

MAVZU: RATSIONAL KASRLAR.
ENG SODDA RATSIONAL
KASRLAR VA ULARNI
INTEGRALLASH.

Reja:

1. Ratsional kasrlar
2. Eng sodda ratsional kasrlar va ularni integrallash

Har qanday ratsional funksiyani ratsional kasr ko'rinishida, ya'ni ikki ko'phadning nisbati ko'rinishida tasvirlash mumkin:

$$\frac{Q(x)}{f(x)} = \frac{B_0x^m + B_1x^{m-1} + \dots + B_m}{A_0x^n + A_1x^{n-1} + \dots + B_n}$$

Muhokamaning umumiyligini cheklamasdan, bu ko'phadlar umumiy ildizga ega emas deb faraz qilamiz.

Agar suratinng darajasi mahrajning darajasidan past bo'lsa, kasr *to'g'ri kasr deb ataladi*, aks holda *noto'g'ri kasr* deb ataladi.

Agar kasr no'to'g'ri bo'lsa, suratni maxrajga (ko'phadni ko'phadga bo'lish qoidasi boyicha) bo'lib, berilgan kasrni ko'phad bilan biror to'g'ri kasrning yig'indisi ko'rinishida tasvirlash mumkin:

$$\frac{Q(x)}{f(x)} = M(x) + \frac{F(x)}{f(x)}$$

bu yerda $M(x)$ – ko'phad, $\frac{F(x)}{f(x)}$ – to'g'ri kasr.

1 – misol. Noto'g'ri ratsional kasr berilgan bo'lsin:

$$\frac{x^4-3}{x^2+2x+1}$$

Suratni maxrajiga (ko'phadlarni bo'lish qoidasi boyicha) bo'lib, shuni hosil qilamiz:

$$\frac{x^4-3}{x^2+2x+1} = x^2 - 2x + 3 - \frac{4x-6}{x^2+2x+1}$$

Ko'phadlarni integrallash hech qanday qiyinchilik tug'dirmagani uchun, ratsional kasrlarni integrallashdagi asosiy qiyinchilik to'g'ri ratsional kasrlarni integrallashdan iboratdir:

T a ' r i f. Ushbu ko'rinishdagi to'g'ri ratsional kasrlar

I. $\frac{A}{x-a}$

II. $\frac{A}{(x-a)^k}$ (k –butun musbat son $k \geq 2$)

III. $\frac{Ax + B}{x^2 + px + q}$ (mahrajning ildizlari kompleks sonlar,
ya'ni $\frac{p^2}{4} - q < 0$)

IV. $\frac{Ax + B}{(x^2 + px + q)^k}$ (k -butun musbat son, $k \geq 2$;
mahrajning ildizlar kompleks sonlar); I, II, III va IV tipdagi
eng sodda ratsional kasrlar deb ataladi.

I, II va III tipdagi eng sodda kasrlarni integrallashda katta qiyinchilikka uchramaymiz, shuning uchun ularning integrallarini hech qanday izohsiz keltiramiz:

$$\text{I. } \int \frac{A}{x-a} dx = A \ln |x-a| + C$$

$$\begin{aligned} \text{II. } \int \frac{A}{(x-a)^k} dx &= A \int (x-a)^{-k} dx = A \frac{(x-a)^{-k+1}}{-k+1} + C = \\ &= \frac{A}{(1-k)(x-a)^{k-1}} + C \end{aligned}$$

$$\begin{aligned}
\text{III. } \int \frac{Ax+B}{x^2+px+q} dx &= \int \frac{\frac{A}{2}(2x+p) + \left(B - \frac{Ap}{2}\right)}{x^2+px+q} dx = \\
&\frac{A}{2} \int \frac{2x+p}{x^2+px+q} dx + \left(B - \frac{Ap}{2}\right) \int \frac{dx}{x^2+px+q} = \\
&\frac{A}{2} \ln|x^2 + px + q| + \left(B - \frac{Ap}{2}\right) \int \frac{dx}{\left(x+\frac{p}{2}\right)^2 + \left(q-\frac{p^2}{4}\right)} = \\
&\frac{A}{2} \ln|x^2 + px + q| + \frac{2B-Ap}{\sqrt{4q-p^2}} \operatorname{arctg} \frac{2x+p}{\sqrt{4q-p^2}} + C.
\end{aligned}$$

IV tipdagi eng sodda kasrlarni integrallash murakkabroq hisoblashish talab qiladi.

$$\text{IV. } \int \frac{Ax+B}{(x^2+px+q)^k} dx.$$

Quyidagicha almashtiramiz:

$$\int \frac{Ax+B}{(x^2+px+q)^k} dx = \int \frac{\frac{A}{2}(2x+p) + \left(B - \frac{Ap}{2}\right)}{(x^2+px+q)^k} dx = \frac{A}{2} \int \frac{2x+p}{(x^2+px+q)^k} dx + \left(B - \frac{Ap}{2}\right) \int \frac{dx}{(x^2+px+q)^k}.$$

Birinchi integral $x^2 + px + q = t$, $(2x + p)dx = dt$ almashtirish yordami bilan olinadi:

$$\int \frac{2x+p}{(x^2+px+q)^k} dx = \int \frac{dt}{t^k} = \int t^{-k} dt = \frac{t^{-k+1}}{1-k} + C = \frac{1}{(1-k)(x^2+px+q)^{k-1}} + C.$$

Ikkinchi integralni (uni I_k bilan berilgilymiz) ushbu ko'inishda yozamiz:

$$I_k = \int \frac{dx}{(x^2+px+q)^k} = \int \frac{dx}{\left[\left(x+\frac{p}{2}\right)^2 + \left(q-\frac{p^2}{4}\right)\right]^k} = \int \frac{dt}{(t^2+m^2)^k},$$

bu yerda

$$x + \frac{p}{2} = t, \quad dx = dt, \quad q - \frac{p^2}{4} = m^2$$

deb faraz qilamiz (maxrajning ildizlari farazga ko'ra kompleks sonlar, demak $q - \frac{p^2}{4} > 0$). Endi quyidagicha kirishamiz:

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2} \int \frac{(t^2+m^2)-t^2}{(t^2+m^2)^k} dt = \frac{1}{m^2} \int \frac{dt}{(t^2+m^2)^{k-1}} - \frac{1}{m^2} \int \frac{t^2}{(t^2+m^2)^k} dt. \quad (1)$$

So'nggi integralni bunday almashtiramiz:

$$\int \frac{t^2 dt}{(t^2+m^2)^k} = \int \frac{t \cdot t dt}{(t^2+m^2)^k} = \frac{1}{2} \int t \frac{d(t^2+m^2)}{(t^2+m^2)^k} =$$
$$- \frac{1}{2(k-1)} \int t d \left(\frac{1}{(t^2+m^2)^{k-1}} \right).$$

Bo'laklab integrallab, quyidagini hosil qilamiz:

$$\int \frac{t^2 dt}{(t^2+m^2)^k} = - \frac{1}{2(k-1)} \left[t \frac{1}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}} \right].$$

Bu ifodani (1) tenglikka qoysak, quyidagi hosil bo'ladi:

$$I_k \int \frac{dt}{(t^2+m^2)^k} =$$
$$\frac{1}{m^2} \int \frac{dt}{(t^2+m^2)^{k-1}} + \frac{1}{m^2} \frac{1}{2(k-1)} \left[\frac{t}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}} \right] =$$
$$\frac{t}{2m^2(k-1)(t^2+m^2)^{k-1}} + \frac{2k-3}{2m^2(k-1)} \int \frac{dt}{(t^2+m^2)^{k-1}}.$$

O'ng tomonda I_k tipidagi integral bor, lekin integral ostidagi funksiya maxrajining daraja ko'rsatkichi uning daraja ko'rsatkichidan bitta birlik past ($k-1$), shunday qilib, I_k ni I_{k-1} orqali ifodaladik.

Shu yo'l bilan davom etib, ma'lum integral

$$I_1 = \int \frac{dt}{t^2+m^2} = \frac{1}{m} \operatorname{arctg} \frac{t}{m} + C$$

ga yetib boramiz. So'gra t va m o'rniga ularning qiymatlarini qo'yib, IV integralning x va berilgan A, B, p, q sonlar orqali ifodasini topamiz.

2 – misol.

$$\int \frac{x-1}{(x^2+2x+3)^2} dx = \int \frac{\frac{1}{2}(2x+2)+(-1-1)}{(x^2+2x+3)^2} dx = \frac{1}{2} \int \frac{2x+2}{(x^2+2x+3)^2} dx -$$
$$2 \int \frac{dx}{(x^2+2x+3)^2} = -\frac{1}{2} \frac{1}{(x^2+2x+3)} - 2 \int \frac{dx}{(x^2+2x+3)^2}.$$

Oxirgi integralga $x+1=t$ almashtirishni qo'llaymiz:

$$\int \frac{dx}{(x^2+2x+3)^2} = \int \frac{dx}{[(x+1)^2+2]^2} = \int \frac{dt}{(t^2+2)^2} = \frac{1}{2} \int \frac{(t^2+2)-t^2}{(t^2+2)^2} dt =$$
$$\frac{1}{2} \int \frac{dt}{t^2+2} - \frac{1}{2} \int \frac{t^2}{(t^2+2)^2} dt = \frac{1}{2} \frac{1}{\sqrt{2}} \arctg \frac{t}{\sqrt{2}} - \frac{1}{2} \int \frac{t^2}{(t^2+2)^2} dt.$$

Oxirgi integralni qaraymiz:

$$\int \frac{t^2}{(t^2+2)^2} dt = \frac{1}{2} \int \frac{td(t^2+2)}{(t^2+2)^2} = -\frac{1}{2} \int td \left(\frac{1}{t^2+2} \right) = -\frac{1}{2} \frac{t}{t^2+2} + \frac{1}{2} \int \frac{dt}{t^2+2} =$$
$$-\frac{t}{2(t^2+2)} + \frac{1}{\sqrt{2}} \arctg \frac{t}{\sqrt{2}}$$

Demak,

$$\int \frac{dx}{(x^2+2x+3)^2} = \frac{1}{2\sqrt{2}} \arctg \frac{x+1}{\sqrt{2}} - \frac{1}{2} \left[-\frac{x+1}{2(x^2+2x+3)} + \frac{1}{2\sqrt{2}} \arctg \frac{x+1}{\sqrt{2}} \right].$$

oxirgi natija quyidagi ko'rinishda bo'ladi:

$$\int \frac{x-1}{(x^2+2x+3)^2} dx = -\frac{x+2}{2(x^2+2x+3)^2} - \frac{\sqrt{2}}{4} \arctg \frac{x+1}{\sqrt{2}} + C.$$

Endi har qanday to'g'ri ratsional kasrni eng sodda kasrlar yig'indisiga ajratish mumkinligini ko'rsatamiz.

Ushbu to'g'ri ratsional kasr

$$\frac{F(x)}{f(x)}$$

berilgan bo'lsin. Bu kasrga kirgan ko'phadlarning koeffisientlari haqiqiy sonlar va berilgan kasr qisqarmaydigan kasr deb faraz qilamiz (bu esa surat va mahraj umumiy ildizga ega emas degan so'zdir).

1 – t e o r e m a. $x=a$ mahrajning k karrali ildizi, ya'ni $f(x)=(x-a)^k f_1(x)$ bo'lsin, bu yerda $f_1(a) \neq 0$; u holda To'ri kasrni boshqa ikki mo'g'ri kasr yig'indisi shaklida quyidagicha tasvirlash mumkin

$$\frac{F(x)}{f(x)} = \frac{A}{(x-a)^k} + \frac{F_1(x)}{(x-a)^{k-1} f_1(x)}$$

bu yerda A nolga teng bo'lmagan o'zgarmas son, $F_1(x)$ ko'phad, buning darajasi $(x-a)^{k-1} f_1(x)$ mahrajning darajasidan past.

2 – t e o r e m a. Agar $f(x)=(x^2+px+q)\varphi_1(x)$ bo'lsa (bu yerda $\varphi_1(x)$ ko'phad x^2+px+q ga bo'linmaydi), to'g'ri ratsional kasrni boshqa ikki too'g'ri kasrning yig'indisi ko'rinishida tasvirlash mumkin:

$$\frac{F(x)}{f(x)} = \frac{Mx + N}{(x^2 + px + q)^k} + \frac{\Phi_1(x)}{(x^2 + px + q)^{\mu-1} \varphi_1(x)}$$