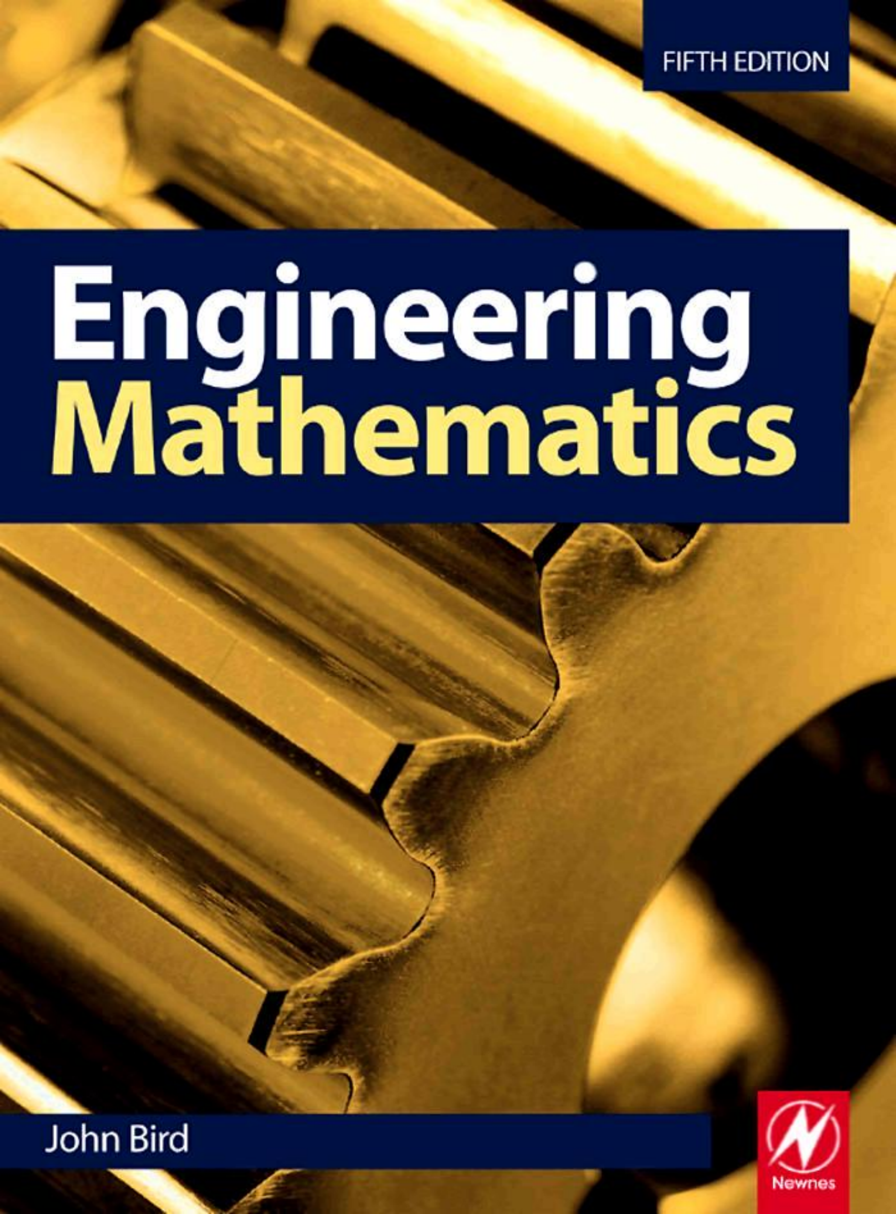
FIFTH EDITION

Engineering Mathematics



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Chapter 51

Integration using partial fractions

51.1 Introduction

The process of expressing a fraction in terms of simpler fractions—called **partial fractions**—is discussed in Chapter 7, with the forms of partial fractions used being summarised in Table 7.1, page 54.

Certain functions have to be resolved into partial fractions before they an be integrated, as demonstrated in the following worked problems.

51.2 Worked problems on integration using partial fractions with linear factors

Problem 1. Determine: $\int \frac{11-3x}{x^2+2x-3}\,dx$

As shown in Problem 1, page 54:

$$\frac{11-3x}{x^2+2x-3} \equiv \frac{2}{(x-1)} - \frac{5}{(x+3)}$$

$$\text{Hence } \int \frac{11-3x}{x^2+2x-3}\,dx = \int \left\{\frac{2}{(x-1)} - \frac{5}{(x+3)}\right\} dx$$

$$= \mathbf{2\ln(x-1) - 5\ln(x+3) + c}$$

(by algebraic substitutions—see chapter 49)

or $\mathbf{\ln\left\{\frac{(x-1)^2}{(x+3)^5}\right\} + c}$ by the laws of logarithms

Problem 2. Find: $\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx$

It was shown in Problem 2, page 55:

$$\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}$$

$$\text{Hence } \int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx \equiv \int \left\{\frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}\right\} dx$$

$$= \mathbf{4\ln(x+1) - 3\ln(x-2) + \ln(x+3) + c}$$

$$\text{or } \mathbf{\ln\left\{\frac{(x+1)^4(x+3)}{(x-2)^3}\right\} + c}$$

Problem 3. Determine: $\int \frac{x^2+1}{x^2-3x+2}\,dx$

By dividing out (since the numerator and denominator are of the same degree) and resolving into partial fractions it was shown in Problem 3, page 55:

$$\frac{x^2+1}{x^2-3x+2} \equiv 1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}$$

$$\text{Hence } \int \frac{x^2+1}{x^2-3x+2}\,dx \equiv \int \left\{1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}\right\} dx$$

$$= \mathbf{x - 2\ln(x-1) + 5\ln(x-2) + c}$$

$$\textbf{or } \mathbf{x + \ln\left\{\frac{(x-2)^5}{(x-1)^2}\right\} + c}$$

Problem 4. Evaluate:

$\displaystyle\int_2^3 \frac{x^3 - 2x^2 - 4x - 4}{x^2 + x - 2}\,dx$, correct to 4 significant figures

By dividing out and resolving into partial fractions, it was shown in Problem 4, page 56:

$$\frac{x^3 - 2x^2 - 4x - 4}{x^2 + x - 2} \equiv x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}$$

Hence $\displaystyle\int_2^3 \frac{x^3 - 2x^2 - 4x - 4}{x^2 + x - 2}\,dx$

$$\equiv \int_2^3 \left\{x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}\right\} dx$$

$$= \left[\frac{x^2}{2} - 3x + 4\ln(x+2) - 3\ln(x-1)\right]_2^3$$

$$= \left(\frac{9}{2} - 9 + 4\ln 5 - 3\ln 2\right) - (2 - 6 + 4\ln 4 - 3\ln 1)$$

$= \mathbf{-1.687}$, correct to 4 significant figures

Now try the following exercise

Exercise 181 Further problems on integration using partial fractions with linear factors

In Problems 1 to 5, integrate with respect to x

1. $\displaystyle\int \frac{12}{(x^2 - 9)}\,dx$

   $\left[\begin{array}{c} 2\ln(x-3) - 2\ln(x+3) + c \\ \text{or } \ln\left\{\frac{x-3}{x+3}\right\}^2 + c \end{array}\right]$

2. $\displaystyle\int \frac{4(x-4)}{(x^2 - 2x - 3)}\,dx$

   $\left[\begin{array}{c} 5\ln(x+1) - \ln(x-3) + c \\ \text{or } \ln\left\{\frac{(x+1)^5}{(x-3)}\right\} + c \end{array}\right]$

3. $\displaystyle\int \frac{3(2x^2 - 8x - 1)}{(x+4)(x+1)(2x-1)}\,dx$

   $\left[\begin{array}{c} 7\ln(x+4) - 3\ln(x+1) - \ln(2x-1) + c \\ \text{or } \ln\left\{\frac{(x+4)^7}{(x+1)^3(2x-1)}\right\} + c \end{array}\right]$

4. $\displaystyle\int \frac{x^2 + 9x + 8}{x^2 + x - 6}\,dx$

   $\left[\begin{array}{c} x + 2\ln(x+3) + 6\ln(x-2) + c \\ \text{or } x + \ln\{(x+3)^2(x-2)^6\} + c \end{array}\right]$

5. $\displaystyle\int \frac{3x^3 - 2x^2 - 16x + 20}{(x-2)(x+2)}\,dx$

   $\left[\begin{array}{c} \frac{3x^2}{2} - 2x + \ln(x-2) \\ -5\ln(x+2) + c \end{array}\right]$

In Problems 6 and 7, evaluate the definite integrals correct to 4 significant figures.

6. $\displaystyle\int_3^4 \frac{x^2 - 3x + 6}{x(x-2)(x-1)}\,dx$ [0.6275]

7. $\displaystyle\int_4^6 \frac{x^2 - x - 14}{x^2 - 2x - 3}\,dx$ [0.8122]

51.3 Worked problems on integration using partial fractions with repeated linear factors

Problem 5. Determine: $\displaystyle\int \frac{2x+3}{(x-2)^2}\,dx$

It was shown in Problem 5, page 57:

$$\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)} + \frac{7}{(x-2)^2}$$

Thus $\displaystyle\int \frac{2x+3}{(x-2)^2}\,dx$

$$\equiv \int \left\{\frac{2}{(x-2)} + \frac{7}{(x-2)^2}\right\} dx$$

$$= \mathbf{2\ln(x-2) - \frac{7}{(x-2)} + c}$$

$\left[\int \frac{7}{(x-2)^2}\,dx$ is determined using the algebraic substitution $u=(x-2)$, see Chapter 49$\right]$

Problem 6. Find: $\displaystyle\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx$

It was shown in Problem 6, page 57:

$$\frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}$$

Hence $\displaystyle\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx$

$$\equiv \int \left\{\frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}\right\} dx$$

$$= \mathbf{2\ln(x+3) + 3\ln(x-1) + \frac{4}{(x-1)} + c}$$

$$\text{or}\quad \mathbf{\ln(x+3)^2(x-1)^3 + \frac{4}{(x-1)} + c}$$

Problem 7. Evaluate:

$\displaystyle\int_{-2}^{1} \frac{3x^2+16x+15}{(x+3)^3}\,dx$, correct to 4 significant figures

It was shown in Problem 7, page 58:

$$\frac{3x^2+16x+15}{(x+3)^3} \equiv \frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}$$

Hence $\displaystyle\int \frac{3x^2+16x+15}{(x+3)^3}\,dx$

$$\equiv \int_{-2}^{1} \left\{\frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}\right\} dx$$

$$= \left[3\ln(x+3) + \frac{2}{(x+3)} + \frac{3}{(x+3)^2}\right]_{-2}^{1}$$

$$= \left(3\ln 4 + \frac{2}{4} + \frac{3}{16}\right) - \left(3\ln 1 + \frac{2}{1} + \frac{3}{1}\right)$$

$= \mathbf{-0.1536}$, correct to 4 significant figures.

Now try the following exercise

Exercise 182 Further problems on integration using partial fractions with repeated linear factors

In Problems 1 and 2, integrate with respect to x.

1. $\displaystyle\int \frac{4x-3}{(x+1)^2}\,dx$ $\quad\left[4\ln(x+1) + \frac{7}{(x+1)} + c\right]$

2. $\displaystyle\int \frac{5x^2-30x+44}{(x-2)^3}\,dx$
$\left[5\ln(x-2) + \frac{10}{(x-2)} - \frac{2}{(x-2)^2} + c\right]$

In Problems 3 and 4, evaluate the definite integrals correct to 4 significant figures.

3. $\displaystyle\int_{1}^{2} \frac{x^2+7x+3}{x^2(x+3)}\,dx$ $\quad$ [1.663]

4. $\displaystyle\int_{6}^{7} \frac{18+21x-x^2}{(x-5)(x+2)^2}\,dx$ $\quad$ [1.089]

51.4 Worked problems on integration using partial fractions with quadratic factors

Problem 8. Find: $\displaystyle\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$

It was shown in Problem 9, page 59:

$$\frac{3+6x+4x^2-2x^2}{x^2(x^2+3)} \equiv \frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{(x^2+3)}$$

Thus $\displaystyle\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$

$$\equiv \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{(x^2+3)}\right) dx$$

$$= \int \left\{\frac{2}{x} + \frac{1}{x^2} + \frac{3}{(x^2+3)} - \frac{4x}{(x^2+3)}\right\} dx$$

$$\int \frac{3}{(x^2+3)}\,dx = 3\int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

from 12, Table 50.1, page 448.

$\int \frac{4x}{x^2+3}\,dx$ is determined using the algebraic substitutions $u=(x^2+3)$.

Hence $\int\left\{\frac{2}{x}+\frac{1}{x^2}+\frac{3}{(x^2+3)}-\frac{4x}{(x^2+3)}\right\}dx$

$$= 2\ln x - \frac{1}{x} + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 2\ln(x^2+3)+c$$

$$= \mathbf{\ln\left(\frac{x}{x^2+3}\right)^2 - \frac{1}{x} + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + c}$$

Problem 9. Determine: $\int \frac{1}{(x^2-a^2)}\,dx$

Let $\frac{1}{(x^2-a^2)} \equiv \frac{A}{(x-a)} + \frac{B}{(x+a)}$

$$\equiv \frac{A(x+a)+B(x-a)}{(x+a)(x-a)}$$

Equating the numerators gives:

$$1 \equiv A(x+a)+B(x-a)$$

Let $x=a$, then $A=\frac{1}{2a}$

and let $x=-a$,

then $B=-\frac{1}{2a}$

Hence $\int \frac{1}{(x^2-a^2)}\,dx \equiv \int \frac{1}{2a}\left[\frac{1}{(x-a)}-\frac{1}{(x+a)}\right]dx$

$$= \frac{1}{2a}[\ln(x-a)-\ln(x+a)]+c$$

$$= \mathbf{\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c}$$

Problem 10. Evaluate: $\int_3^4 \frac{3}{(x^2-4)}\,dx$, correct to 3 significant figures

From Problem 9,

$$\int_3^4 \frac{3}{(x^2-4)}dx = 3\left[\frac{1}{2(2)}\ln\left(\frac{x-2}{x+2}\right)\right]_3^4$$

$$= \frac{3}{4}\left[\ln\frac{2}{6}-\ln\frac{1}{5}\right]$$

$$= \frac{3}{4}\ln\frac{5}{3} = \mathbf{0.383},\ \text{correct to 3 significant figures.}$$

Problem 11. Determine: $\int \frac{1}{(a^2-x^2)}\,dx$

Using partial fractions, let

$$\frac{1}{(a^2-x^2)} \equiv \frac{1}{(a-x)(a+x)} \equiv \frac{A}{(a-x)}+\frac{B}{(a+x)}$$

$$\equiv \frac{A(a+x)+B(a-x)}{(a-x)(a+x)}$$

Then $1 \equiv A(a+x)+B(a-x)$

Let $x=a$ then $A=\frac{1}{2a}$. Let $x=-a$ then $B=\frac{1}{2a}$

Hence $\int \frac{1}{(a^2-x^2)}\,dx$

$$= \int \frac{1}{2a}\left[\frac{1}{(a-x)}+\frac{1}{(a+x)}\right]dx$$

$$= \frac{1}{2a}[-\ln(a-x)+\ln(a+x)]+c$$

$$= \mathbf{\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c}$$

Problem 12. Evaluate: $\int_0^2 \frac{5}{(9-x^2)}\,dx$, correct to 4 decimal places

From Problem 11,

$$\int_0^2 \frac{5}{(9-x^2)}\,dx = 5\left[\frac{1}{2(3)}\ln\left(\frac{3+x}{3-x}\right)\right]_0^2$$

$$= \frac{5}{6}\left[\ln\frac{5}{1}-\ln 1\right] = \mathbf{1.3412},$$

correct to 4 decimal places

Now try the following exercise

Exercise 183 Further problems on integration using partial fractions with quadratic factors

1. Determine $\int \frac{x^2 - x - 13}{(x^2+7)(x-2)}\,dx$

$$\left[\begin{array}{c} \ln(x^2+7) + \frac{3}{\sqrt{7}} \tan^{-1} \frac{x}{\sqrt{7}} \\ -\ln(x-2) + c \end{array} \right]$$

In Problems 2 to 4, evaluate the definite integrals correct to 4 significant figures.

2. $\int_5^6 \frac{6x-5}{(x-4)(x^2+3)}\,dx$ [0.5880]

3. $\int_1^2 \frac{4}{(16-x^2)}\,dx$ [0.2939]

4. $\int_4^5 \frac{2}{(x^2-9)}\,dx$ [0.1865]