = \frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{z_1 \overline{z_2}}{\left|z_2\right|^2}$$

EXAMPLE Express $\frac{3+4i}{1-2i}$ in the form $a + bi$.

Solution $$\frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)}$$

$$= \frac{3+6i+4i+8i^2}{1-4i^2} = \frac{-5+10i}{5} = -1+2i$$

THEOREM 2 Properties of the Conjugate

For any complex numbers z, z_1, and z_2:

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(b) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(c) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

(d) $\overline{z_1 / z_2} = \overline{z_1} / \overline{z_2}$

(e) $\overline{\overline{z}} = z$

Polar Form (тригонометрический вид)

$z = x + iy,\ z \neq 0, \quad r = |z|.$

Then

$$x = r\cos\theta, \qquad y = r\sin\theta$$

so that

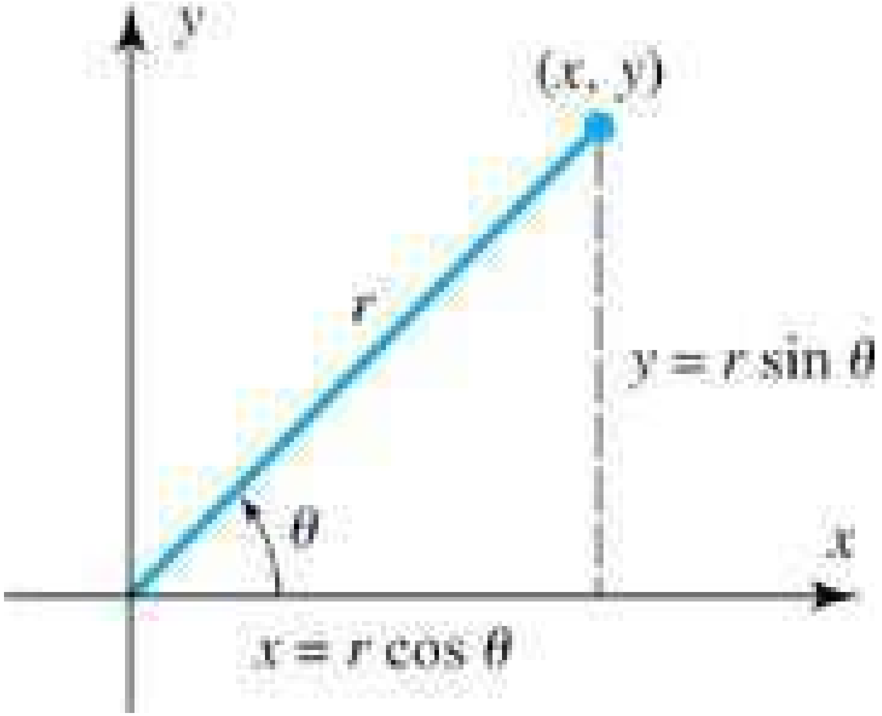
$$z = x + iy = r\cos\theta + ir\sin\theta$$

or

$$z = r(\cos\theta + i\sin\theta)$$

This is a ***polar form of z***.

The angle θ is called an ***argument of z*** and is denoted by $\theta = \arg z$



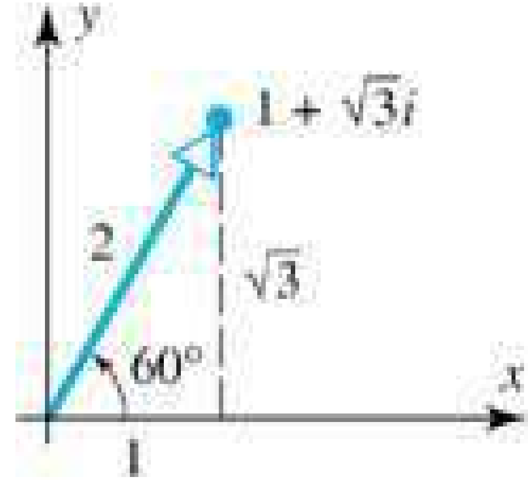
EXAMPLE 1 Express in polar form

$$z = 1 + \sqrt{3}\, i$$

Solution

$$r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$z = 1 + \sqrt{3}\, i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$



(a)

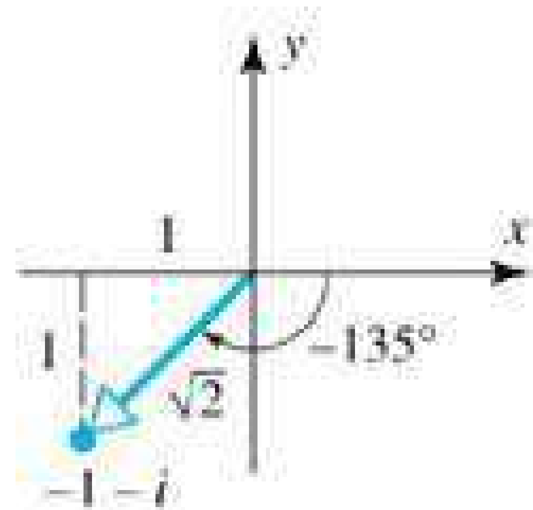
EXAMPLE 2 Express in polar form

$$z = -1 - i$$

Solution

$$r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$z = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$$



(b)

Multiplication and Division in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplying, we obtain

$$z_1 z_2 = r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)]$$

$\cos(\theta_1 + \theta_2)$

$\sin(\theta_1 + \theta_2)$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Similarly,

if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

EXAMPLE

$$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \qquad z_2 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z_1 z_2 = 2\cdot 3\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right]$$

$$= 6\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 6i$$

$$\frac{z_1}{z_2} = \frac{2}{3}\left[\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]$$

$$= \frac{2}{3}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = \frac{1}{3}\left(\sqrt{3} + i\right)$$

DeMoivre's Formula (формула Муавра)

If n is a positive integer and $z = r\,(\cos\theta + i\sin\theta)$, then

$$z^n = z \cdot z \cdot z \cdot \ldots \cdot z$$

$$= r^n\,[\cos\,(\theta + \theta + \ldots + \theta) + i\sin\,(\theta + \theta + \ldots + \theta)]$$

or

$$z^n = r^n\,(\cos n\theta + i\sin n\theta)$$

Moreover, this also holds for negative integers if $z \neq 0$.

Finding *n*th Roots

If n is a positive integer and $z \neq 0$ is any complex number, then we define an ***nth root of z*** to be any complex number w that satisfies

$$w^n = z$$

We denote an nth root of z by $z^{1/n}$. Let

$$w = \rho\,(\cos\alpha + i\sin\alpha) \text{ and } z = r\,(\cos\theta + i\sin\theta)$$

Then

$$\rho^n\,(\cos n\alpha + i\sin n\alpha) = r\,(\cos\theta + i\sin\theta)$$

We see that $\rho^n = r$ or

$$\rho = \sqrt[n]{r}$$

Moreover, since

$$\cos n\alpha = \cos\theta \text{ and } \sin n\alpha = \sin\theta,$$

we have

$$n\alpha = \theta + 2\pi k \quad \text{or} \quad \alpha = \frac{\theta}{n} + \frac{2\pi k}{n}, \quad k \in \mathbb{Z}$$

Since only 0, 1, 2, …, $n - 1$ produce distinct values of α, there are n different nth roots of $z = r(\cos\theta + i\sin\theta)$:

$$w = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right],$$

$$k = 0, 1, 2, \ldots, n - 1$$

EXAMPLE Find all cube roots of −8.

Solution An argument of −8 is $\theta = \pi$.

Moreover, $r = |z| = |-8| = 8$, so

$$-8 = 8\,(\cos\pi + i\sin\pi)$$

We have

$$(-8)^{1/3} = \sqrt[3]{8}\left[\cos\left(\frac{\pi}{3}+\frac{2\pi k}{3}\right)+i\sin\left(\frac{\pi}{3}+\frac{2\pi k}{3}\right)\right],\ k = 0, 1, 2$$

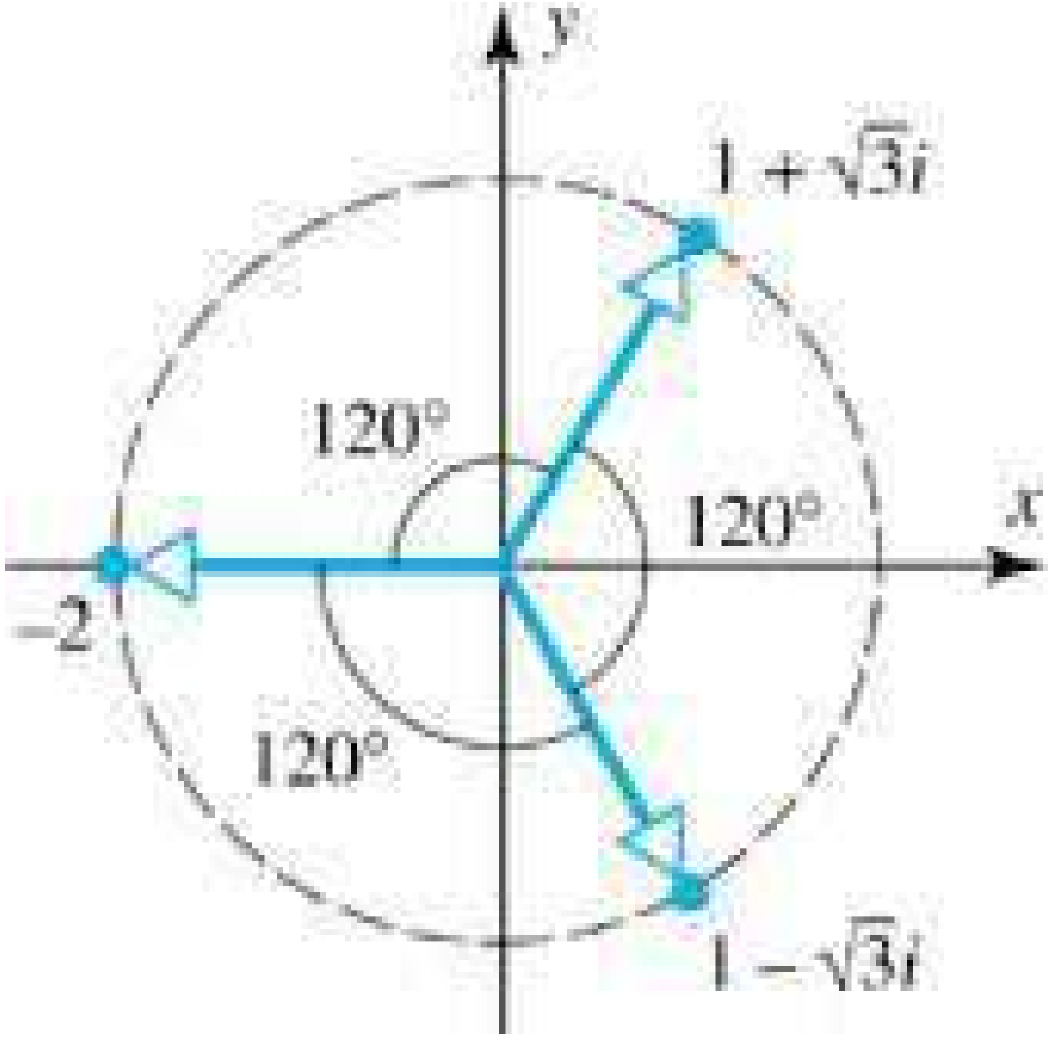
Thus the cube roots of −8 are

$$2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 1+\sqrt{3}i$$

$$2(\cos\pi + i\sin\pi) = 2(-1) = -2$$

$$2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 1-\sqrt{3}i$$

The cube roots of –8

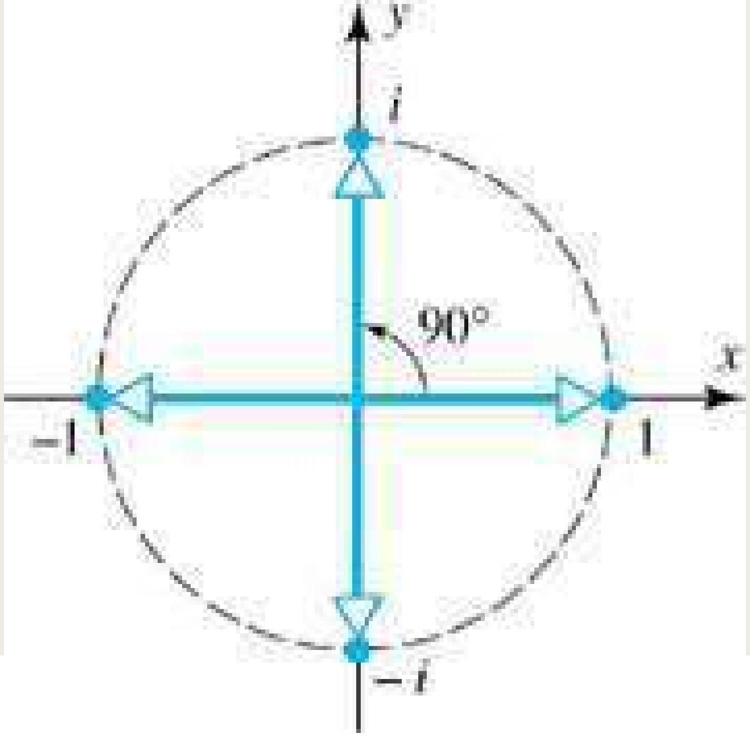


EXAMPLE Find all fourth roots of 1.

Solution Observe that

$z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)\ (z - i)(z + i)$

So the fourth roots of 1 are 1, -1, i, $-i$.



THEOREM 3 (Fundamental Theorem of Algebra) *Let $f(z)$ be a polynomial of degree $n \geq 1$. Then $f(z)$ always has a root $z_0 \in \mathbb{C}$.*

COROLLARY *Let $f(z)$ be a polynomial of degree $n \geq 1$. Then there exist complex numbers $z_1, z_2, \ldots, z_n$, such that*

$$f(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

THEOREM 4 *Suppose*

$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0$

has real coefficients. If z_0 *is a root of* $f(z)$*, then the conjugate* $\overline{z_0}$ *is also a root of* $f(z)$.

THEOREM 5 *If*

$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0$

is a polynomial with real coefficients, then it could be factorized in factors with real coefficients of degree one or two.

EXAMPLE Find a polynomial of 2nd degree with real coefficients that has the root $z_1 = 2 + 5i$.

Solution By theorem 4, this polynomial has another root $z_2 = 2 - 5i$. Thus

$$f(z) = (z - z_1)(z - z_2)$$

$$= (z - (2 + 5i))(z - (2 - 5i))$$

$$= (z - 2)^2 - 25i^2$$

$$= z^2 - 4z + 4 + 25 = z^2 - 4z + 29$$

Functions from $\mathbb{R}^n$ to $\mathbb{R}$

DEFINITION A ***function*** $f : A \to B$ is a rule

each element a in a set A → one and only one element b in a set B

Write: $b = f(a)$ or $a \mapsto b$

EXAMPLE

$f(x, y) = x^2 + y^2$ is a function from $\mathbb{R}^2$ to $\mathbb{R}$

Terminology

- b is the ***image (образ)*** of a under f
- $f(a)$ is the ***value (значение)*** of f at a
- the set A is the ***domain (область определения)*** of f
- the set B is the ***codomain*** of f
- the subset of B consisting of all values $f(a)$ as a varies over A is the ***range (область значений)*** of f
- A and B are sets of real numbers $\Longrightarrow$ f is a ***real-valued function of a real variable***

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

DEFINITION If $A = \mathbb{R}^n$ and $B = \mathbb{R}^m$ (m and n possibly the same), then f is called a ***map (отображение)*** or a ***transformation (преобразование)*** from $\mathbb{R}^n$ to $\mathbb{R}^m$, and is usually denoted by *T*. We say that *T* ***maps*** $\mathbb{R}^n$ into $\mathbb{R}^m$.

Notation $T: \mathbb{R}^n \to \mathbb{R}^m$

DEFINITION If $m = n$, the transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is called an ***operator*** on $\mathbb{R}^n$.

Suppose that $f_1, f_2, \ldots, f_m$ are real-valued functions of n real variables, say

$$
\begin{aligned}
w_1 &= f_1(x_1, x_2, \ldots, x_n) \\
w_2 &= f_2(x_1, x_2, \ldots, x_n) \\
&\vdots \\
w_m &= f_m(x_1, x_2, \ldots, x_n)
\end{aligned}
$$

These m equations assign a unique point $(w_1, w_2, \ldots, w_m)$ in $\mathbb{R}^m$ to each point $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and thus define a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(x_1, x_2, \ldots, x_n) = (w_1, w_2, \ldots, w_m)$$

EXAMPLE A Transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$

The equations

$$w_1 = x_1 + x_2$$

$$w_2 = 3x_1x_2$$

$$w_3 = x_1^2 - x_2^2$$

define a transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that

$$T(x_1, x_2) = (x_1 + x_2, 3x_1x_2, x_1^2 - x_2^2)$$

Thus, for example, $T(1, -2) = (-1, -6, -3)$

Lecture 11

Linear Transformations

Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

DEFINITION A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is called ***linear*** if it is specified by linear equations of the form

$$\begin{cases} w_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \qquad\vdots \\ w_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{cases}$$

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be written in matrix notation by

$$\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

or more briefly by $\mathbf{w} = A\mathbf{x}$.

The matrix $A = [a_{ij}]$ is called the ***standard matrix*** for the linear transformation T.

Notation: $T = T_A$, $A = [T]$.

EXAMPLE The linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$

$$\begin{cases} w_1 = 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 = 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 = 5x_1 - x_2 + 4x_3 \end{cases}$$

can be expressed in matrix form as

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

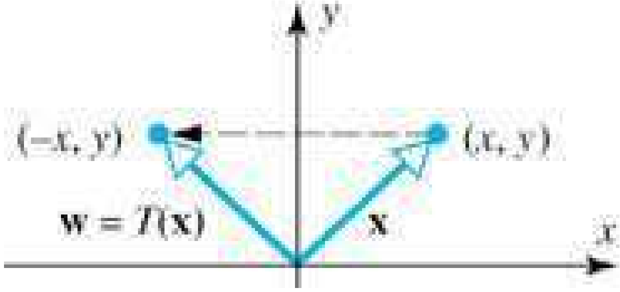
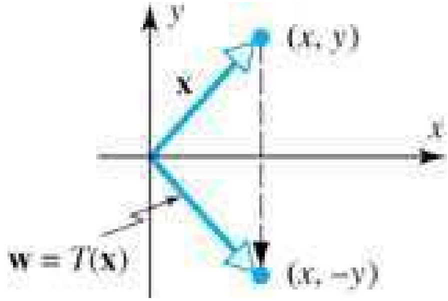
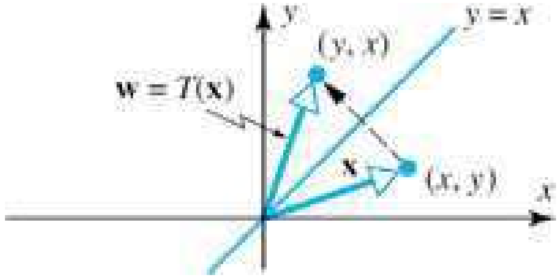
so the standard matrix for T is

$$A = \begin{pmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{pmatrix}$$

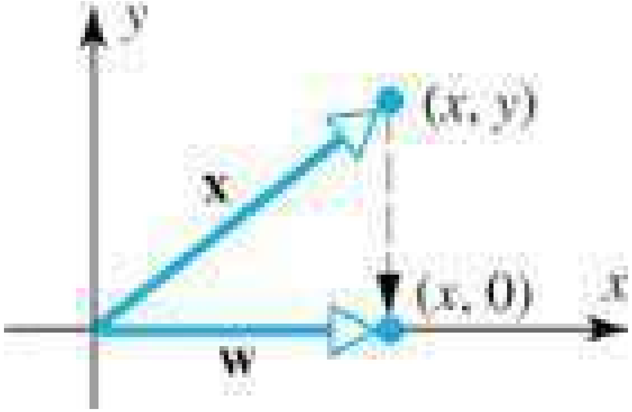
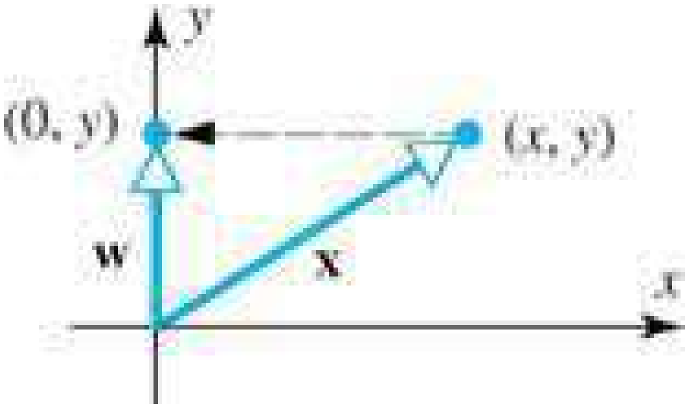
$T(1, -3, 0, 2) = ?$

$(1, 3, 8)$

Reflection Operators

Operator	Illustration	Equations	Standard Matrix
Reflection about the *y*-axis		$w_1 = -x$ $w_2 = y$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection about the *x*-axis		$w_1 = x$ $w_2 = -y$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection about the line *y* = *x*		$w_1 = y$ $w_2 = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Projection Operators

Operator	Illustration	Equations	Standard Matrix
Orthogonal projection on the *x*-axis		$w_1 = x$ $w_2 = 0$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Orthogonal projection on the *y*-axis		$w_1 = 0$ $w_2 = y$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Rotation Operators

Let ϕ be the angle from x-axis to **x** and $r = |\mathbf{x}| = |\mathbf{w}|$.

Then $\mathbf{x} = (x, y) = (r\cos\phi, r\sin\phi)$

and $\mathbf{w} = (w_1, w_2) = (r\cos(\phi + \theta), r\sin(\phi + \theta))$

Since $w_1 = \boxed{r\cos\phi}\cos\theta - \boxed{r\sin\phi}\sin\theta$

$w_2 = \boxed{r\cos\phi}\sin\theta + \boxed{r\sin\phi}\cos\theta$

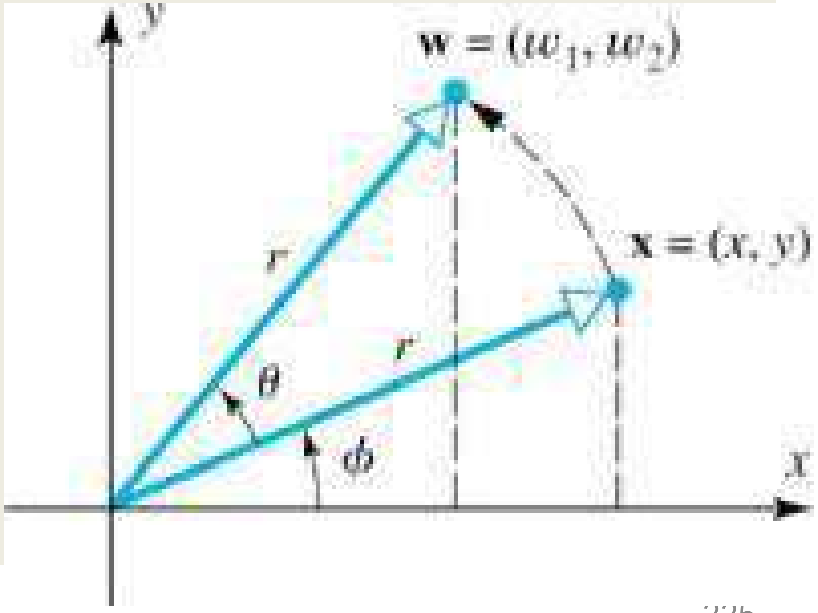
we have

$w_1 = x\cos\theta - y\sin\theta$

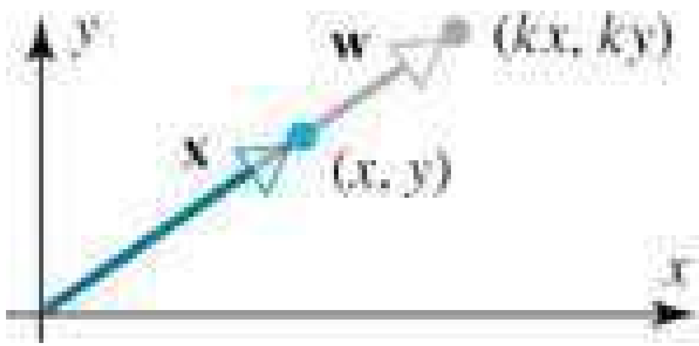
$w_2 = x\sin\theta + y\cos\theta$

Then

$$[T] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Dilation and Contraction Operators (операторы сжатия и растяжения)

Operator	Illustration	Equations	Standard Matrix
Contraction with factor *k* $(0 \le k \le 1)$			
Dilation with factor *k* $(k \ge 1)$			

$$w_1 = kx \qquad w_2 = ky$$

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Compositions of Linear Transformations

If T_A: $\mathbb{R}^n \rightarrow \mathbb{R}^k$ and T_B: $\mathbb{R}^k \rightarrow \mathbb{R}^m$ are linear transformations, then

each $\mathbf{x}$ → $T_A(\mathbf{x})$ → $T_B(T_A(\mathbf{x}))$

This is called the ***composition of*** T_B ***with*** T_A and is denoted by $T_B \circ T_A$ (read "T_B circle T_A"). Thus

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$

The composition $T_B \circ T_A$ is linear since

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = B(A\mathbf{x}) = (BA)\mathbf{x}$$

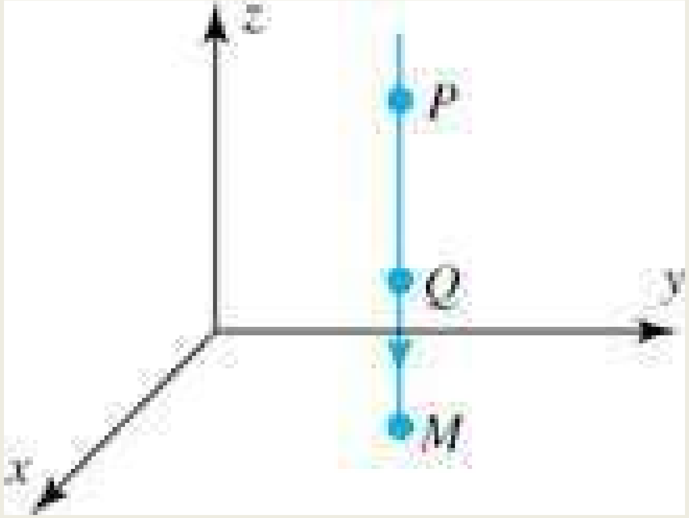
Therefore, $T_B \circ T_A$ is multiplication by BA:

$$T_B \circ T_A = T_{BA}$$

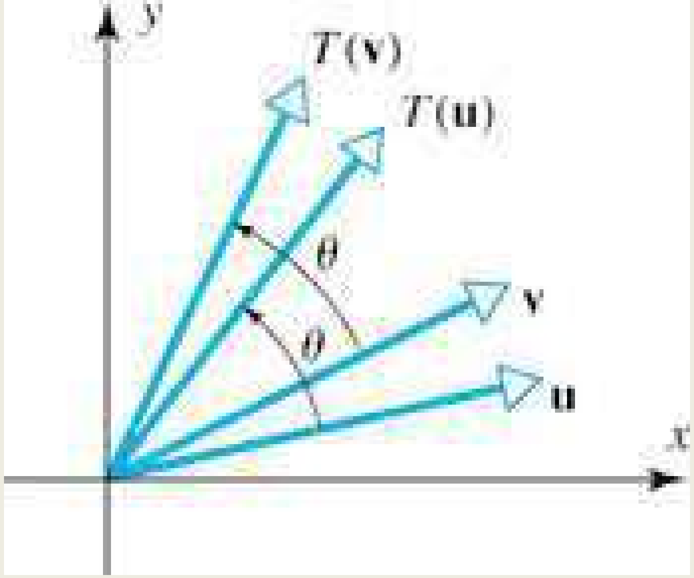
One-to-One Linear Transformations

DEFINITION A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be ***one-to-one*** if T maps distinct vectors in $\mathbb{R}^n$ into distinct vectors in $\mathbb{R}^m$.

EXAMPLE



Projection: not one-to-one



Rotation: one-to-one

THEOREM 1 Equivalent Statements

If A is an n x n matrix and T_A: $\mathbb{R}^n \to \mathbb{R}^n$ is multiplication by A, then the following statements are equivalent.

(a) *A is invertible.*

(b) *The range of T_A is $\mathbb{R}^n$.*

(c) *T_A is one-to-one.*

EXAMPLE Rotation operator *T* is one-to-one, since

$$\det[T] = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Inverse of a One-to-One Linear Operator

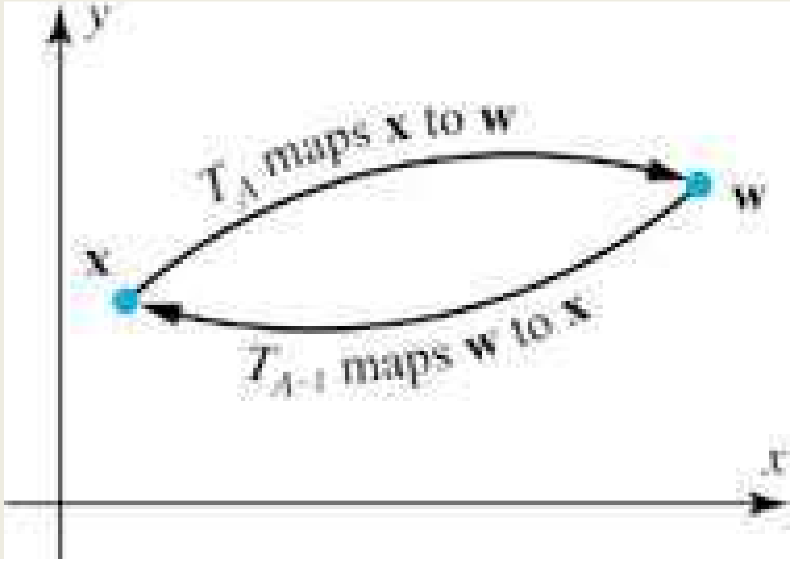
If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear operator,

$$\mathbf{x} \mapsto \mathbf{w} = T(\mathbf{x}),$$

$\Longrightarrow$ there is the operator $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ that maps **w** back to **x**.

Operator T^{-1} is called the ***inverse of*** T.

$$[T^{-1}] = [T]^{-1}$$



THEOREM 2 Properties of Linear Transformations

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ *is linear if and only if the following relationships hold for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$ *and for every scalar c.*

(a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(b) $T(c\mathbf{u}) = cT(\mathbf{u})$

Proof 1) Assume first that T is a linear transformation, and let A be the standard matrix for T. Then

$$T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$$

2) Conversely, assume that properties (*a*) and (*b*) hold.

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the ***standard basis*** vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and let A be

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \ldots \mid T(\mathbf{e}_n)]$$

If $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$ is any vector in $\mathbb{R}^n$, then

$$A\mathbf{x} = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \ldots + x_n T(\mathbf{e}_n)$$

$$= T(x_1 \mathbf{e}_1) + T(x_2 \mathbf{e}_2) + \ldots + T(x_n \mathbf{e}_n)$$

$$= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n) = T(\mathbf{x})$$

This shows that T is multiplication by A and therefore linear.

□

THEOREM 3

If T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $\mathbf{e}_1$, $\mathbf{e}_2, \ldots, \mathbf{e}_n$ *are the standard basis vectors for* $\mathbb{R}^n$, *then the standard matrix for T is*

$$[T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \ldots \mid T(\mathbf{e}_n)]$$

EXAMPLE Suppose T: $\mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\3\end{pmatrix},\quad T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix},\quad T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\6\end{pmatrix}$$

Then

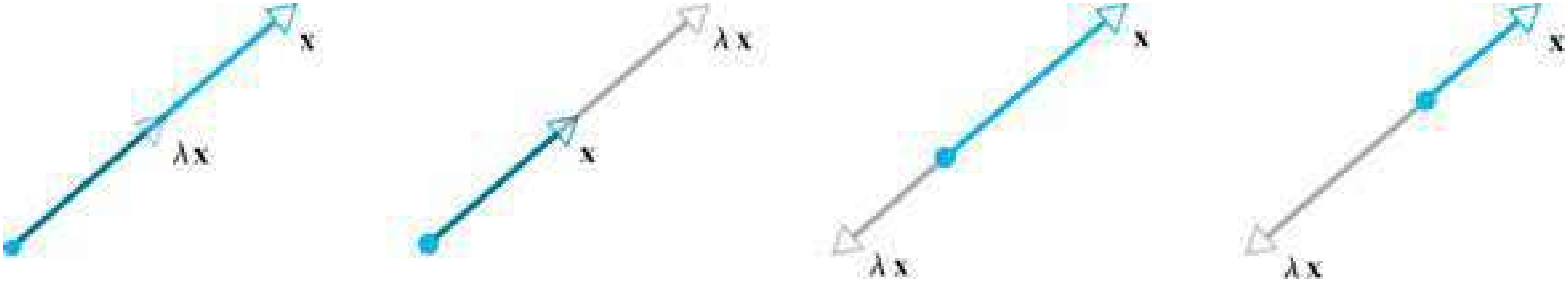
$$[T]=\begin{pmatrix}-1 & 2 & 1\\3 & 0 & 6\end{pmatrix}$$

Eigenvalues and Eigenvectors

DEFINITION If A is an n x n matrix, then a nonzero vector **x** in $\mathbb{R}^n$ is called an ***eigenvector*** (***собственный вектор***) of A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** (***собственное значение***) of A, and **x** is said to be an eigenvector of A ***corresponding*** to λ.



(a) $0 \le \lambda \le 1$ (b) $\lambda \ge 1$ (c) $-1 \le \lambda \le 0$ (d) $\lambda \le -1$

EXAMPLE Verify whether the vectors

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } y = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

are eigenvectors of

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

Solution

$$A\mathbf{x} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\mathbf{x}$$

x is an eigenvector of *A* corresponding to the eigenvalue λ = 3

$$A\mathbf{y} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \end{pmatrix}$$

y is not an eigenvector of *A*

Rewrite $A\mathbf{x} = \lambda\mathbf{x}$ as

$$A\mathbf{x} - \lambda\mathbf{x} = A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = 0$$

I is the identity matrix

nonzero solutions ⟺ $\det(A - \lambda I) = 0$

- $p(\lambda) = \det(A - \lambda I)$ is the ***characteristic polynomial*** (***характеристический многочлен***) of *A*
- $p(\lambda) = 0$ is the ***characteristic equation*** (***характеристическое уравнение***) of *A*

$\deg(p) = n$ ⟺ $p(\lambda) = 0$ has at most *n* distinct solutions

⟺

n x *n* matrix *A* has at most *n* distinct eigenvalues

EXAMPLE Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

Solution

1) The characteristic equation of *A* is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0$$

Therefore, the eigenvalues of *A* are

$\lambda_1 = -2$ and $\lambda_2 = 5$.

2) **x** is an eigenvector of A ⇔ **x** is a nontrivial solution of $(A - \lambda I)\mathbf{x} = 0$; that is,

$$\begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda_1 = -2$

$$\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix}, t \neq 0$$

$\lambda_2 = 5$

$$\begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3t/4 \\ t \end{pmatrix}, t \neq 0$$

THEOREM 4 *If A is a triangular matrix, then the eigenvalues of A are the entries on the main diagonal of A.*

Proof for a 3 x 3 upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$$

and the eigenvalues are $\lambda = a_{11}$, $\lambda = a_{22}$, $\lambda = a_{33}$. □

THEOREM 5 *If k is a positive integer, λ is an eigenvalue of a matrix A, and* **x** *is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and* **x** *is a corresponding eigenvector.*

Proof If λ is an eigenvalue of *A* and ***x*** is a corresponding eigenvector, then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$

$$A^k\mathbf{x} = A(A^{k-1}\mathbf{x}) = A(\lambda^{k-1}\mathbf{x}) = \lambda^{k-1}(A\mathbf{x}) = \lambda^{k-1}(\lambda\mathbf{x}) = \lambda^k\mathbf{x}$$

$\Longrightarrow$ λ^k is an eigenvalue of A^k and **x** is an eigenvector. □

Lecture 12

Linear Transformations – 2

Change of Basis

Change-of-Basis Problem If we change the basis for a vector space V from some old basis B to some new basis B', how is the old coordinate vector $(\mathbf{v})_B$ of a vector $\mathbf{v}$ related to the new coordinate vector $(\mathbf{v})_{B'}$?

Solution for dimension 2 Let

- $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be the old basis
- $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ be the new basis

Suppose

$$(\mathbf{u}'_1)_B = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad (\mathbf{u}'_2)_B = \begin{pmatrix} c \\ d \end{pmatrix}$$

That is,

$$\mathbf{u}'_1 = a\mathbf{u}_1 + b\mathbf{u}_2 \qquad \text{and} \qquad \mathbf{u}'_2 = c\mathbf{u}_1 + d\mathbf{u}_2$$

Let $\mathbf{v} = k_1\mathbf{u}'_1 + k_2\mathbf{u}'_2$ be any vector in V.

Then

$$\begin{aligned}\mathbf{v} &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2)\\ &= (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2\end{aligned}$$

Thus the old coordinate vector for **v** is

$$(\mathbf{v})_B = \begin{pmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{pmatrix}$$

which can be written as

$$(\mathbf{v})_B = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \quad \text{or} \quad (\mathbf{v})_B = \begin{pmatrix} a & c \\ b & d \end{pmatrix} (\mathbf{v})_{B'}$$

The old coordinate vector $(\mathbf{v})_B$ results when we multiply the new coordinate vector $(\mathbf{v})_{B'}$ on the left by the matrix

$$P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

The columns of this matrix are the coordinates of the new basis vectors relative to the old basis.

Solution of the Change-of-Basis Problem in General Case

If we change the basis from the old basis

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

to the new basis

$$B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\},$$

then for any **v** the old coordinate vector $(\mathbf{v})_B$ is related to the new one $(\mathbf{v})_{B'}$ by the equation

$$(\mathbf{v})_B = P \cdot (\mathbf{v})_{B'}$$

where

$$P = \left[(\mathbf{u}'_1)_B \,|\, (\mathbf{u}'_2)_B \,|\, \ldots \,|\, (\mathbf{u}'_n)_B \right]$$

is the ***transition matrix*** (***матрица перехода***) from *B*’ to *B*.

EXAMPLE Let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ be bases for $\mathbb{R}^2$, where $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (0, 1)$, $\mathbf{u}'_1 = (1, 1)$, $\mathbf{u}'_2 = (2, 1)$

(a) Find the transition matrix from B' to B.

(b) Find $(\mathbf{v})_B$ if $(\mathbf{v})_{B'} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

Solution (a) We have

$$(\mathbf{u}'_1)_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (\mathbf{u}'_2)_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus the transition matrix from B' to B is $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

Solution (b)

$$(\mathbf{v})_B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

THEOREM 1 *If P is the transition matrix from a basis B′ to a basis B for a vector space V, then P is invertible, and P^{-1} is the transition matrix from B to B′.*

Summary If P is the transition matrix from a basis B' to a basis B, then for every vector **v**:

$$(\mathbf{v})_B = P \cdot (\mathbf{v})_{B'}$$

$$(\mathbf{v})_{B'} = P^{-1} \cdot (\mathbf{v})_B$$

THEOREM 2 *If B and B' are bases for a vector space V, and if $I: V \to V$ is the identity operator, then the transition matrix from B' to B is $[I]_{B,B'}$*

Proof Suppose that $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$ are bases for V. Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V, we have

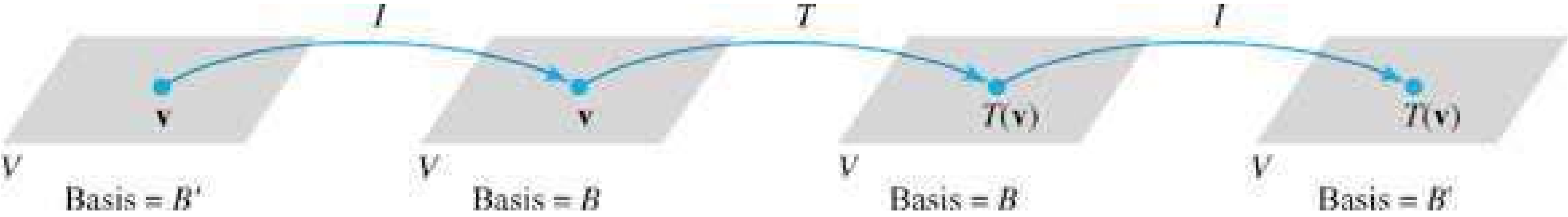
$$[I]_{B,B'} = \left[[I(\mathbf{u}'_1)]_B \mid [I(\mathbf{u}'_2)]_B \mid \ldots \mid [I(\mathbf{u}'_n)]_B \right]$$

$$= \left[(\mathbf{u}'_1)_B \mid (\mathbf{u}'_2)_B \mid \ldots \mid (\mathbf{u}'_n)_B \right] = P$$

□

Effect of Changing Bases on Matrices of Linear Operators

Problem If B and B' are two bases for a vector space V, and if T: $V \rightarrow V$ is a linear operator, what relationship, if any, exists between the matrices $[T]_B$ and $[T]_{B'}$?



We have

$$[T]_{B'} = [I \circ T \circ I]_{B'} = [I]_{B',B} \cdot [T]_B \cdot [I]_{B,B'}$$

Let $P = [I]_{B,B'}$, then $P^{-1} = [I]_{B',B}$, so

$$[T]_{B'} = P^{-1} \cdot [T]_B \cdot P$$

THEOREM 3 *Let T: $V \to V$ be a linear operator on a vector space V, and let B and B' be bases for V. Then*

$$[T]_{B'} = P^{-1} \cdot [T]_B \cdot P$$

where P is the transition matrix from B' to B.

EXAMPLE Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$$

Find the matrix of T with respect to the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$; then find the matrix of T with respect to the basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$, where

$$\mathbf{u}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u}'_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution Since

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$$

then

$$[T]_B = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

Find the transition matrix

$$P = [I]_{B,B'} = \left[(\mathbf{u}'_1)_B \mid (\mathbf{u}'_2)_B \right] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and its inverse

$$P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

so the matrix of *T* relative to the basis *B'* is

$$[T]_{B'} = P^{-1}[T]_B P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Orthogonal Matrices

DEFINITION A square matrix A with the property

$$A^{-1} = A^T$$

is said to be an ***orthogonal matrix*** (***ортогональная***).

A square matrix A is orthogonal if and only if

$$A A^T = A^T A = I$$

EXAMPLE A rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal for all choices of θ, since

$$A^T A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that the row vectors

$$\mathbf{r}_1 = (\cos\theta, -\sin\theta), \qquad \mathbf{r}_2 = (\sin\theta, \cos\theta)$$

are orthogonal, since $(\mathbf{r}_1, \mathbf{r}_2) = 0$.

THEOREM 4 *The following are equivalent for an $n \times n$ matrix A.*

(a) *A is orthogonal.*

(b) *The row vectors of A form an orthonormal set in $\mathbb{R}^n$.*

(c) *The column vectors of A form an orthonormal set in $\mathbb{R}^n$.*

COROLLARY *If P is the transition matrix from one orthonormal basis to another orthonormal basis for $\mathbb{R}^n$, then P is an orthogonal matrix.*

THEOREM 5

(a) *The inverse of an orthogonal matrix is orthogonal.*

(b) *A product of orthogonal matrices is orthogonal.*

(c) *If A is orthogonal, then*

$$\det(A) = 1 \qquad or \qquad \det(A) = -1.$$

EXAMPLE For a rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

we have

$$\det(A) = \cos^2\theta + \sin^2\theta = 1$$

The Matrix Diagonalization Problem

The Eigenvector Problem Given an n x n matrix A, does there exist a basis for $\mathbb{R}^n$ consisting of eigenvectors of A?

The Diagonalization Problem **(*Matrix Form*)** Given an n x n matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?

DEFINITION A square matrix A is called ***diagonalizable*** if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix; the matrix P is said to ***diagonalize*** A.

THEOREM 6 *If A is an n* x *n matrix, then the following are equivalent.*

(a) *A is diagonalizable.*

(b) *A has n linearly independent eigenvectors.*

Lecture 13

Diagonalization and Quadratic Forms

Procedure for Diagonalizing a Matrix

Step 1. Find n linearly independent eigenvectors of A, say $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.

Step 2. Form the matrix P having $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ as its column vectors.

Step 3. The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to $\mathbf{p}_i$ for $i = 1, 2, \ldots, n$.

EXAMPLE Find a matrix *P* that diagonalizes

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$$

Solution The characteristic equation of *A* is

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0$$

and we get the following eigenvalues and eigenvectors:

$$\lambda_1 = 1: \ \mathbf{p}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = 2: \ \mathbf{p}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So the matrix *A* is diagonalizable and

$$P = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

diagonalizes *A*. As a check,

$$P^{-1}AP = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

THEOREM 2 *If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *are eigenvectors of A corresponding to distinct eigenvalues* $\lambda_1, \lambda_2, \ldots \lambda_k$*, then* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a linearly independent set.*

COROLLARY *If an* $n \times n$ *matrix A has n distinct eigenvalues, then A is diagonalizable.*

THEOREM 3 *If A is a symmetric matrix, then*

(a) *The eigenvalues of A are all real numbers.*

(b) *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*

THEOREM 4 *If A is an n* x *n matrix, then the following are equivalent.*

(a) *A is orthogonally diagonalizable.*

(b) *A has an orthonormal set of n eigenvectors.*

(c) *A is symmetric.*

EXAMPLE Find an orthogonal matrix *P* that diagonalizes

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$$

Solution The characteristic equation of *A* is

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 11\lambda + 24 = 0$$

Solving, we get

$$\lambda_1 = 3\text{:}\ \ \mathbf{p}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \lambda_2 = 8\text{:}\ \ \mathbf{p}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

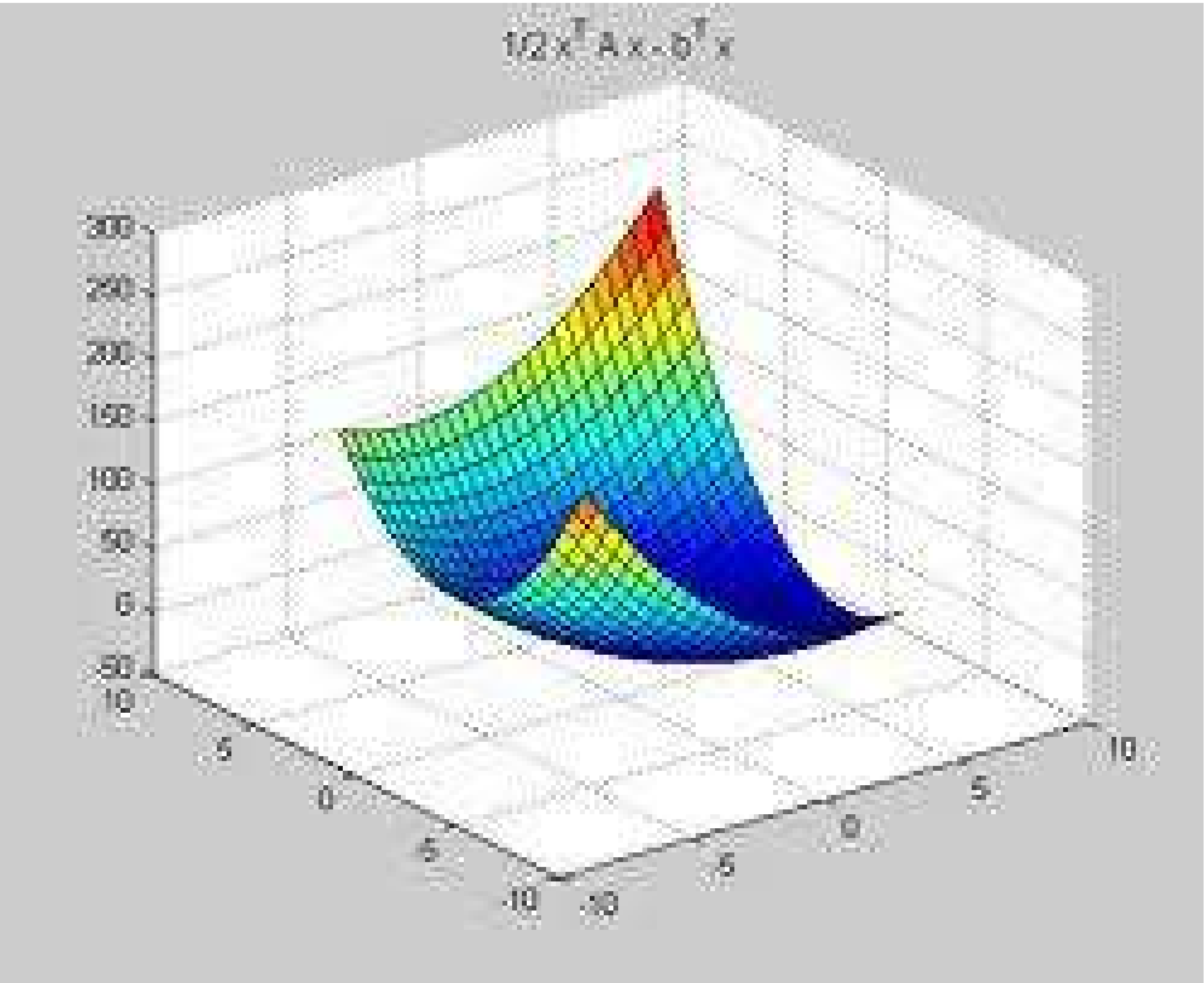
These vectors are orthogonal. Normalizing them yields

$$\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Finally, using $\mathbf{v}_1$ and $\mathbf{v}_2$ as column vectors, we get

$$P = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \text{ and } D = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

Quadratic Forms



- A ***linear form***:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

- A ***quadratic form***:

$$\Phi = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 +$$ all possible terms of the form $a_{ij}x_ix_j$ for $i < j$

EXAMPLE

- A quadratic form in the variables x_1 and x_2 is

$$\Phi = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2$$

- A quadratic form in the variables x_1, x_2 and x_3 is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2$$
$$+a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$$

squared terms ***cross-product terms***

Matrix Representation of Quadratic Forms

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2 \Longleftrightarrow \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$$

$$\Updownarrow$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}/2 & a_{13}/2 \\ a_{12}/2 & a_{22} & a_{23}/2 \\ a_{13}/2 & a_{23}/2 & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

EXAMPLE 1 Find matrix representation of the quadratic form

$$2x^2 + 6xy - 7y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

EXAMPLE 2 Find the quadratic form from its matrix representation

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 10xy + 4y^2$$

- General form of matrix representation:

$$\Phi(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

where **x** is the column vector of variables, and

$$A = \begin{pmatrix} a_{11} & a_{12}/2 & \cdots & a_{1n}/2 \\ a_{12}/2 & a_{22} & \cdots & a_{2n}/2 \\ \vdots & \vdots & & \vdots \\ a_{1n}/2 & a_{2n}/2 & \cdots & a_{nn} \end{pmatrix}$$

coefficients of the squared terms

half the coefficients of the cross-product terms

- A quadratic form in terms of the Euclidean inner product:

$$\mathbf{x}^T A \mathbf{x} = (A\mathbf{x}, \mathbf{x}) = (\mathbf{x}, A\mathbf{x})$$

Definiteness

DEFINITION A quadratic form $\Phi(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is

- ***positive definite*** (***положительно определенная***) if $\Phi(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ (then A is a ***positive definite matrix***)
- ***negative definite*** (***отрицательно определенная***) if $\Phi(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$
- ***indefinite*** (***неопределенная***) if $\Phi(\mathbf{x})$ has both positive and negative values

EXAMPLES

- the form $\Phi(\mathbf{x}) = 2x_1^2 + 5x_2^2$ is **positive definite**
- the form $\Phi(\mathbf{x}) = -3x_1^2 - x_2^2$ is **negative definite**
- the form $\Phi(\mathbf{x}) = 7x_1^2 - 11x_2^2$ is **indefinite**

THEOREM 5 *A symmetric matrix A is positive definite if and only if all the eigenvalues of A are positive.*

THEOREM 6 *A symmetric matrix A is negative definite if and only if all the eigenvalues of A are negative.*

Criterion for Definiteness

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

DEFINITION The ***leading principal submatrices*** of *A* are

$$A_1 = (a_{11}), A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ldots, A_n = A$$

The ***leading principle minors*** **(*угловые миноры*)** are the determinants of these submatrices.

THEOREM 7 (Sylvester's Criterion for Definiteness) *A symmetric matrix A is*

- *positive definite if and only if* det $(A_1) > 0$, det $(A_2) > 0$, …
- *negative definite if and only if* det $(A_1) < 0$, det $(A_2) > 0$, …
- *otherwise, A is indefinite*

EXAMPLE Test for definiteness the matrix

$$A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{pmatrix}$$

Solution

$$\Delta_1 = |\,2\,| = 2 > 0$$

$$\Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{vmatrix} = 1 > 0$$

Answer: the matrix *A* is positive definite.

Diagonalization of Quadratic Forms

THEOREM 8 *Let* $\Phi = \mathbf{x}^T A\mathbf{x}$ *be a quadratic form in the variables* $x_1, x_2, \ldots, x_n$*. If* P *orthogonally diagonalizes* A*, and if* $\mathbf{x} = P\mathbf{y}$*, then*

$$\Phi = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of* A *and*

$$D = P^T AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

DEFINITION The matrix *P* in this theorem is said to ***orthogonally diagonalize*** the quadratic form or ***reduce the quadratic form to a sum of squares***.

EXAMPLE Find a change of variables that will reduce the quadratic form $\Phi = x_1^2 + x_2^2 + 4x_1x_2$ to a sum of squares, and express the quadratic form in terms of the new variables.

Solution

$$\Phi = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation of this matrix is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

so the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -1$.

Then

$$\lambda_1 = 3:\ \mathbf{p}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \lambda_2 = -1:\ \mathbf{p}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

or, equivalently,

$$x_1 = 1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2$$
$$x_2 = 1/\sqrt{2}\, y_1 - 1/\sqrt{2}\, y_2$$

The new quadratic form is $\Phi = 3y_1^2 - y_2^2$.

Indices of Inertia

DEFINITION Suppose $D = P^T A P$ is a diagonal matrix. Then

- the ***positive index of inertia*** i_+ : the number of positive entries of D
- the ***negative index of inertia*** i_- : the number of negative entries of D
- the number of zeros of D i_0
- the ***rank*** of A: $\text{rank}(A) = i_+ + i_-$

THEOREM 9 (Law of Inertia for Quadratic Forms)

The positive and the negative indices of inertia, and the rank of the quadratic form are uniquely defined, i.e. are invariants of Φ.

Useful relationship:

$$i_+ + i_- + i_0 = n$$

Lagrange's Reduction

EXAMPLE $\Phi = x_1^2 + 6x_1x_2 + 5x_2^2$

$= x_1^2 + 6x_1x_2 + 9x_2^2 - 9x_2^2 + 5x_2^2$

$= (x_1 + 3x_2)^2 - 9x_2^2 + 5x_2^2$

$= (x_1 + 3x_2)^2 - 4x_2^2 = (x_1 + 3x_2)^2 - (2x_2)^2$

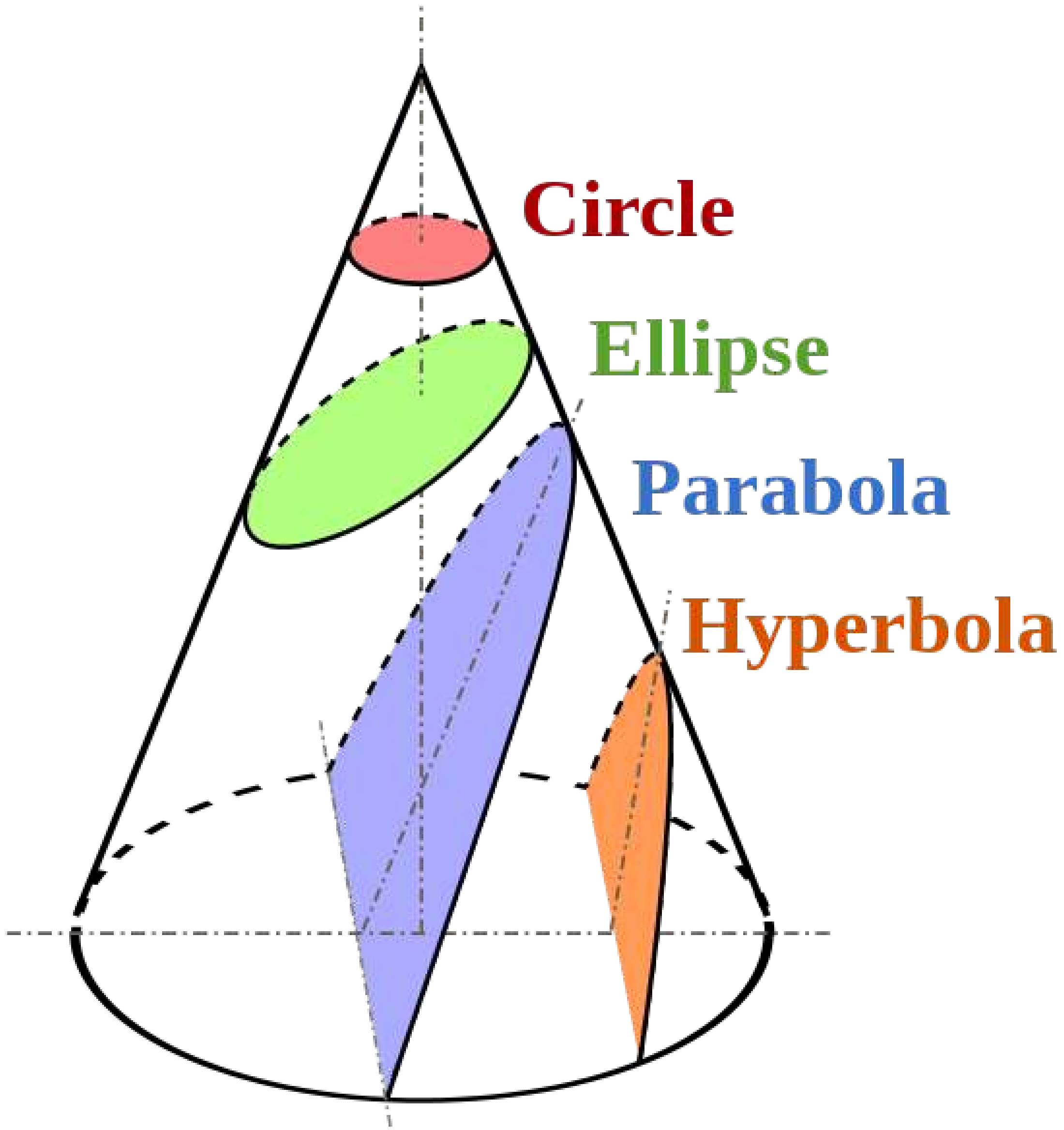
$(x_1 + 3x_2)^2$
$= x_1^2 + 6x_1x_2 + 9x_2^2$

Let $\begin{cases} y_1 = x_1 + 3x_2, \\ y_2 = \quad\quad 2x_2 \end{cases}$

Then $\Phi = y_1^2 - y_2^2$

Lecture 14

Conic Sections and Lines


Circle
Ellipse
Parabola
Hyperbola

DEFINITION Equations of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

where a, b, …, f are real numbers, and at least one of the numbers a, b, c is not zero is called a ***quadratic equation*** in x and y, and

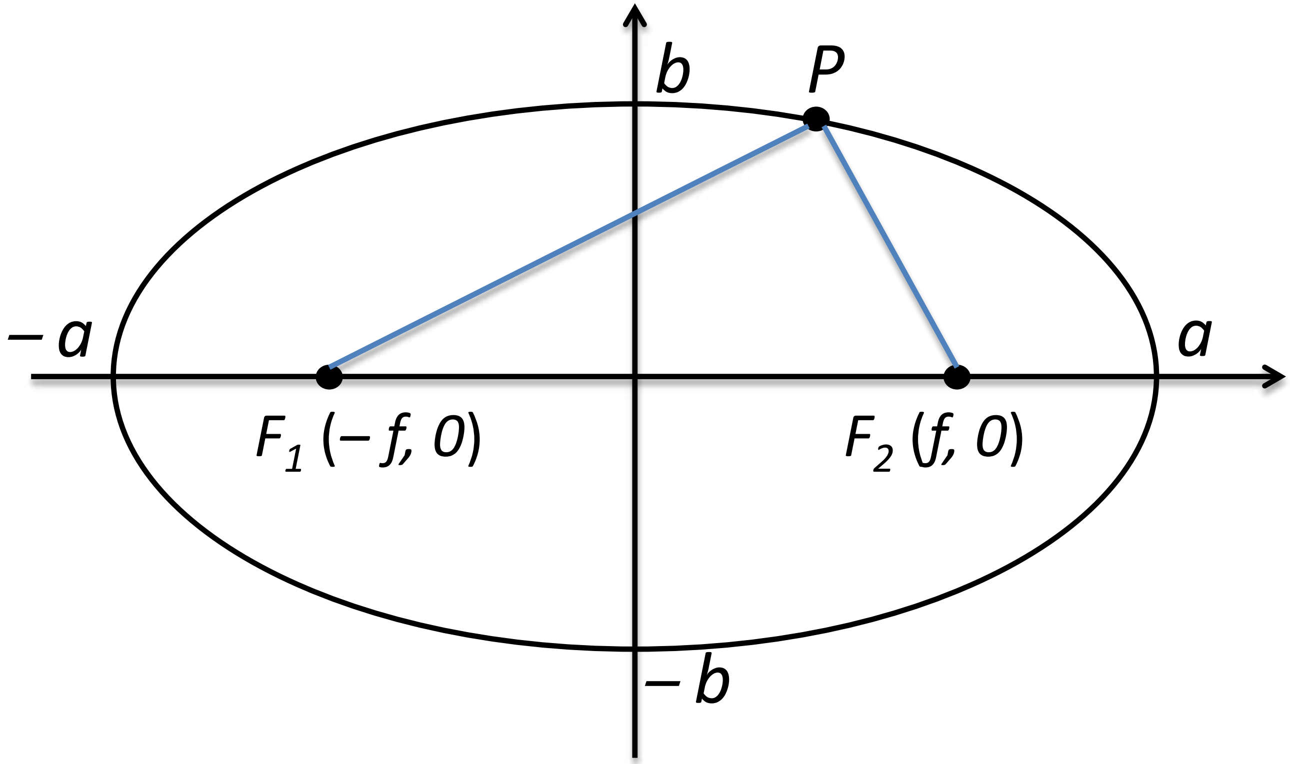
$$ax^2 + 2bxy + cy^2$$

is called the ***associated quadratic form***.

Graphs of quadratic equations are called ***conics*** (***кривые второго порядка***) or ***conic sections***.

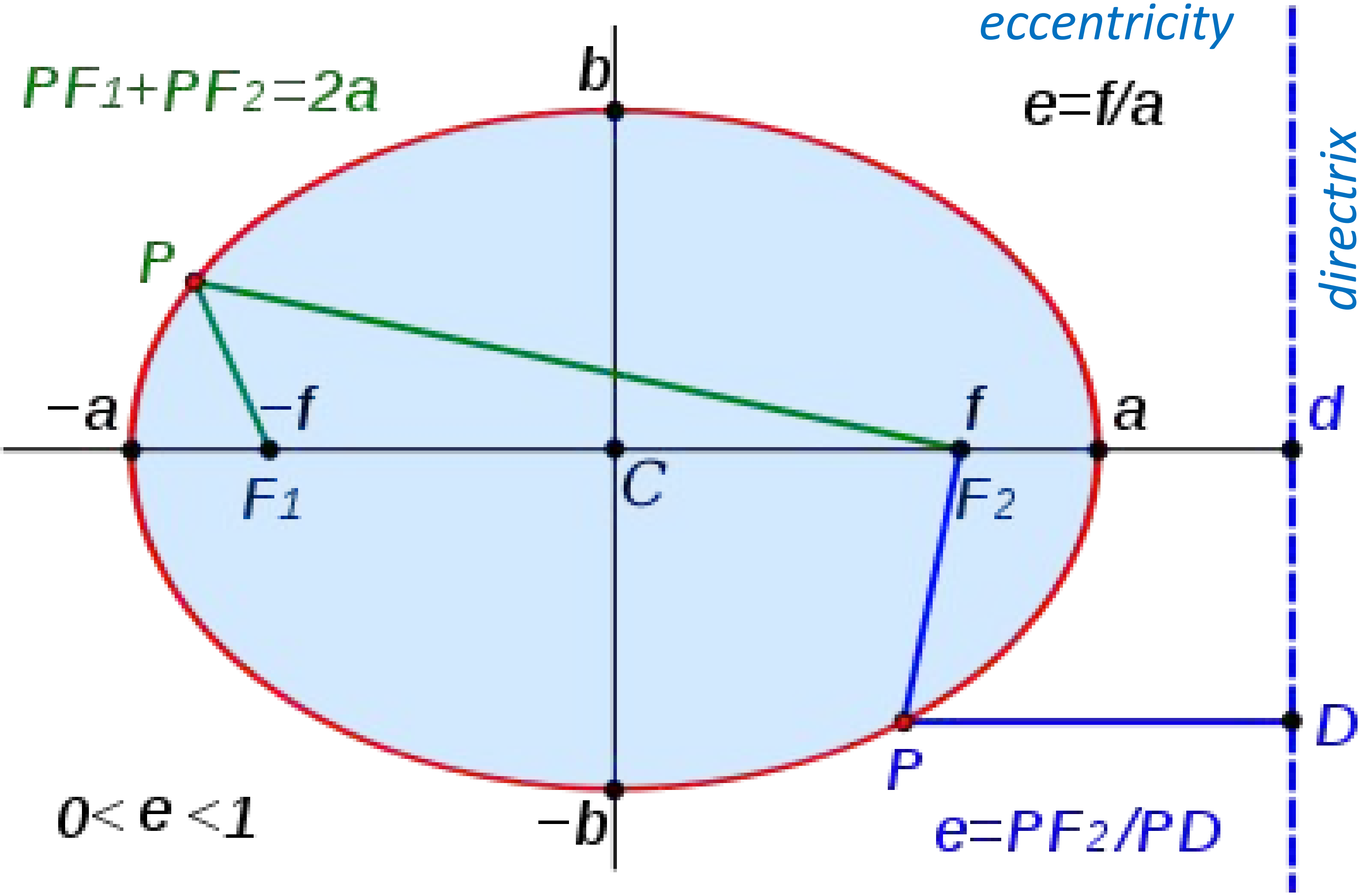
Real non-degenerate	Real degenerate	Imaginary
Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Intersecting lines: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	Imaginary ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$
Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Parallel lines: $y^2-b^2=0$	Imaginary parallel lines: $y^2+b^2=0$
Parabola: $y^2=2px$	Coincident lines: $y^2=0$	Imaginary lines, intersecting at a real point: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

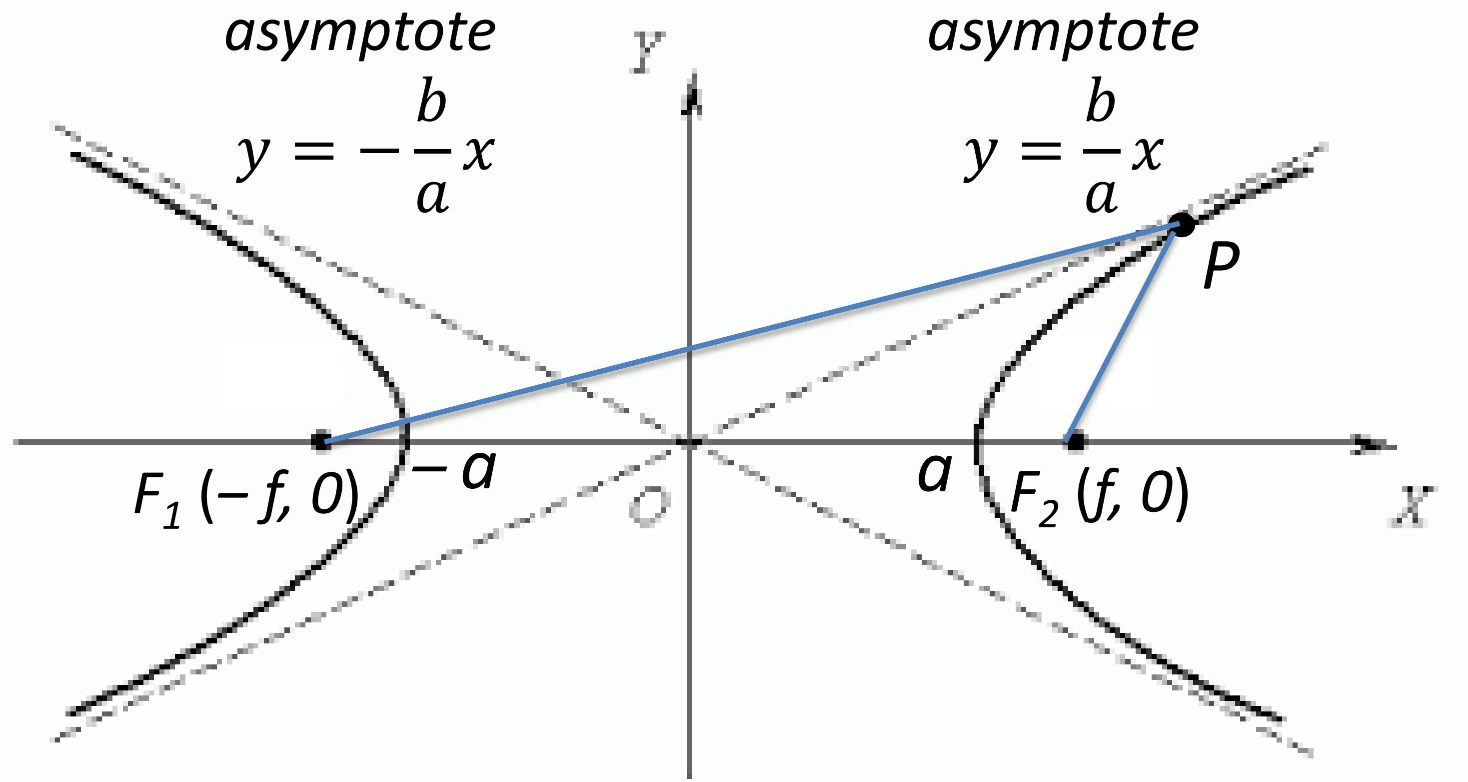


$PF_1 + PF_2 = 2a$

$f^2 = a^2 - b^2$

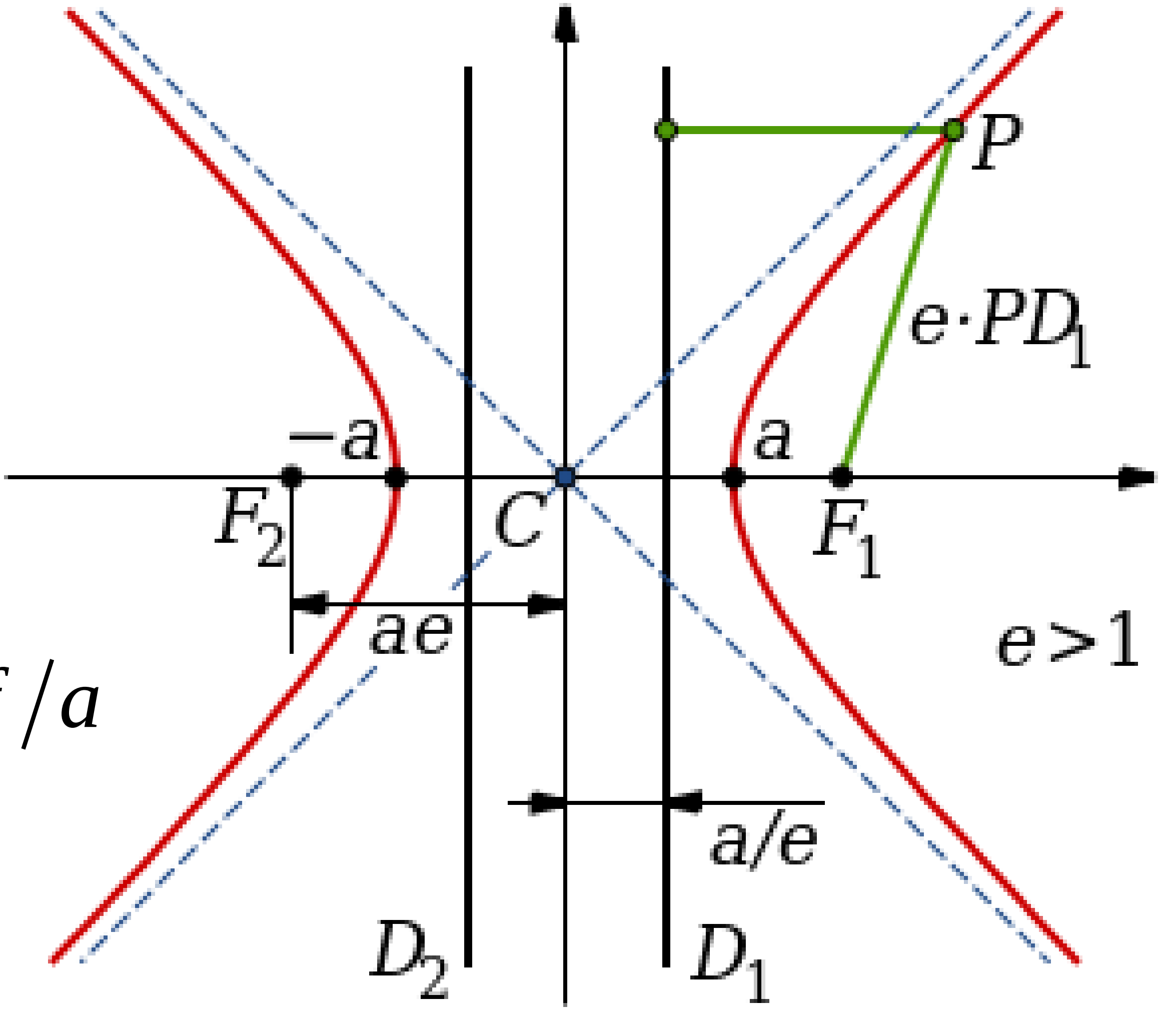

eccentricity
e=f/a
PF1+PF2=2a
b
directrix
P
−a
−f
f
a
d
F1
C
F2
D
P
0< e <1
−b
e=PF2/PD

Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



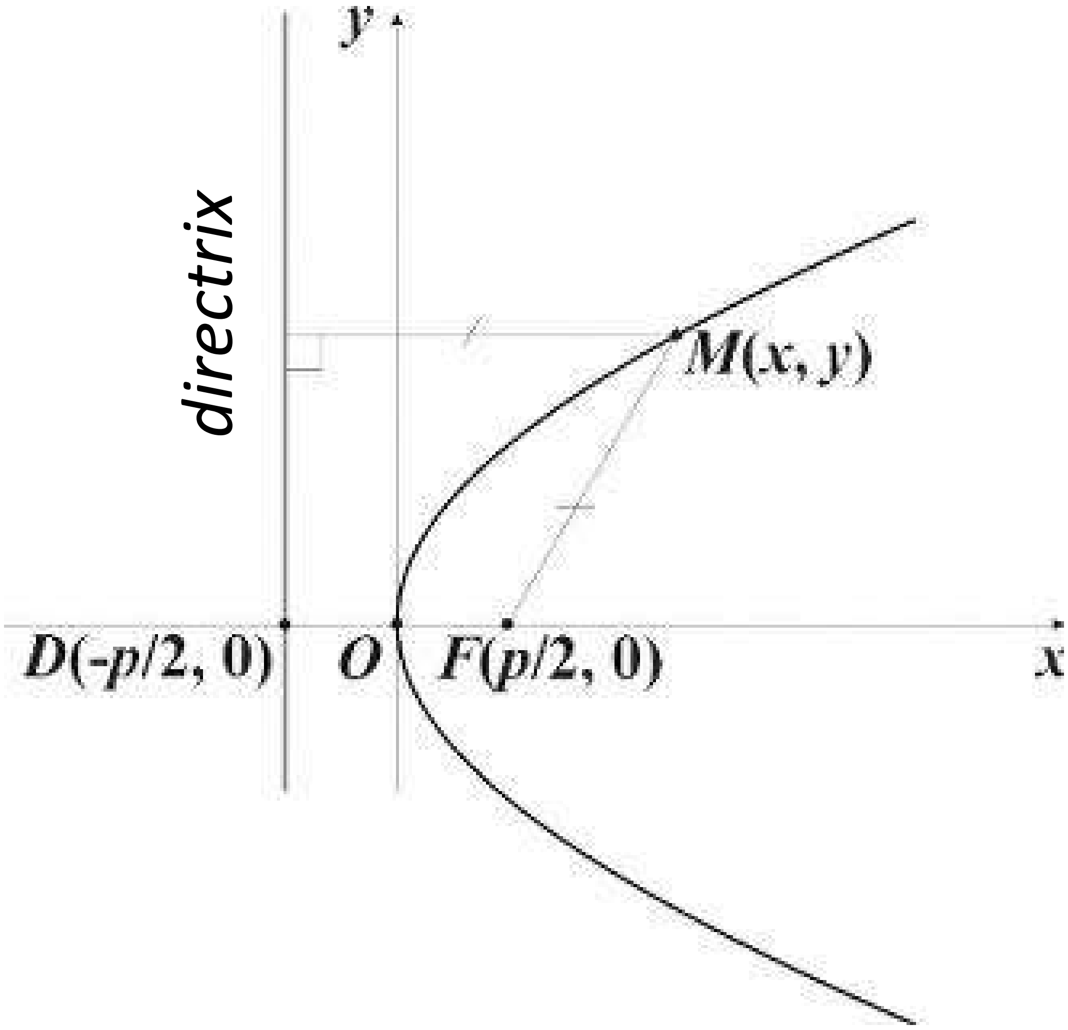
$|PF_1 - PF_2| = 2a$

$f^2 = a^2 + b^2$



$$e = f / a$$

Parabola: $y^2 = 2px$



$e = 1$

Principal Axes Theorem for $\mathbb{R}^2$

THEOREM 1 *Let*

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

be the equation of a conic C, and let

$$\mathbf{x}^T A\mathbf{x} = ax^2 + 2bxy + cy^2$$

be the associated quadratic form. Then the coordinate axes can be rotated so that the equation for C in the new x'y'-coordinate system has the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + d'x' + e'y' + f = 0$$

where λ_1 *and* λ_2 *are the eigenvalues of A. The rotation can be accomplished by the substitution* $\mathbf{x} = P\mathbf{x}'$, *where P orthogonally diagonalizes A and* $\det(P) = 1$.

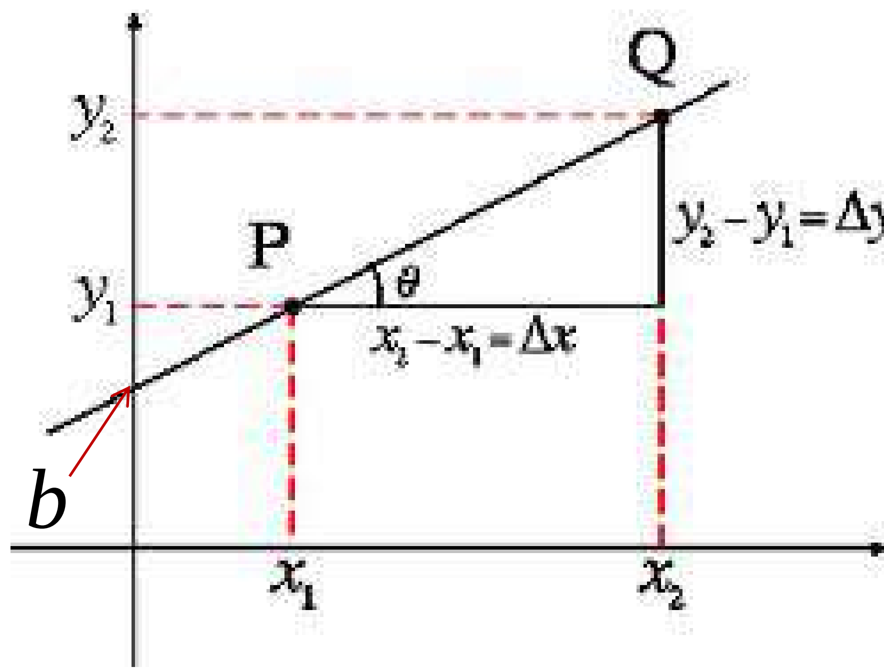
Slope-Intercept Equation of a Line

An equation of a line is

$$y = mx + b,$$

where *m* and *b* are constants (parameters):

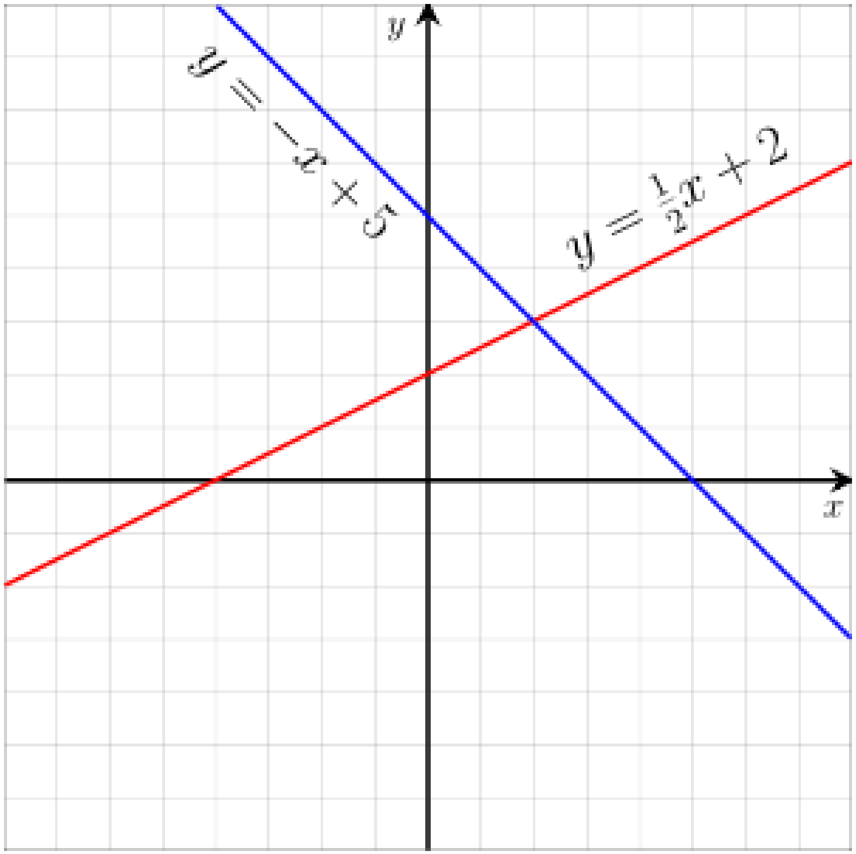
- *m* determines the ***slope*** (***угловой коэффициент***) of that line,
- *b* determines the point at which the line crosses the *y*-axis, known as the y-***intercept***.



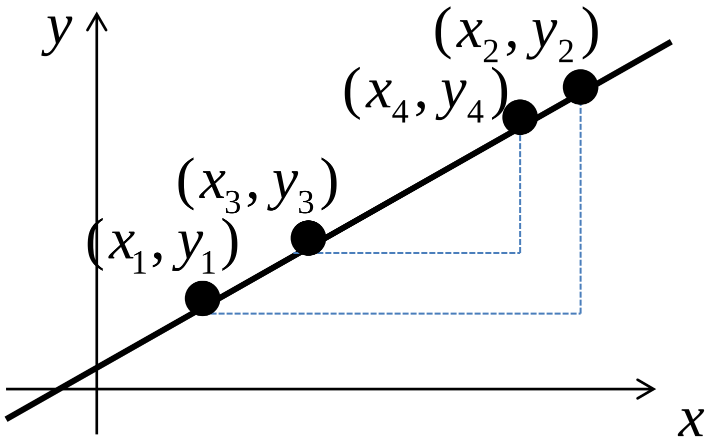
$$m = \frac{\Delta y}{\Delta x} = \tan(\theta)$$

Properties of a Slope:

- If $m > 0$, then the line rises from left to right
- If $m = 0$, then the line is horizontal
- If $m < 0$, then the line falls from left to right
- If m is undefined, then the line is vertical
- The greater the absolute value of the slope of a line, the steeper the line is.



The Point-Slope Equation of a Line



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

Suppose we are given the slope *m* and the point (x_1, y_1).
If (x, y) is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m$$

We get the **point-slope equation of a line**:

$$y - y_1 = m(x - x_1)$$

Principal Axes Theorem for $\mathbb{R}^2$

THEOREM 1 *Let*

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

be the equation of a conic C, and let

$$\mathbf{x}^T A\mathbf{x} = ax^2 + 2bxy + cy^2$$

be the associated quadratic form. Then the coordinate axes can be rotated so that the equation for C in the new x'y'-coordinate system has the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + d'x' + e'y' + f = 0$$

where λ_1 and λ_2 are the eigenvalues of A. The rotation can be accomplished by the substitution $\mathbf{x} = P\mathbf{x}'$, where P orthogonally diagonalizes A and $\det(P) = 1$.

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An equation of a line is

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where m and b are constants (parameters):

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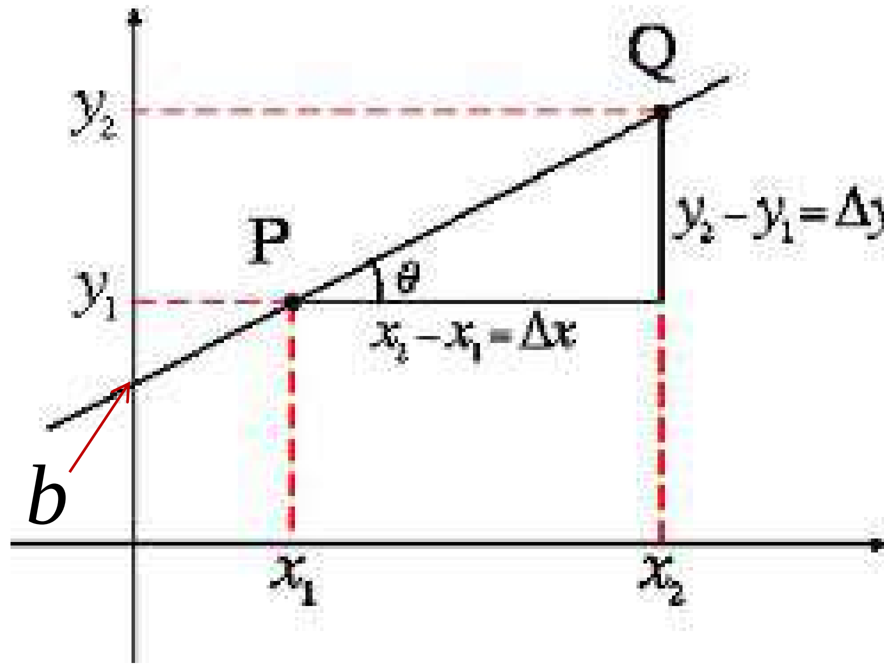

$$m = \frac{\Delta y}{\Delta x} = \tan(\theta)$$

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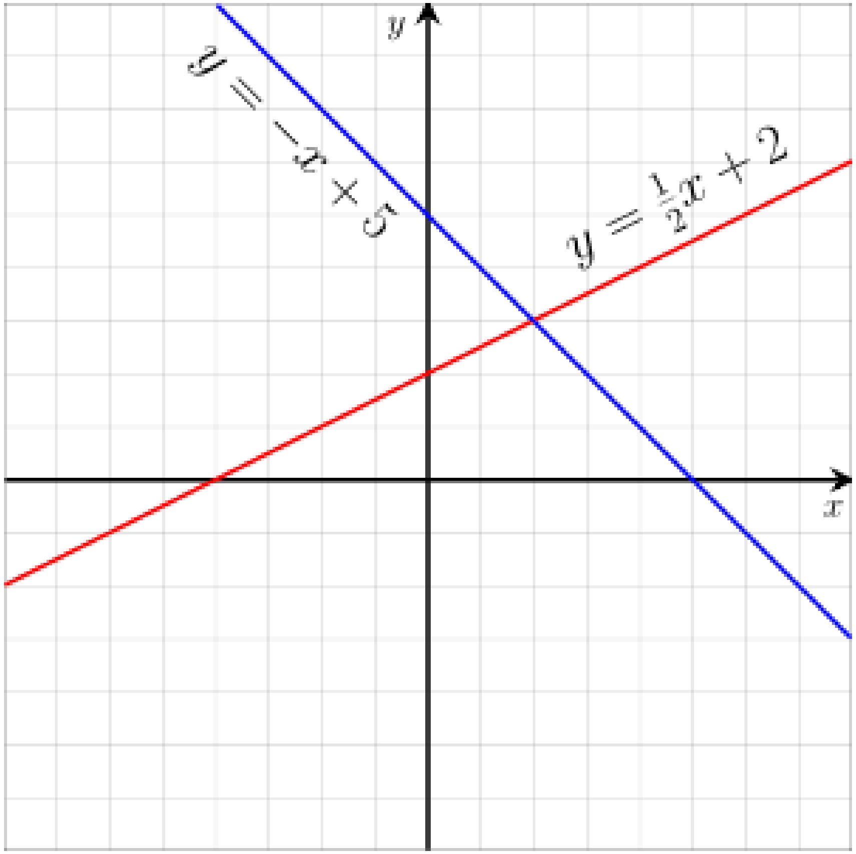
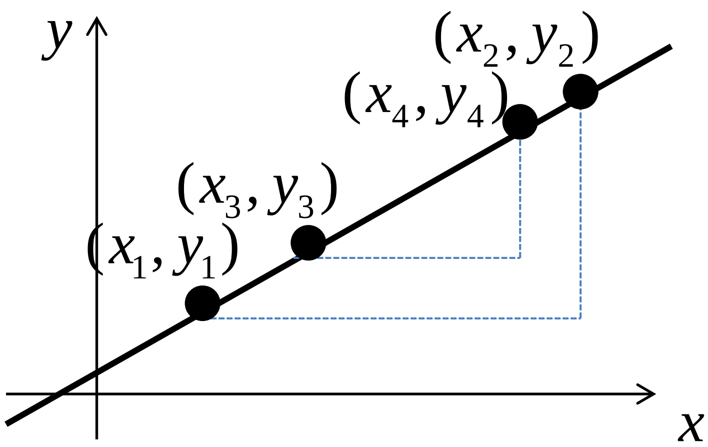


The Point-Slope Equation of a Line



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

Suppose we are given the slope *m* and the point (x_1, y_1).
If (x, y) is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m$$

We get the **point-slope equation of a line**:

$$y - y_1 = m(x - x_1)$$

Suppose a line contains a point $P_0(x_0, y_0)$ and has a ***direction*** vector $\mathbf{p} = (p_1, p_2)$. Then

Vector Equation of a Line:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{p}$$

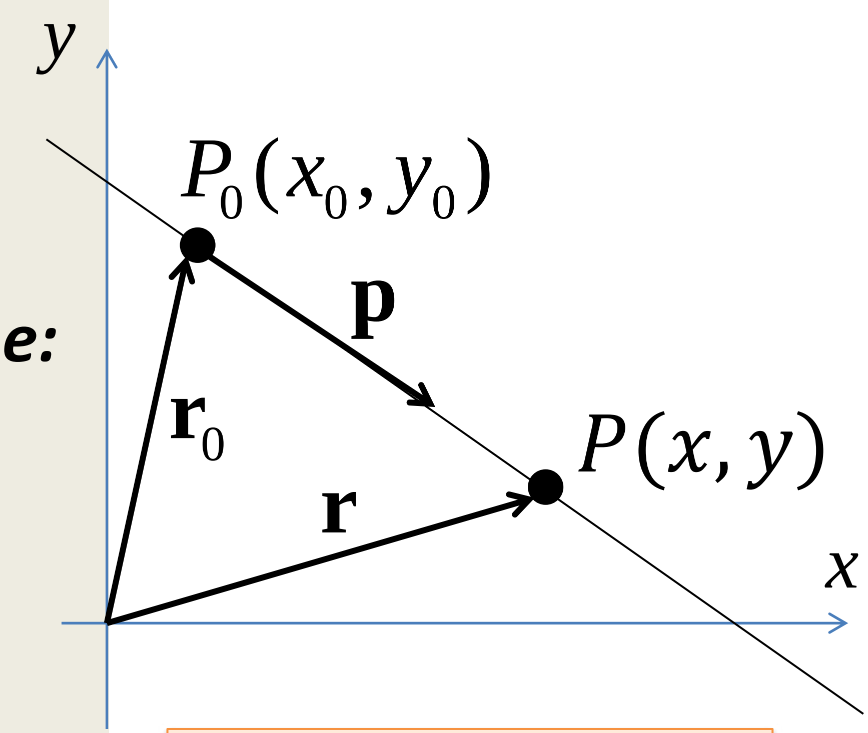
Parametric Equations of a Line:

$$\begin{cases} x = x_0 + tp_1 \\ y = y_0 + tp_2 \end{cases}$$

(t is a parameter, $t \in \mathbb{R}$)

Symmetric Equation of a Line:

$$\frac{x - x_0}{p_1} = \frac{y - y_0}{p_2}$$



EXAMPLE

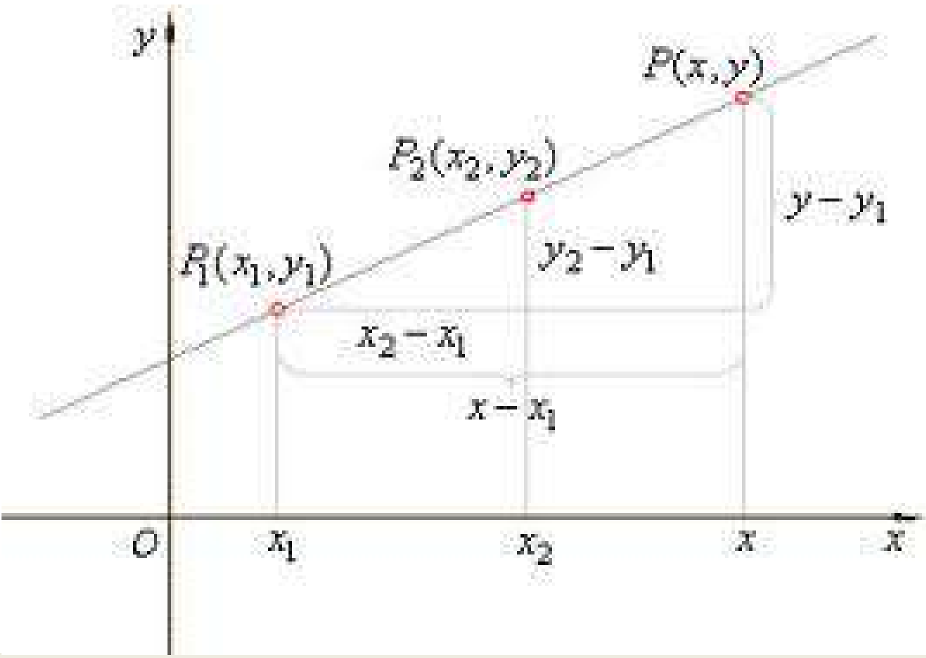
Find the equations of the line passing through the point (7, 9) and having the direction vector **p** = (–3, 5).

Solution

- parametric equations: $\begin{cases} x = 7 - 3t \\ y = 9 + 5t \end{cases}$

- symmetric equation: $\dfrac{x-7}{-3} = \dfrac{y-9}{5}$

Two-Point Equation of a Line

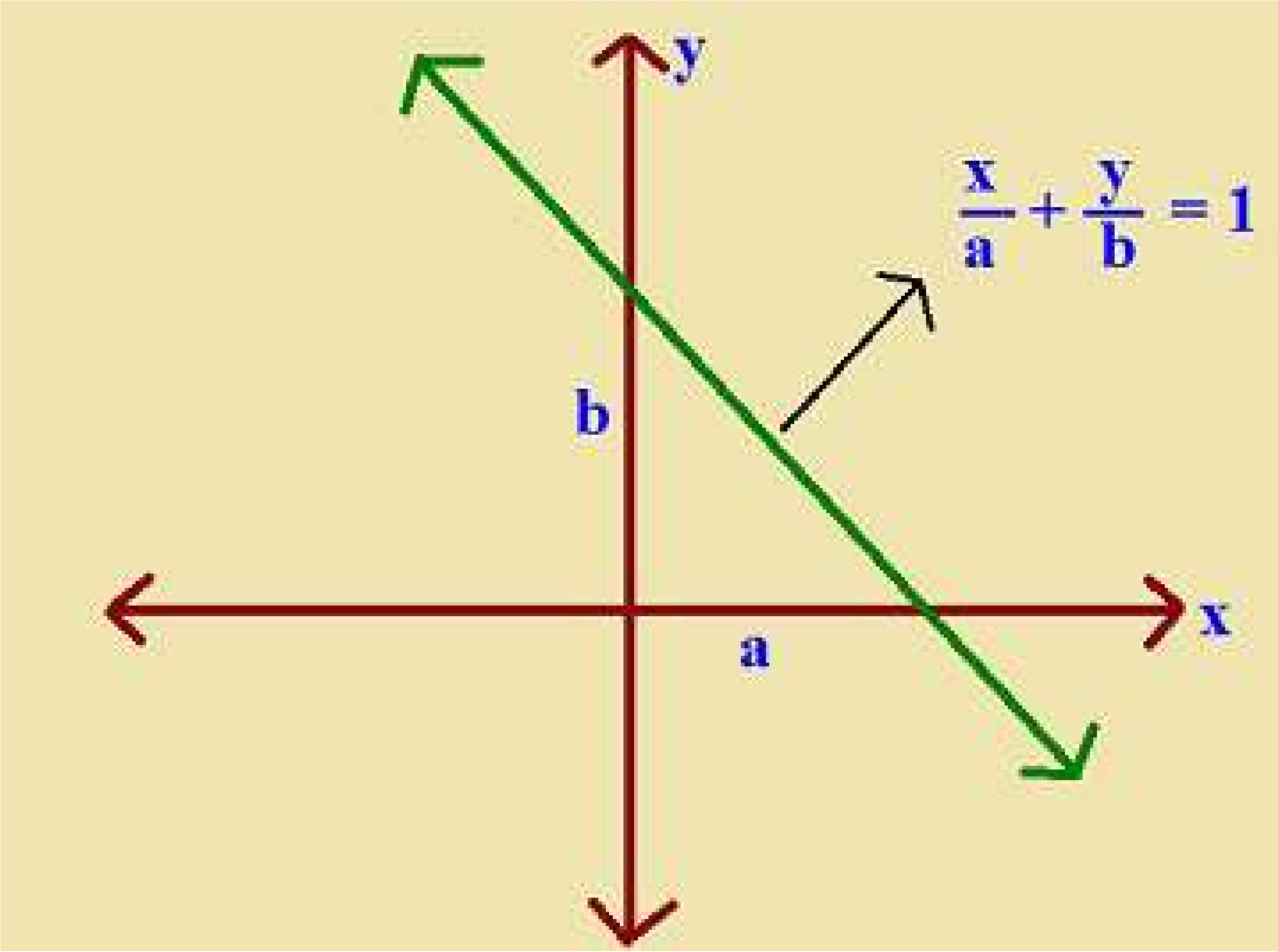


$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

EXAMPLE Find the equation of the line passing through $P_1(2, -1)$ and $P_2(4, 5)$

$$\frac{x-2}{4-2} = \frac{y+1}{5+1} \Leftrightarrow \frac{x-2}{2} = \frac{y+1}{6}$$

Intercept Equation of a Line



Distance Between a Point and a Line

THEOREM 3 *The distance D between a point* $P_0(x_0, y_0)$ *and the line* l: $ax + by + c = 0$ *can be found using the formula*

$$D(P_0, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Using the Distance Formula

EXAMPLE

Find the distance *d* from the point (1, –2) to the line $3x + 4y - 6 = 0$.

Solution

$$D = \frac{|3 \cdot 1 + 4 \cdot (-2) - 6|}{\sqrt{3^2 + 4^2}} = \frac{|-11|}{\sqrt{25}} = 2.2$$

Lecture 15

Conic Sections and Lines

Parallel Lines

Lines l_1 and l_2 are parallel ⟺ their direction vectors **v** and **w** are collinear

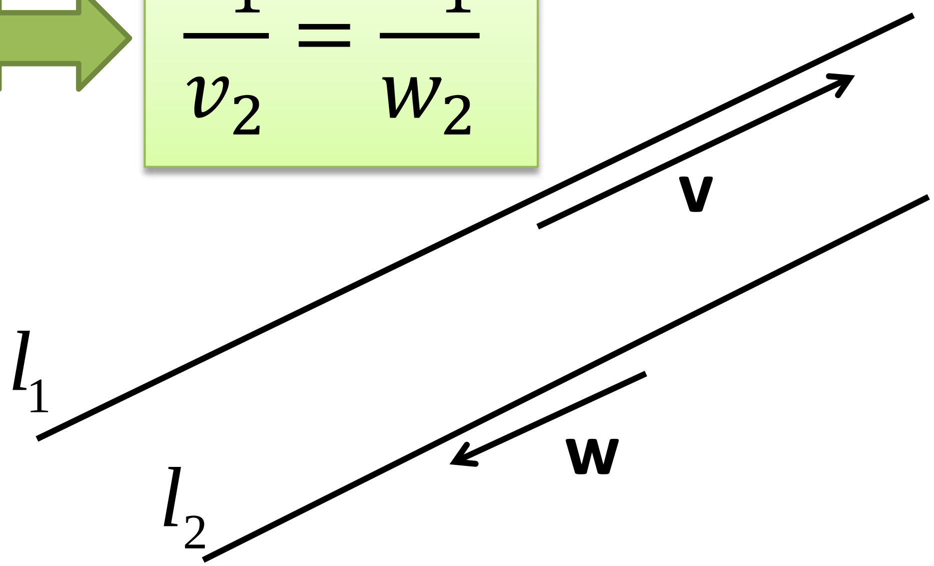
$$l_1: \frac{x - x_1}{v_1} = \frac{y - y_1}{v_2}$$

is parallel to

$$l_2: \frac{x - x_2}{w_1} = \frac{y - y_2}{w_2}$$

⟺

$$\frac{v_1}{v_2} = \frac{w_1}{w_2}$$



EXAMPLE Find the symmetric equation of a line through the point $P_0(2, 6)$ that is parallel to

$$\frac{x-1}{3} = \frac{y+7}{5}$$

Answer:

$$\frac{x-2}{3} = \frac{y-6}{5}$$

EXAMPLE Find a pair of parallel lines:

$$l_1: \frac{x+11}{-2} = \frac{y-9}{4}$$

$$l_2: \frac{x+1}{-1} = \frac{y-2}{3}$$

$$l_3: \frac{x-2}{2} = \frac{y+4}{1}$$

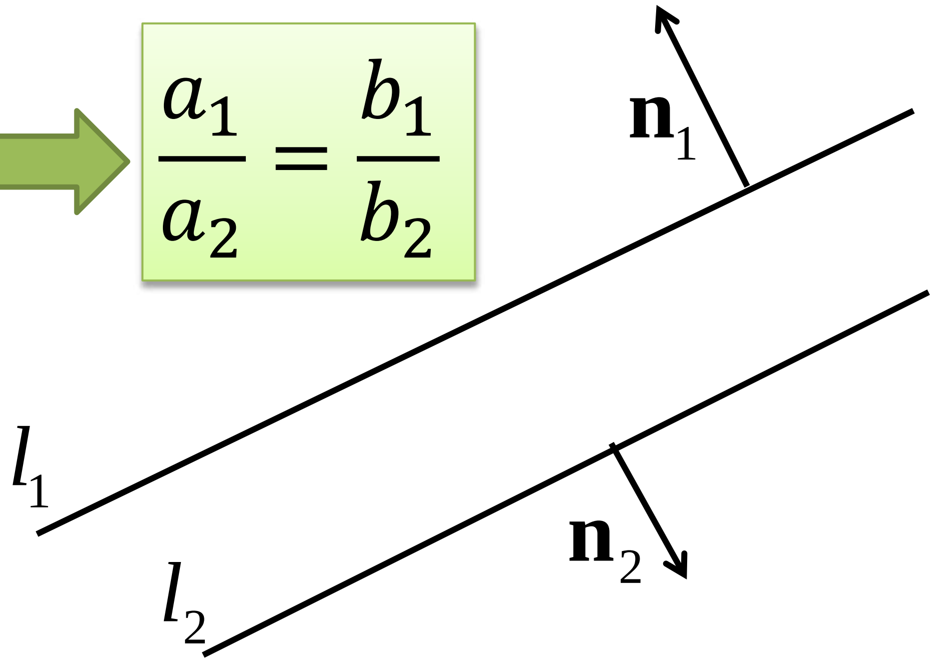
$$l_4: \frac{x-5}{3} = \frac{y+1}{-6}$$

Answer: l_1 and l_4

Parallel Lines

Lines l_1 and l_2 are parallel ⟺ their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are collinear

Lines
$l_1: a_1x + b_1y + c_1 = 0$
and
$l_2: a_2x + b_2y + c_2 = 0$
are parallel ⟺ $$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$



EXAMPLE Find the standard equation of a line through the point $P_0(-7, 2)$ that is parallel to the line $3x - 2y + 15 = 0$.

Answer: Point-normal form: $3(x + 7) - 2(y - 2) = 0$

Standard form: $3x - 2y + 25 = 0$

EXAMPLE Find a pair of parallel lines:

l_1: $2x + 5y = 11$, l_2: $-x + 3y + 7 = 0$,

l_3: $2(x + 1) - 6(y - 2) = -5$, l_4: $5x - 2y = 1$

Answer: l_2 and l_3

Perpendicular Lines

Lines l_1 and l_2 are perpendicular $\Leftrightarrow$ their direction vectors **v** and **w** are orthogonal

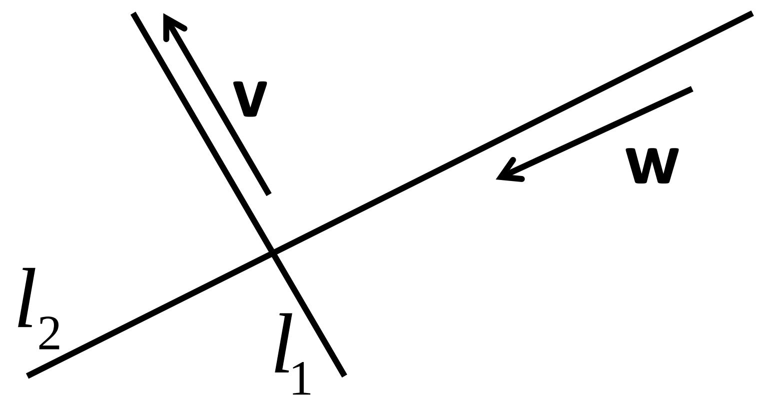
$$\frac{x - x_1}{v_1} = \frac{y - y_1}{v_2}$$

is perpendicular to

$$\frac{x - x_2}{w_1} = \frac{y - y_2}{w_2}$$

$\Leftrightarrow$

$$(\mathbf{v}, \mathbf{w}) = v_1 w_1 + v_2 w_2 = 0$$



EXAMPLE Given the line

$$l: \frac{x-3}{7} = \frac{y-4}{-9}$$

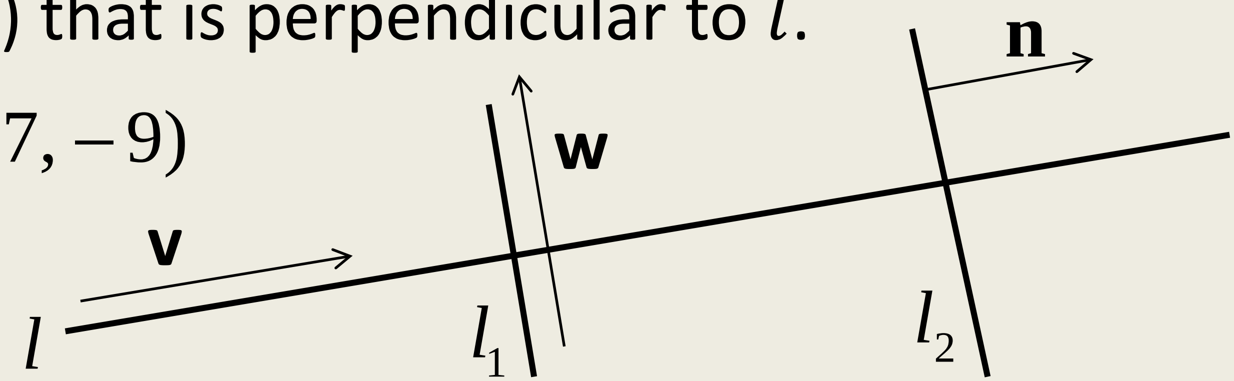
- find the symmetric equation of a line l_1 through the point $P_1(5, -2)$ that is perpendicular to l.

Solution: $\mathbf{v} = (7, -9)$, $\mathbf{w} = (9, 7)$

Answer: $\dfrac{x-5}{9} = \dfrac{y+2}{7}$

- find the standard equation of a line l_2 through the point $P_2(1, 1)$ that is perpendicular to l.

Solution: $\mathbf{n} = (7, -9)$



Answer: $7(x-1) - 9(y-1) = 0 \leftrightarrow 7x - 9y + 2 = 0$

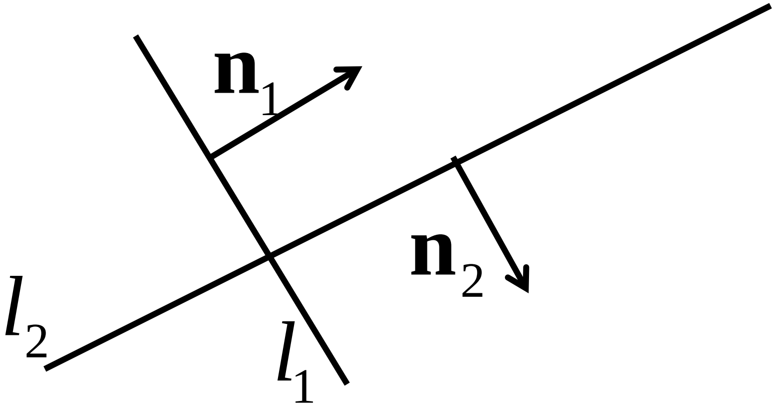
Perpendicular Lines

Lines l_1 and l_2 are perpendicular ⟺ their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal

Lines
$l_1: a_1x + b_1y + c_1 = 0$
and
$l_2: a_2x + b_2y + c_2 = 0$
are perpendicular

⟺

$(\mathbf{n}_1, \mathbf{n}_2) = a_1a_2 + b_1b_2 = 0$



EXAMPLE Given the line l: $4x + y + 7 = 0$,

- find the standard equation of a line l_1 through the point $P_1(-5, -8)$ that is perpendicular to l.

Solution: *Answer*:

$\mathbf{n}_1 = (4, 1)$

Point-normal form: $1(x + 5) - 4(y + 8) = 0$

$\mathbf{n}_2 = (1, -4)$

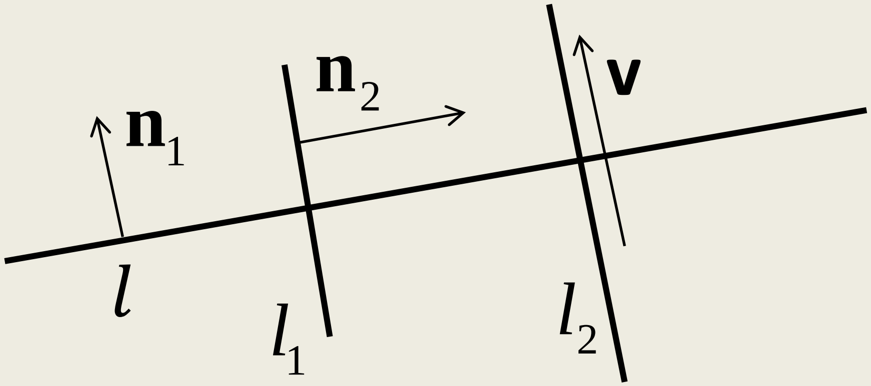
Standard form: $x - 4y - 27 = 0$

- find the symmetric equation of a line l_2 through the point $P_2(3, -12)$ that is perpendicular to l.

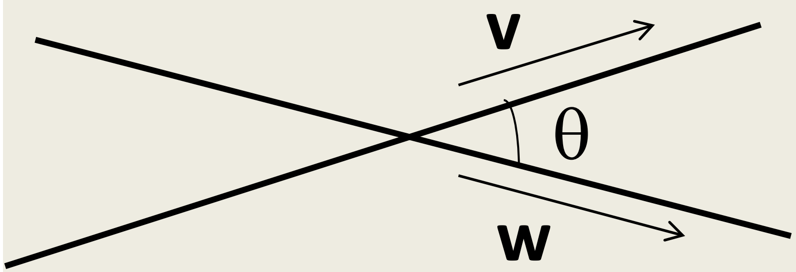
Solution: $\mathbf{v} = (4,1)$

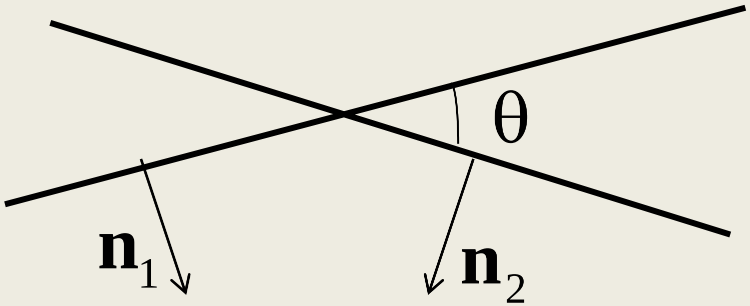
Answer:

$$\frac{x-3}{4} = \frac{y+12}{1}$$



Angle between Two Lines





$$\cos\theta = \frac{|(\mathbf{v}, \mathbf{w})|}{||\mathbf{v}|| \cdot ||\mathbf{w}||} \qquad \cos\theta = \frac{|(\mathbf{n}_1, \mathbf{n}_2)|}{||\mathbf{n}_1|| \cdot ||\mathbf{n}_2||}$$

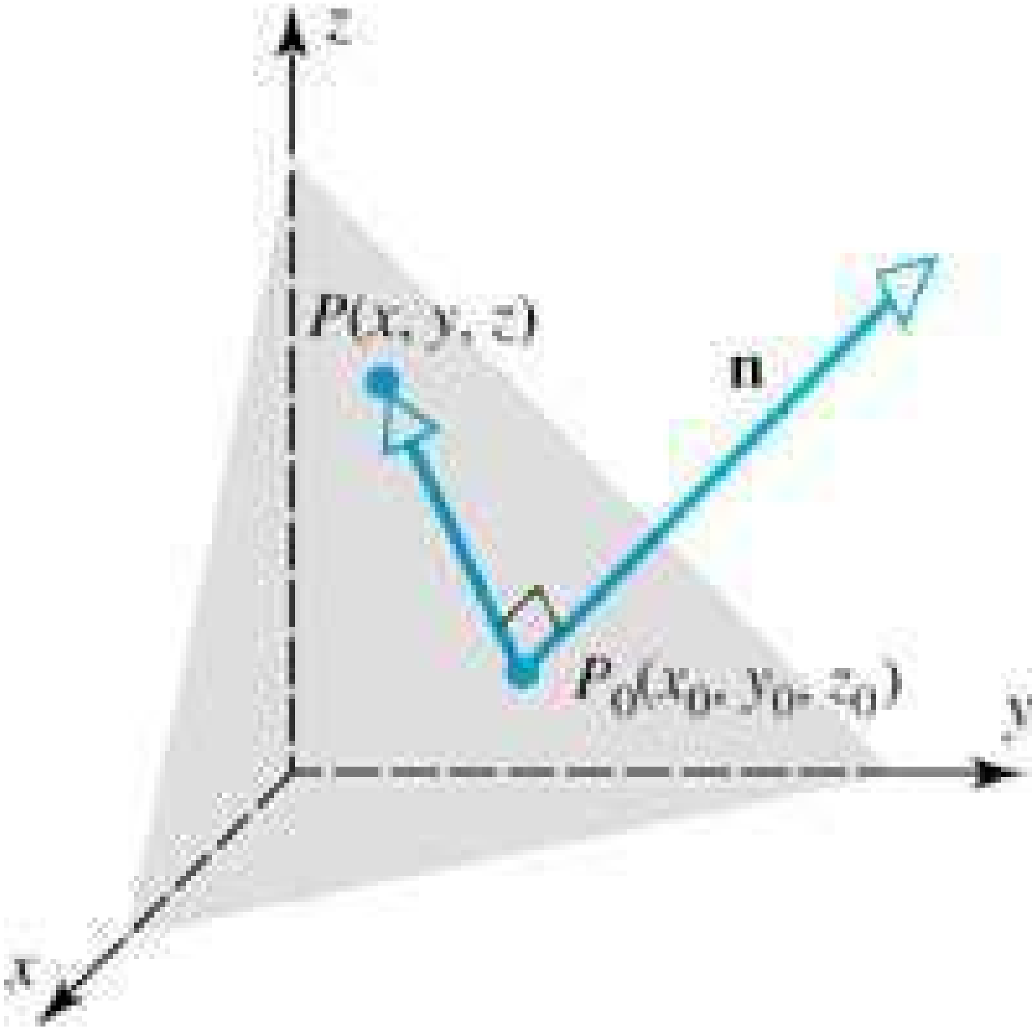
EXAMPLE Find the cosine of an angle between the lines

l_1: $6x + 5y - 3 = 0$ and l_2: $2x - 7y + 1 = 0$

Answer:

$$\cos\theta = \frac{|6\cdot 2 + 5\cdot(-7)|}{\sqrt{36+25}\cdot\sqrt{4+49}} = \frac{23}{\sqrt{61}\sqrt{53}}$$

Planes in 3-Space



Task Find the equation of the plane passing through the point $P_0(x_0, y_0, z_0)$ and having the nonzero vector $\mathbf{n} = (a, b, c)$ as a normal.

The plane consists of points $P(x, y, z)$ for which

$$\left(\mathbf{n}, \overrightarrow{P_0P}\right) = 0$$

Since $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$, we have

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is the ***point-normal*** equation of a plane.

EXAMPLE Find an equation of the plane passing through the point (3, –1, 7) and perpendicular to the vector **n** = (4, 2, –5).

Solution A point-normal form is

$$4(x - 3) + 2(y + 1) - 5(z - 7) = 0 \Leftrightarrow$$

$$4x + 2y - 5z + 25 = 0$$

DEFINITION The equation of the form

$$ax + by + cz + d = 0$$

where *a*, *b*, *c*, and *d* are constants, which are not all zero, is called the ***general*** **(*standard*)** ***form*** of the equation of a plane.

This plane has the vector **n** = (*a*, *b*, *c*) as a normal.

Equation of a Plane Through Three Points

THEOREM 1 *The equation of the plane passing through three points* $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, *and* $P_3(x_3, y_3, z_3)$ *that do not lie in the same line is*

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Proof

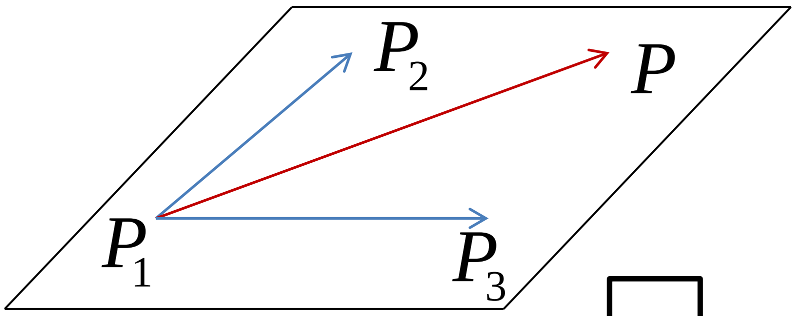
A point $P(x, y, z)$ lies in the plane

⇕

The vectors $\overrightarrow{P_1P} = (x - x_1, y - y_1, z - z_1)$
$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$, and
$\overrightarrow{P_1P_3} = (x_3 - x_1, y_3 - y_1, z_3 - z_1)$ are linearly dependent

⇕

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$



□

EXAMPLE Find the equation of the plane passing through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

Solution

$$\begin{vmatrix} x-1 & y-2 & z+1 \\ 2-1 & 3-2 & 1+1 \\ 3-1 & -1-2 & 2+1 \end{vmatrix} = \begin{vmatrix} x-1 & y-2 & z+1 \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 0$$

$$(x-1)\begin{vmatrix} 1 & 2 \\ -3 & 3 \end{vmatrix} - (y-2)\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + (z+1)\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 0$$

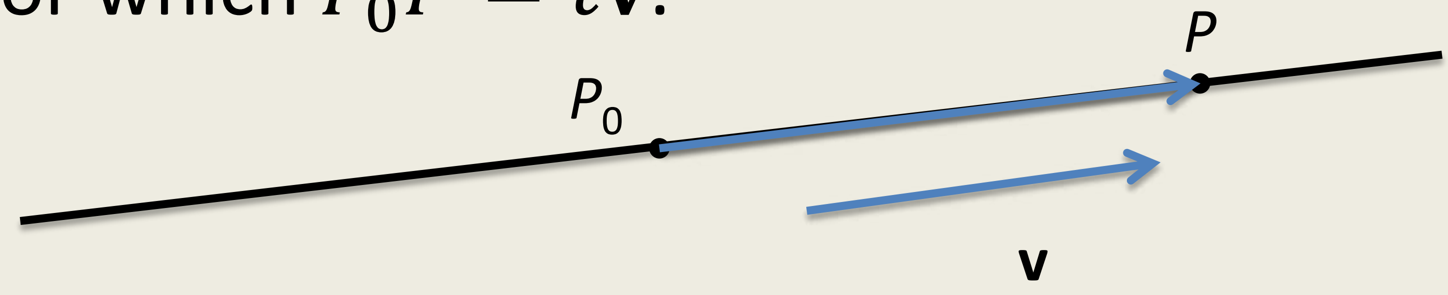
$$9(x-1) + (y-2) - 5(z+1) = 0$$

$$9x + y - 5z - 16 = 0$$

Parametric Equations of a Line in 3-Space

Suppose that l is a line in 3-space through the point $P_0(x_0, y_0, z_0)$ and parallel to the nonzero vector $\mathbf{v} = (v_1, v_2, v_3)$, which is called the ***direction* (*направляющий*) *vector***.

Then l consists precisely of those points $P(x, y, z)$ for which $\overrightarrow{P_0P} = t\mathbf{v}$.



In terms of components, this can be written as

$$(x - x_0, y - y_0, z - z_0) = (tv_1, tv_2, tv_3)$$

So $x - x_0 = tv_1$, $y - y_0 = tv_2$, and $z - z_0 = tv_3$

We get the equations

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}, \qquad -\infty < t < +\infty$$

which are called the ***parametric equations*** for *l*.

EXAMPLE

The line through the point (1, 2, –3) and parallel to the vector **v** = (4, 5, –7) has the parametric equations

$$\begin{cases} x = 1 + 4t \\ y = 2 + 5t \,, \quad -\infty < t < +\infty \\ z = -3 - 7t \end{cases}$$

Symmetric Equations of a Line

Consider the parametric equations for a line

$$x = x_0 + tv_1,\ y = y_0 + tv_2, \text{ and } z = z_0 + tv_3$$

If v_1, v_2, and v_3 are all nonzero, we solve for t:

$$\frac{x - x_0}{v_1} = t, \qquad \frac{y - y_0}{v_2} = t, \qquad \frac{z - z_0}{v_3} = t$$

Symmetric equations (***канонические уравнения***):

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

If, for example, $v_1 = 0$, the symmetric equations become

$$x - x_0 = 0 \quad \text{and} \quad \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

Two-Point Form Equations of a Line

If the line *l* passes through two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$, then the vector

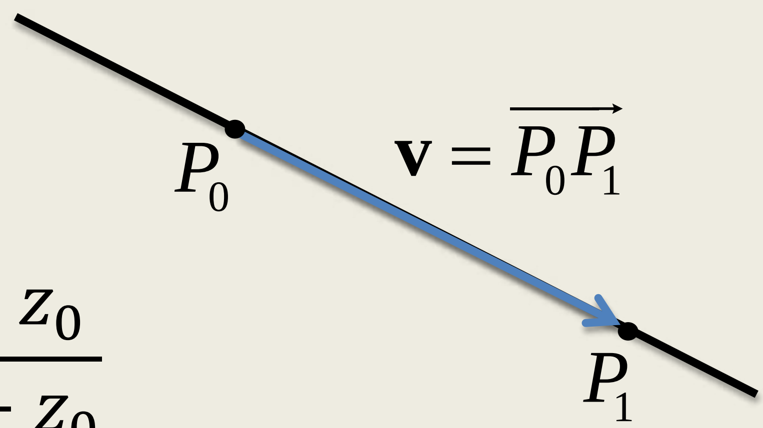
$$\mathbf{v} = \overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

is parallel to the line. We get the ***two-point equations***:

$$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \\ z = z_0 + t(z_1 - z_0) \end{cases}$$

or ***symmetric equations***:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$



EXAMPLE

The line passing through the points (1, 2, 3) and (4, 6, 9) has the two-point form equations:

$$\begin{cases} x = 1 + t(4-1) \\ y = 2 + t(6-2) \\ z = 3 + t(9-3) \end{cases} \Longleftrightarrow \begin{cases} x = 1 + 3t \\ y = 2 + 4t \\ z = 3 + 6t \end{cases}$$

and the symmetric equations:

$$\frac{x-1}{3} = \frac{y-2}{4} = \frac{z-3}{6}$$

EXAMPLE Intersection of a Line and a Plane

(a) Find parametric equations for the line *l* passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

(b) Where does the line intersect the *xy*-plane?

(c) Where does the line intersect the plane $\pi: 2x - y + 3z - 65 = 0$?

Solution **(*a*)** Since $\overrightarrow{P_1P_2} = (3, -4{,}8) || l$ and $P_1(2, 4, -1) \in l$,

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t \ (-\infty < t < +\infty)$$

Solution **(*b*)** The line *l* intersects the *xy*-plane at the point where

$$z = -1 + 8t = 0 \quad \Leftrightarrow \quad t = 1/8$$

Substituting this value of *t* for *l* yields

$$(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right)$$

Solution **(*c*)** Suppose the line *l* intersects π at the point $P_0(2 + 3t, 4 - 4t, -1 + 8t)$ for some *t*. Since $P_0 \in \pi$,

$$2(2 + 3t) - (4 - 4t) + 3(-1 + 8t) - 65 = 0 \ \Leftrightarrow \ t = 2$$

Thus,

$$P_0(2 + 3\cdot2, 4 - 4\cdot2, -1 + 8\cdot2) = (8, -4, 15)$$

EXAMPLE Line of Intersection of Two Planes

Find parametric equations for the line of intersection of the planes $3x + 2y - 4z - 6 = 0$ and $x - 3y - 2z - 4 = 0$

Solution

$$\begin{cases} 3x + 2y - 4z - 6 = 0 \\ x - 3y - 2z - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = \dfrac{26}{11} + \dfrac{16}{11}t \\ y = -\dfrac{6}{11} - \dfrac{2}{11}t \\ z = t \end{cases}$$

These are the parametric equations of the line.

Distance Between a Point and a Plane

THEOREM 3 *The distance D between a point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

EXAMPLE Find the distance *D* between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$.

Solution

First rewrite the equation of the plane in the form

$$2x - 3y + 6z + 1 = 0$$

Then

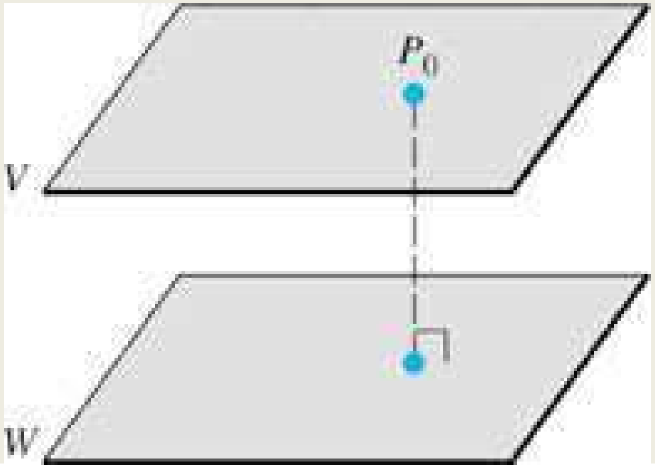
$$D = \frac{|2 \cdot 1 + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

EXAMPLE Find the distance between parallel planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$.

Solution

Idea: select an arbitrary point in one plane and compute the distance to the other plane.

$y = z = 0$ in
$x + 2y - 2z = 3$ → $P_0(3, 0, 0)$



$$D = \frac{|2 \cdot 3 + 4 \cdot 0 + (-4) \cdot 0 - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$

Lecture 16

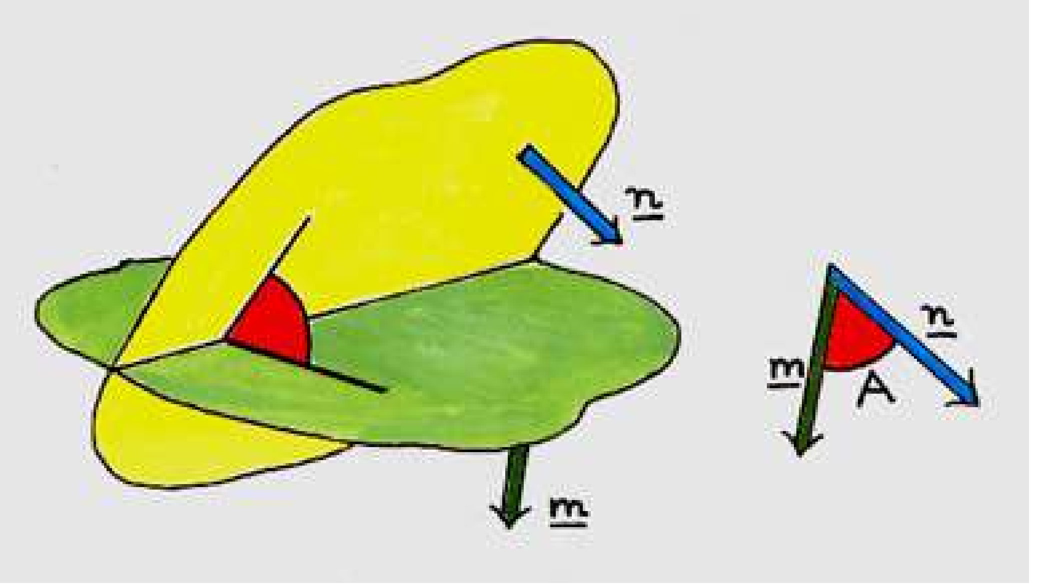
Planes and Convex Hulls

The Angle between Two Planes

Dihedral angle (двугранный угол) between the planes = the smaller (acute) angle between them

The angle between the two planes is equal to

- the angle between their normals if it is acute
- the supplementary angle otherwise



EXAMPLE Find the dihedral angle between the planes having equations $2x + 3y - 4z = 5$ and $4x - 2y + 3z = 6$.

Solution A pair of normals to the given planes are $\mathbf{n}_1 = (2, 3, -4)$ and $\mathbf{n}_2 = (4, -2, 3)$. The cosine of the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

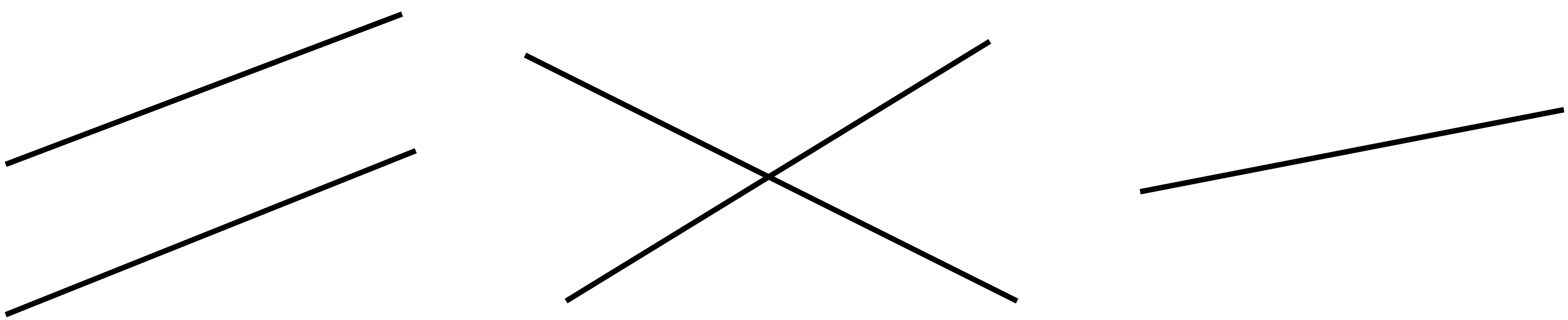
$$\cos\theta = \frac{(\mathbf{n}_1, \mathbf{n}_2)}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|} = \frac{8 - 6 - 12}{\sqrt{4 + 9 + 16}\sqrt{16 + 4 + 9}} = -\frac{10}{29}$$

Since the cosine is negative, then the angle is obtuse. Therefore, the angle between two planes is $\arccos(10/29) = \cos^{-1}(10/29)$

Relative Position of Two Lines

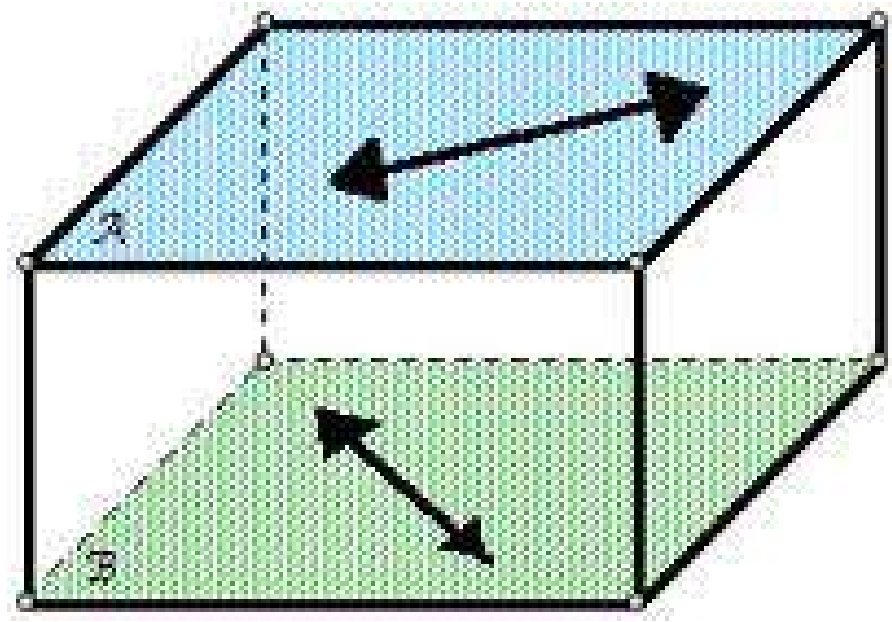
two lines in $\mathbb{R}^2$

parallel | intersecting | coincident



two lines in $\mathbb{R}^3$

- parallel
- coincident
- intersecting
- skew (скрещивающиеся)



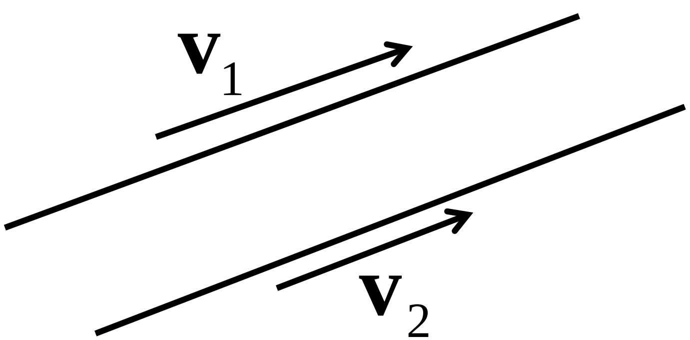
Determining the Relative Position of Two Lines

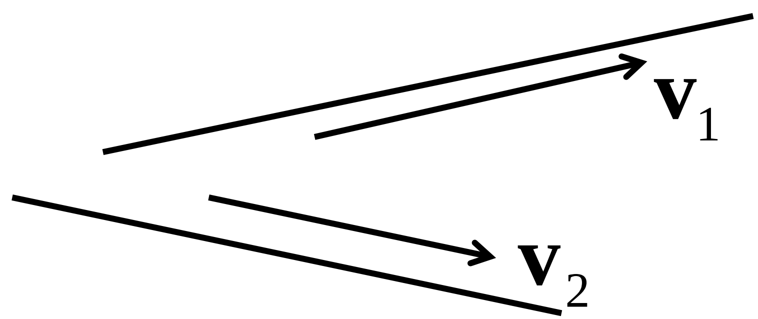
the direction vectors are

linearly dependent | linearly independent

the lines are parallel or equal

the lines are skew or intersecting





EXAMPLE Verify whether two lines

$$\frac{x-2}{-1}=\frac{y+3}{2}=\frac{z-7}{3} \quad \text{and} \quad \frac{x+1}{2}=\frac{y-5}{-4}=\frac{z-2}{-6}$$

are parallel, coincident, intersecting or skew.

Solution The direction vectors

$$\mathbf{v}_1 = (-1, 2, 3) \text{ and } \mathbf{v}_2 = (2, -4, -6)$$

are collinear $\Rightarrow$ l_1 and l_2 are ***parallel or coincident***

Take a point $P_1(2, -3, 7)$ on the line l_1 and check if it also belongs to l_2.

$P_1 \notin l_2 \Longrightarrow$ the lines are ***parallel***

EXAMPLE Verify whether two lines

$$\frac{x-2}{-1}=\frac{y+3}{2}=\frac{z-7}{3} \quad \text{and} \quad \frac{x+1}{3}=\frac{y-5}{-1}=\frac{z-2}{7}$$

are parallel, coincident, intersecting or skew.

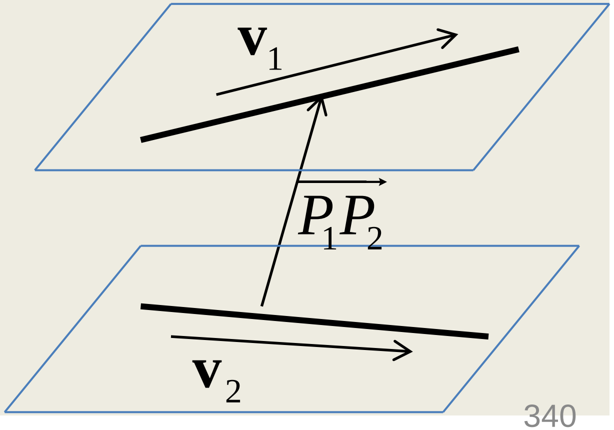
Solution The direction vectors

$$\mathbf{v}_1 = (-1, 2, 3) \text{ and } \mathbf{v}_2 = (3, -1, 7)$$

are not collinear ⇒ l_1 and l_2 are ***intersecting or skew***.

Create the vector from $P_1(2, -3, 7) \in l_1$ to $P_2(-1, 5, 2) \in l_2$. If $\overrightarrow{P_1P_2} = (-3, 8, -5)$ is linearly dependent with $\mathbf{v}_1$ and $\mathbf{v}_2$, then the lines are intersecting. If not, they are skew.

$$\begin{vmatrix} -1 & 2 & 3 \\ 3 & -1 & 7 \\ -3 & 8 & -5 \end{vmatrix} = 102 \neq 0$$



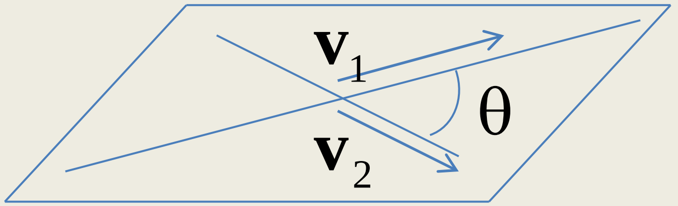
$\Longrightarrow$ the lines ***are skew***

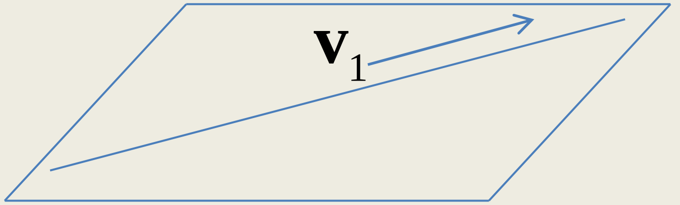
The Angle between Skew Lines

EXAMPLE Find the angle between

l_1: $x = 2 + 3t, \quad y = 4 - t, \quad z = 3 + 2t$

l_2: $x = 1 - t, \quad y = 5 + 2t, \quad z = 6 + 3t$





Solution

$\mathbf{v}_1 = (3, -1, 2) \parallel l_1$ and $\mathbf{v}_2 = (-1, 2, 3) \parallel l_2$.

The angle θ between the lines = the angle between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

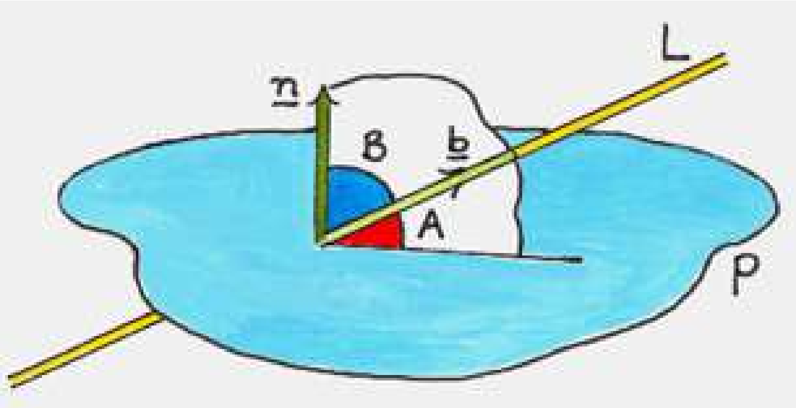
$$\cos\theta = \frac{(\mathbf{v}_1, \mathbf{v}_2)}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|} = \frac{-3 - 2 + 6}{\sqrt{9+1+4} \cdot \sqrt{1+4+9}} = \frac{1}{14}$$

So the angle is $\theta = \arccos(1/14)$.

The Angle between a Line and a Plane

Line → direction vector **v**

Plane → normal vector **n**



$\theta = \pi/2 -$ (angle between **v** and **n**)

EXAMPLE Find the angle between the line $x = 5 - 2t$, $y = 4 + 7t$, $z = 1 + 4t$ and the plane $3x - y + 2z + 21 = 0$.

Solution The direction vector is **v** = (–2, 7, 4), the normal vector is **n** = (3, –1, 2). Then

$$\frac{(\mathbf{v}, \mathbf{n})}{\|\mathbf{v}\| \cdot \|\mathbf{n}\|} = \frac{-6 - 7 + 8}{\sqrt{4 + 49 + 16} \cdot \sqrt{9 + 1 + 4}} = \frac{-5}{\sqrt{966}}$$

So the angle is $\theta = \frac{\pi}{2} - \arccos\left(\frac{5}{\sqrt{966}}\right) = \arcsin\left(\frac{5}{\sqrt{966}}\right)$.

Lines in $\mathbb{R}^n$

DEFINITION Given vectors $\mathbf{p}$ and $\mathbf{v} \neq 0$ in $\mathbb{R}^n$, the set of all points $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}, \quad \text{where } -\infty < t < \infty,$$

is called the ***line*** through $\mathbf{p}$ in the direction of $\mathbf{v}$.

This equation is called a ***vector equation*** for the line.

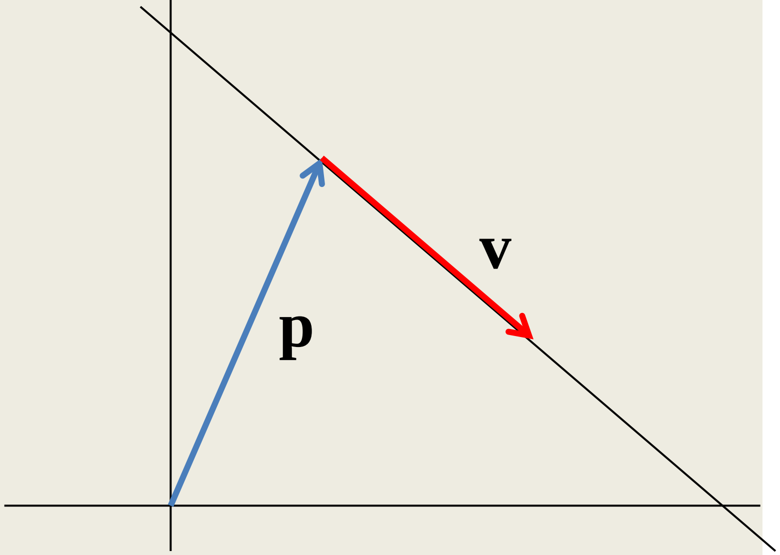
If $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$, and $\mathbf{p} = (p_1, \ldots, p_n)$, then

$$(x_1, \ldots, x_n) = (p_1, \ldots, p_n) + t(v_1, \ldots, v_n)$$

which holds if and only if

$$x_1 = p_1 + tv_1$$
$$\ldots$$
$$x_n = p_n + tv_n$$

These are ***parametric equations*** for the line in $\mathbb{R}^n$.



EXAMPLE Suppose *l* is the line in $\mathbb{R}^4$ through $P_0(1, 2, -7, 8)$ in the direction of $\mathbf{v} = (1, -3, 2, 11)$

Then the parametric equations of *l* are

$$x_1 = 1 + t$$
$$x_2 = 2 - 3t$$
$$x_3 = -7 + 2t$$
$$x_4 = 8 + 11t$$

Note that if we solve for *t* in both of these equations and exclude this variable, we get

$$\frac{x_1 - 1}{1} = \frac{x_2 - 2}{-3} = \frac{x_3 + 7}{2} = \frac{x_4 - 8}{11}$$

These are just the symmetric equations of a line.

Segments in $\mathbb{R}^n$

Suppose we have two distinct points $A = \mathbf{p}$ and $B = \mathbf{q}$. Consider the vector $\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Then all points $\mathbf{x} = \mathbf{p} + t\mathbf{v}$, where $0 \le t \le 1$, lie on the segment joining $\mathbf{p}$ and $\mathbf{q}$. Thus the equation

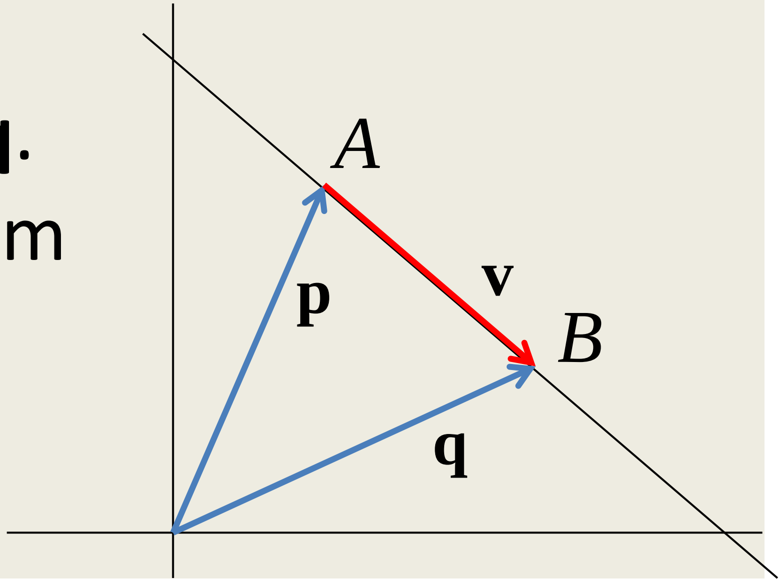
$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

where $0 \le t \le 1$, determines the ***segment*** joining $\mathbf{p}$ and $\mathbf{q}$.

It can be rewritten in the form

$$\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$$

or $$\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$$



If $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_n)$, and $\mathbf{q} = (q_1, \ldots, q_n)$, then

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = (q_1 - p_1, \ldots, q_n - p_n),$$

and the segment equation may be written as

$$(x_1, \ldots, x_n) = (p_1, \ldots, p_n) + t(q_1 - p_1, \ldots, q_n - p_n)$$

which holds if and only if

$$x_1 = p_1 + t(q_1 - p_1)$$

$$\ldots$$

$$x_n = p_n + t(q_n - p_n)$$

These equations are called ***parametric equations*** for the segment in $\mathbb{R}^n$. They can be rewritten in the form

$$x_1 = (1 - t)p_1 + tq_1$$

$$\ldots$$

$$x_n = (1 - t)p_n + tq_n$$

EXAMPLE Find a parametric equation of the segment joining the points $A(1, 2, 3, 4)$ and $B(-1, 3, -5, 2)$.

Solution

$$x_1 = (1-t) - t \qquad\qquad x_1 = 1 - 2t$$

$$x_2 = 2(1-t) + 3t \qquad\qquad x_2 = 2 + t$$

$$x_3 = 3(1-t) - 5t \quad \text{or} \quad x_3 = 3 - 8t$$

$$x_4 = 4(1-t) + 2t \qquad\qquad x_4 = 4 - 2t$$

$t = 0 \implies$ we get A

$t = 1 \implies$ we get B

EXAMPLE Verify whether the point *Q*(8, 1, –4, –2) belongs to the segment joining *A*(5, 2, –1, –4) and *B*(20, –3, –16, 6).

Solution The parametric equations for the segment:

$$\begin{cases} x_1 = 5 + 15t \\ x_2 = 2 - 5t \\ x_3 = -1 - 15t \\ x_4 = -4 + 10t \end{cases}$$

Then solve the system of equations

$$\begin{cases} 5 + 15t = 8 \\ 2 - 5t = 1 \\ -1 - 15t = -4 \\ -4 + 10t = -2 \end{cases} \Rightarrow \quad t = \frac{1}{5}$$

Since 0 ≤ 1/5 ≤ 1, then *Q* lies on the segment.

Hyperplanes in $\mathbb{R}^n$

DEFINITION Suppose **n** and **p** are vectors in $\mathbb{R}^n$ with $\mathbf{n} \neq \mathbf{0}$. The set of all vectors **x** in $\mathbb{R}^n$ which satisfy the equation

$$\mathbf{n}\cdot(\mathbf{x} - \mathbf{p}) = 0 \qquad (1)$$

is called a ***hyperplane*** through the point **p**. We call **n** a normal vector for the hyperplane and we call this equation a ***normal equation*** for the hyperplane.

Let $\mathbf{n} = (a_1, a_2, \ldots, a_n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then (1) $\Longrightarrow$

$$a_1(x_1 - p_1) + a_2(x_2 - p_2) + \ldots + a_n(x_n - p_n) = 0$$

or

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n + d = 0$$

Hyperplanes in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$

Space	Hyperplane	Equation
$\mathbb{R}^1$	a point	$ax + b = 0 \Leftrightarrow x = -b/a$
$\mathbb{R}^2$	a line	$ax + bx + c = 0$
$\mathbb{R}^3$	a plane	$ax + bx + cz + d = 0$
$\mathbb{R}^n$	an $(n-1)$-dimensional affine subspace	$a_1x_1 + a_2x_2 + \ldots + a_nx_n + d = 0$

EXAMPLE The set of all points (x, y, z, w) in $\mathbb{R}^4$ which satisfy

$$-x + 4y + 2z + 3w = 5$$

is a 3-dimensional hyperplane with the normal vector $\mathbf{n} = (3, -1, 4, 2)$.

Angles between Hyperplanes

DEFINITION Let G and H be hyperplanes in $\mathbb{R}^n$ with normal equations

$$\mathbf{m} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{and} \quad \mathbf{n} \cdot (\mathbf{x} - \mathbf{q}) = 0$$

respectively, chosen so that $\mathbf{m} \cdot \mathbf{n} \geq 0$. Then the angle between G and H is the angle between **m** and **n**.

Moreover, we will say that

- G and H are orthogonal if **m** and **n** are perpendicular
- G and H are parallel if **m** and **n** are parallel

EXAMPLE 1 Find the angle θ between the two planes in $\mathbb{R}^4$ with equations

$$x + 2y - z + 5w + 7 = 0 \quad \text{and} \quad x - 3y - z - 2w = 5$$

Solution First note that **m** = (1, 2, –1, 5) and **n** = (1, –3, –1, –2). Since **m** · **n** = –14, we will compute the angle between **m** and – **n**. Hence

$$\cos\theta = \frac{(\mathbf{m}, -\mathbf{n})}{\|\mathbf{m}\| \cdot \|\mathbf{n}\|} = \frac{14}{\sqrt{31}\sqrt{15}} \Rightarrow \theta = \arccos\left(\frac{14}{\sqrt{31}\sqrt{15}}\right)$$

EXAMPLE 2 The planes in $\mathbb{R}^4$ with equations

$$3x + y - 2z + 7w = 3 \quad \text{and} \quad 6x + 2y - 4z + 14w = 13$$

are parallel since their normal vectors **m** = (3, 1, –2, 7) and **n** = (6, 2, –4, 14) are parallel.

DEFINITION A **half-space** is either of the two parts into which a hyperplane divides $\mathbb{R}^n$.

It may be specified by a non-strict inequality:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n + d \geq 0$$

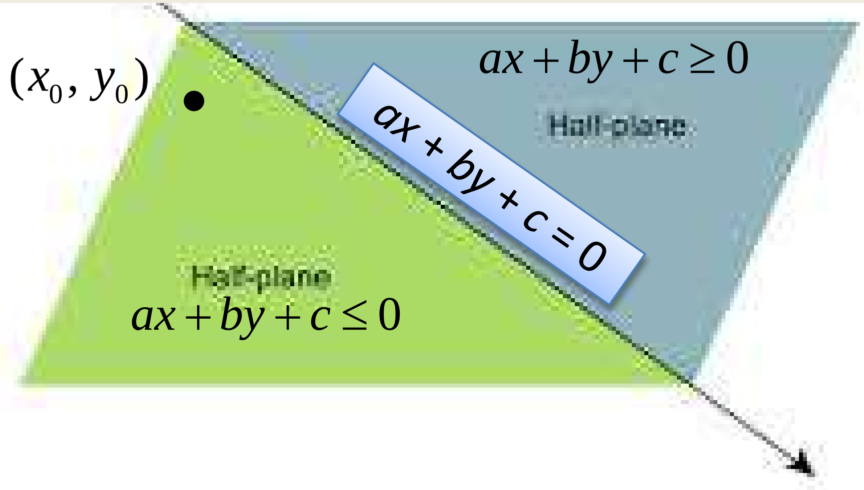
where not all of the real numbers $a_1, a_2, \ldots, a_n$ are zero.

Space	Half-Space	Equation
$\mathbb{R}^1$	a ray	$ax + b \geq 0$
$\mathbb{R}^2$	a half-plane	$ax + bx + c \geq 0$
$\mathbb{R}^3$	a half-space	$ax + bx + cz + d \geq 0$

How to graph a half-plane $ax + by + c \geq 0$?

1) Graph the line $ax + by + c = 0$. This line divides the plane into two halves.
2) Choose a test point (x_0, y_0) *not on the line.*

- The coordinates of (x_0, y_0) satisfy the *inequality* $\Longrightarrow$ so do all points on the same side of the line $\Longrightarrow$ shade that whole area.
- The coordinates of (x_0, y_0) do not satisfy the original inequality $\Longrightarrow$ shade the portion of the plane on the side of the line in which (x_0, y_0) does not lie.



Any line that lies in a plane separates the plane into two half-planes.

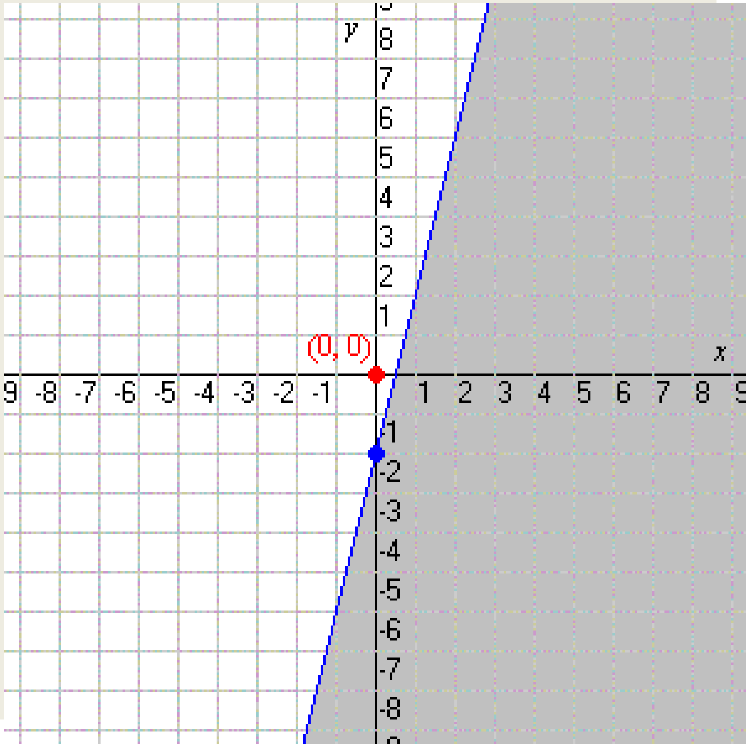
EXAMPLE Graph $4x - y - 2 \geq 0$.

Solution We first graph the line $4x - y - 2 = 0$.

We now choose a convenient point not on the line, say (0, 0), and plug its coordinates into our original inequality:

$$4 \cdot 0 - 0 - 2 \geq 0, \quad \text{(False!)}$$

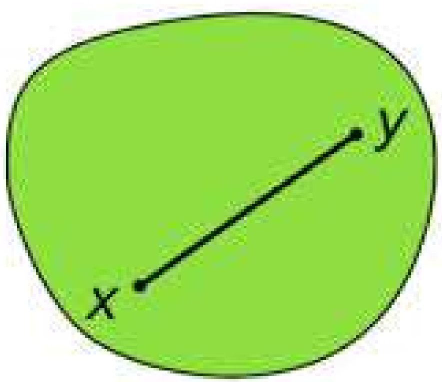
Since the inequality does not hold, we shade in the portion of the plane lying on the other side then (0, 0).



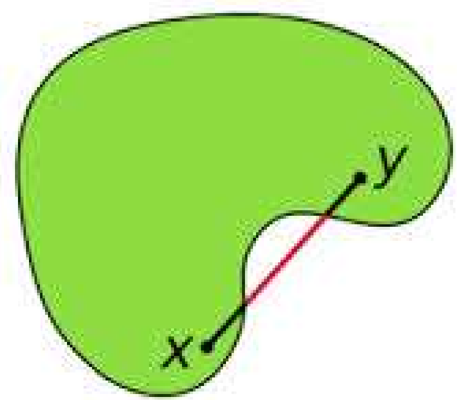
Convex Sets

DEFINITION A set C in $\mathbb{R}^n$ is said to be ***convex*** if for every $\mathbf{p}$, $\mathbf{q} \in C$ and every real number t, $0 < t < 1$, the point $(1 - t)\mathbf{p} + t\mathbf{q} \in C$.

Geometrically: a set is convex if, given two points in the set, every point on the line segment joining these two points is also a member of the set.



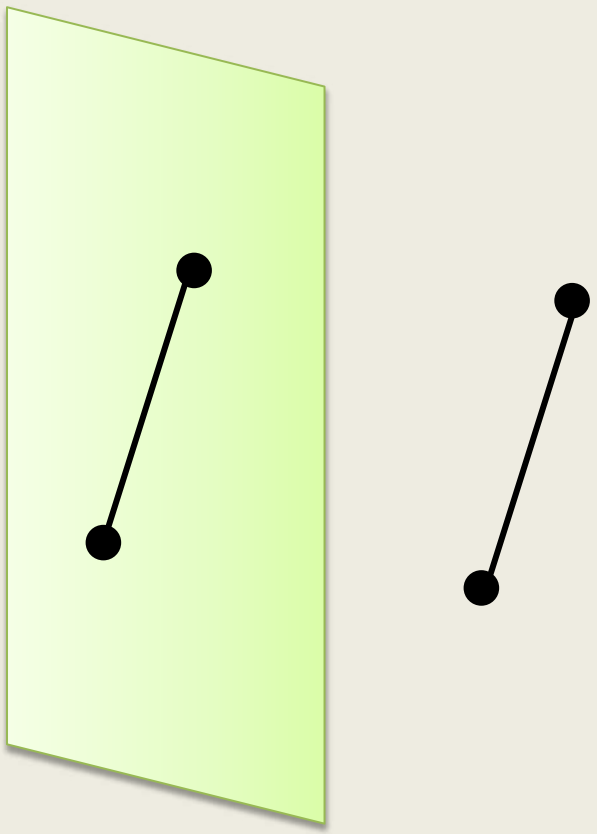
convex



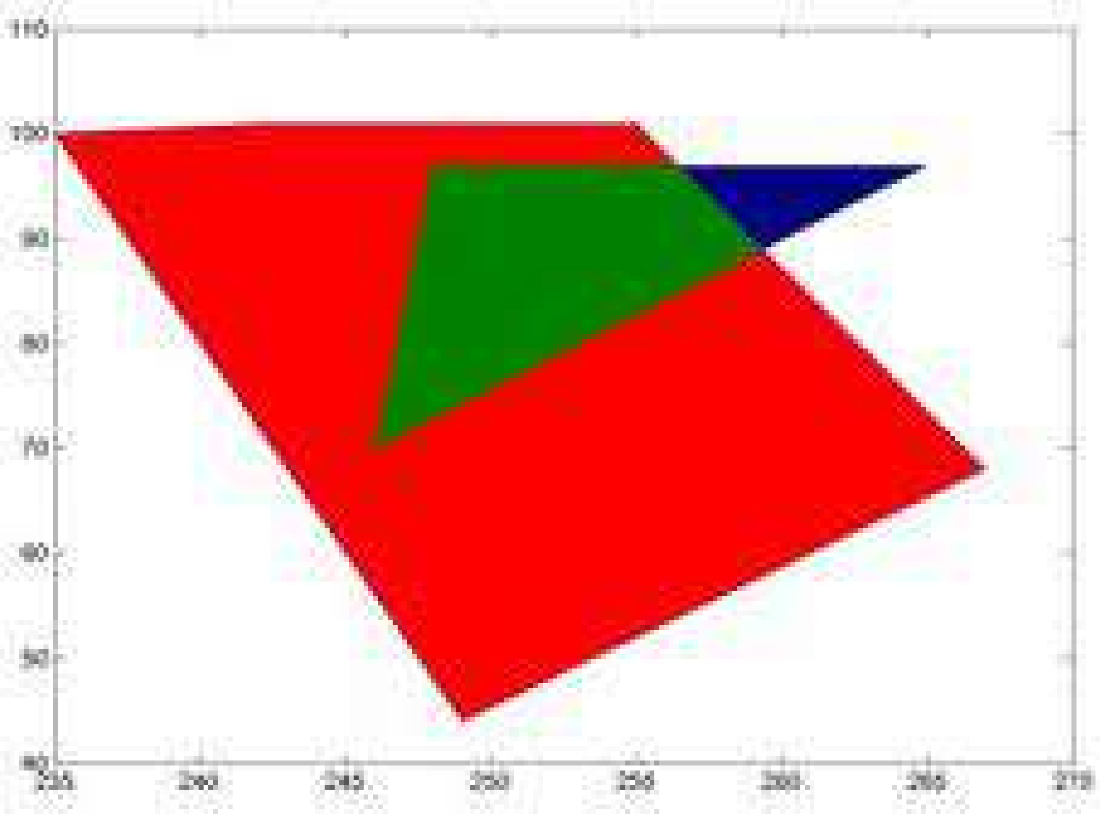
nonconvex

EXAMPLES of convex sets:

- **a point**
- **a line**
- **a half-line (a ray)**
- **a plane**
- **a half-plane**
- **a half-space**
- **a hyperplane**



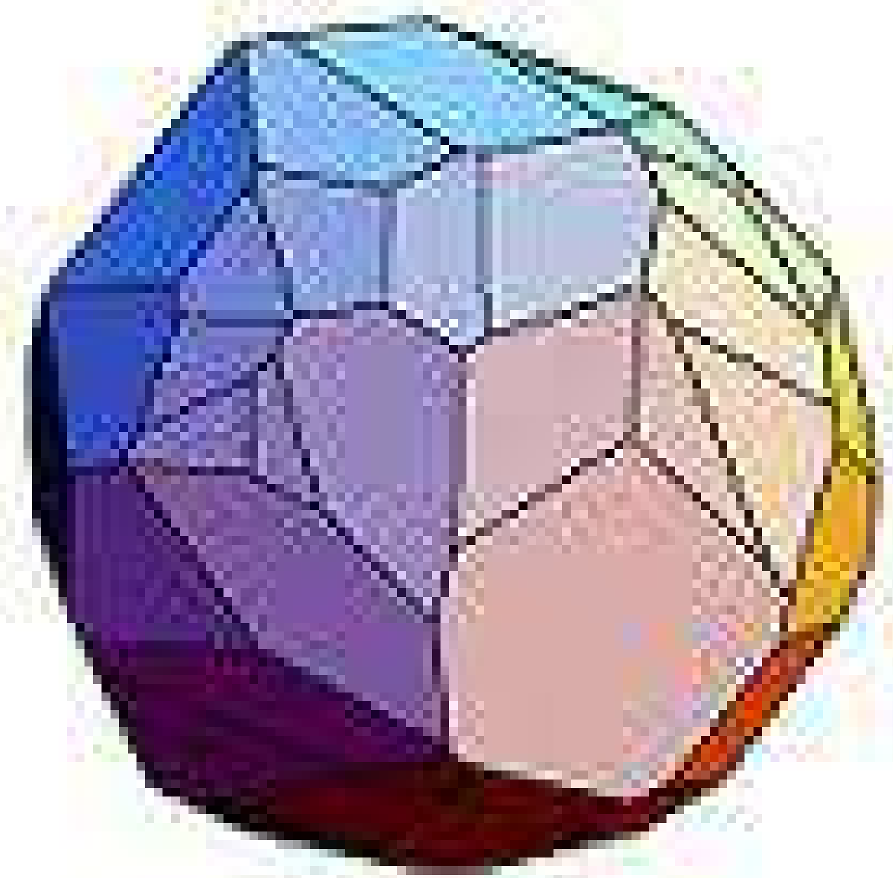
THEOREM *The intersection of any collection of convex sets is convex.*



DEFINITION A set which can be expressed as the intersection of a finite number of closed spaces is said to be a ***polytope (многогранник).***

DEFINITION A nonempty bounded polytope is called a ***polyhedron (ограниченный многогранник).***

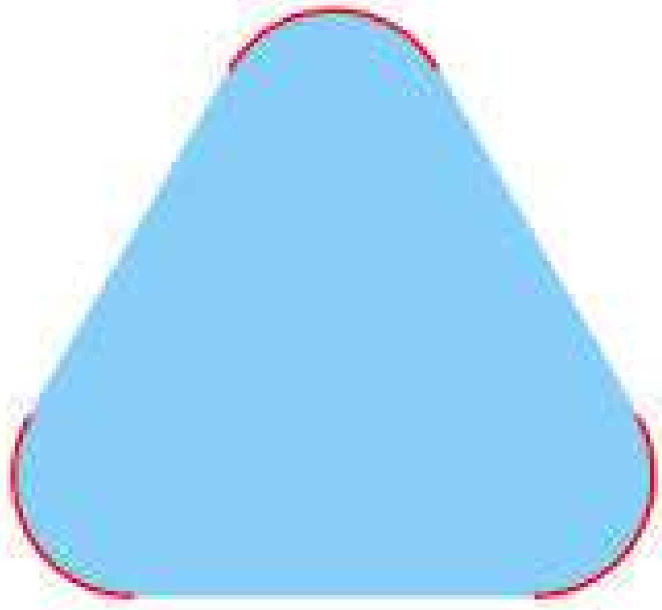
COROLLARY *Any polytope is a convex set.*



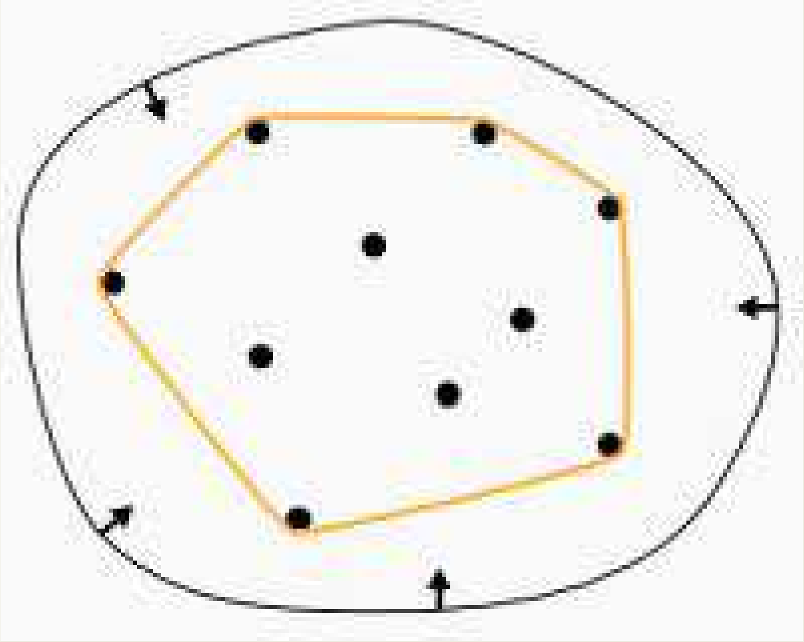
Convex Hulls

DEFINITION The ***convex hull*** or ***convex envelope (выпуклая оболочка точек)*** of a set *X* of points is the smallest convex set that contains *X*.

Formally, the convex hull may be defined as the intersection of all convex sets containing *X* or as the set of all convex combinations of points in *X*.

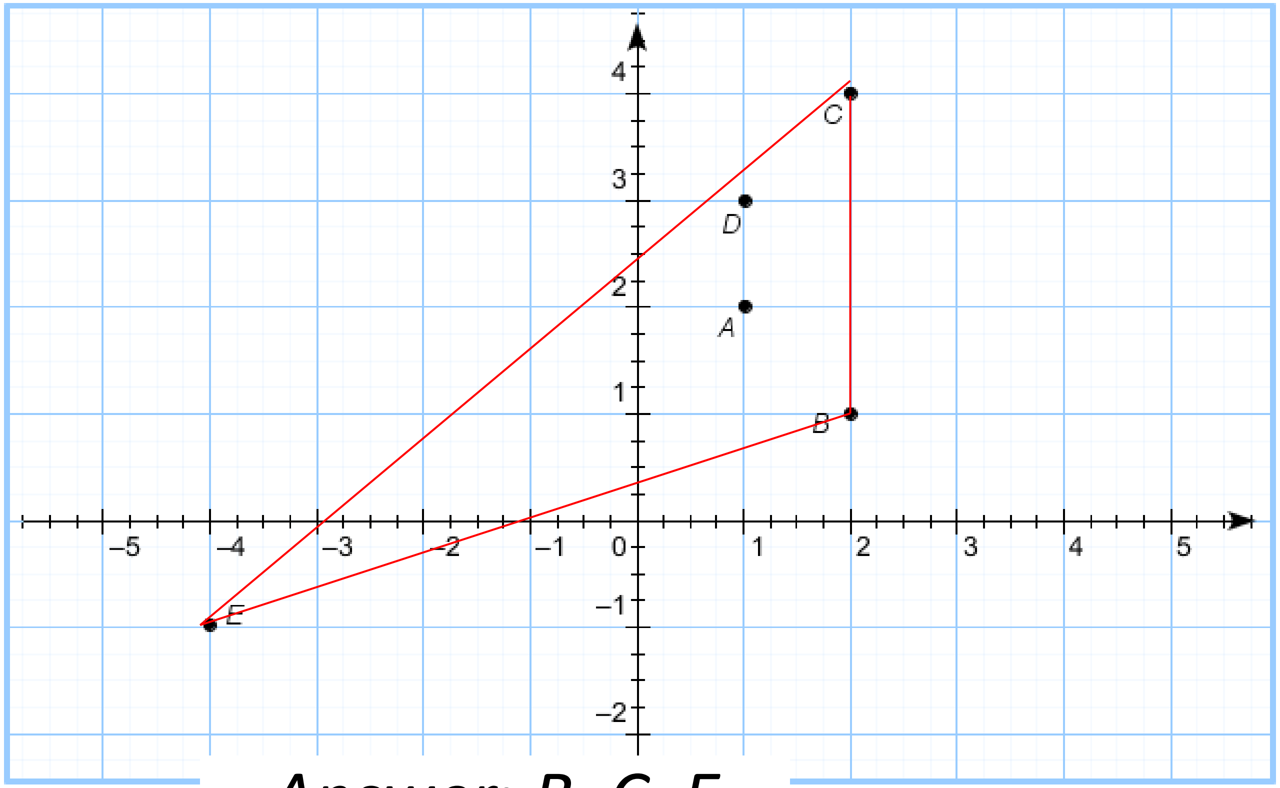


When *X* is a finite point set on the plane, the convex hull may be visualized as the shape formed by a rubber band stretched around *X*.



Convex Hull of a Finite Point Set

EXAMPLE Point out all vertices of a convex hull of the following finite point set: $A(1, 2)$, $B(2, 1)$, $C(2, 4)$, $D(1, 3)$, and $E(-4, -1)$.



Answer: *B*, *C*, *E*

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