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CHAPTER 8

Analytic Geometry in Two and Three Dimensions

The oval-shaped lawn behind the White House in Washington, D.C. is called *the Ellipse*. It has views of the Washington Monument, the Jefferson Memorial, the Department of Commerce, and the Old Post Office Building. The Ellipse is 616 ft long, 528 ft wide, and is in the shape of a conic section. Its shape can be modeled using the methods of this chapter. See page 652.

8.6 Three-Dimensional Cartesian Coordinate System

What you'll learn about

- Three-Dimensional Cartesian Coordinates
- Distance and Midpoint Formulas
- Equation of a Sphere
- Planes and Other Surfaces
- Vectors in Space
- Lines in Space

. . . and why

This is the analytic geometry of our physical world.

OBJECTIVE

Students will be able to draw three-dimensional figures and analyze vectors in space.

MOTIVATE

Ask students why it might be desirable to create a coordinate system representing three-dimensional space.

LESSON GUIDE

Day 1: Three-Dimensional Cartesian Coordinates; Distance and Midpoint Formulas; Equation of a Sphere
Day 2: Planes and Other Surfaces; Vectors in Space; Lines in Space

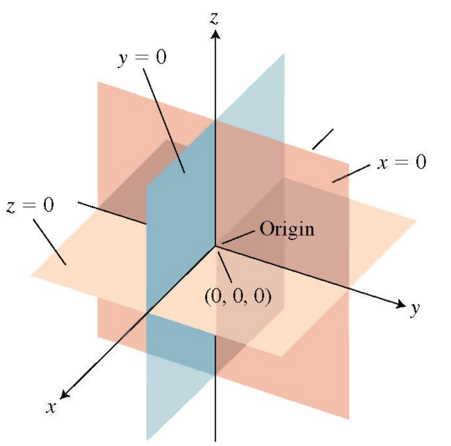


FIGURE 8.47 The coordinate planes divide space into eight octants.

Three-Dimensional Cartesian Coordinates

In Sections P.2 and P.4, we studied Cartesian coordinates and the associated basic formulas and equations for the two-dimensional plane; we now extend these ideas to *three-dimensional space*. In the plane, we used two axes and ordered pairs to name points; in space, we use three mutually perpendicular axes and ordered triples of real numbers to name points. See Figure 8.46.

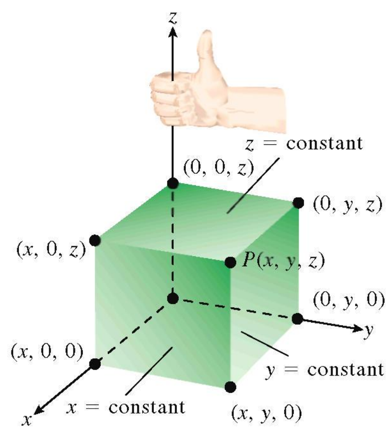


FIGURE 8.46 The point $P(x, y, z)$ in Cartesian space.

Notice that Figure 8.46 exhibits several important features of the *three-dimensional Cartesian coordinate system*:

- The axes are labeled x, y, and z, and these three **coordinate axes** form a **right-handed coordinate frame**: When you hold your right hand with fingers curving from the positive x-axis toward the positive y-axis, your thumb points in the direction of the positive z-axis.
- A point P in space uniquely corresponds to an ordered triple (x, y, z) of real numbers. The numbers x, y, and z are the **Cartesian coordinates of P**.
- Points on the axes have the form $(x, 0, 0)$, $(0, y, 0)$, or $(0, 0, z)$, with $(x, 0, 0)$ on the x-axis, $(0, y, 0)$ on the y-axis, and $(0, 0, z)$ on the z-axis.

In Figure 8.47, the axes are paired to determine the **coordinate planes**:

- The coordinate planes are the ***xy*-plane**, the ***xz*-plane**, and the ***yz*-plane**, and have equations $z = 0$, $y = 0$, and $x = 0$, respectively.
- Points on the coordinate planes have the form $(x, y, 0)$, $(x, 0, z)$, or $(0, y, z)$, with $(x, y, 0)$ on the xy-plane, $(x, 0, z)$ on the xz-plane, and $(0, y, z)$ on the yz-plane.
- The coordinate planes meet at the **origin**, $(0, 0, 0)$.
- The coordinate planes divide space into eight regions called **octants**, with the **first octant** containing all points in space with three positive coordinates.

EXAMPLE 1 Locating a Point in Cartesian Space

Draw a sketch that shows the point (2, 3, 5).

SOLUTION To locate the point (2, 3, 5), we first sketch a right-handed three-dimensional coordinate frame. We then draw the planes $x = 2$, $y = 3$, and $z = 5$, which parallel the coordinate planes $x = 0$, $y = 0$, and $z = 0$, respectively. The point (2, 3, 5) lies at the intersection of the planes $x = 2$, $y = 3$, and $z = 5$, as shown in Figure 8.48. ***Now try Exercise 1.***

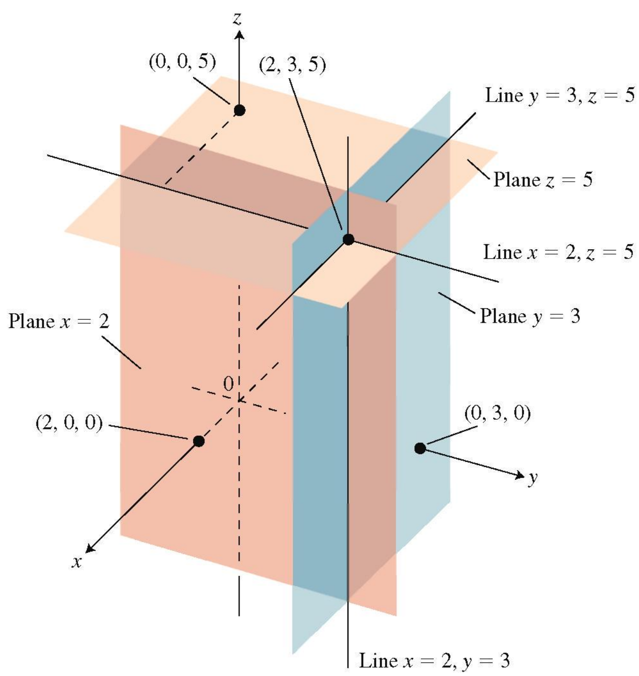


FIGURE 8.48 The planes $x = 2$, $y = 3$, and $z = 5$ determine the point (2, 3, 5). (Example 1)

Distance and Midpoint Formulas

The distance and midpoint formulas for space are natural generalizations of the corresponding formulas for the plane.

Distance Formula (Cartesian Space)

The distance $d(P, Q)$ between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space is

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Just as in the plane, the coordinates of the midpoint of a line segment are the averages for the coordinates of the endpoints of the segment.

Midpoint Formula (Cartesian Space)

The midpoint M of the line segment PQ with endpoints $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

EXAMPLE 2 Calculating a Distance and Finding a Midpoint

Find the distance between the points $P(-2, 3, 1)$ and $Q(4, -1, 5)$, and find the midpoint of line segment PQ.

SOLUTION The distance is given by

$$\begin{aligned} d(P, Q) &= \sqrt{(-2-4)^2 + (3+1)^2 + (1-5)^2} \\ &= \sqrt{36 + 16 + 16} \\ &= \sqrt{68} \approx 8.25 \end{aligned}$$

The midpoint is

$$M = \left(\frac{-2+4}{2}, \frac{3-1}{2}, \frac{1+5}{2}\right) = (1, 1, 3).$$

Now try Exercises 5 and 9.

TEACHING NOTE

Many of the results discussed in this section are fairly intuitive extensions of previously studied two-dimensional results. You may wish to present a comparison between the two-dimensional results and their three-dimensional counterparts.

Equation of a Sphere

A *sphere* is the three-dimensional analogue of a circle: In space, the set of points that lie a fixed distance from a fixed point is a **sphere**. The fixed distance is the **radius**, and the fixed point is the **center** of the sphere. The point $P(x, y, z)$ is a point of the sphere with center (h, k, l) and radius r if and only if

$$\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r.$$

Squaring both sides gives the standard equation shown below.

Standard Equation of a Sphere

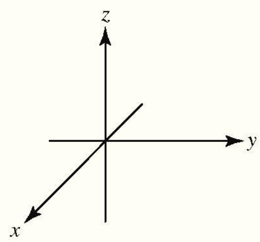
A point $P(x, y, z)$ is on the sphere with center (h, k, l) and radius r if and only if

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2.$$

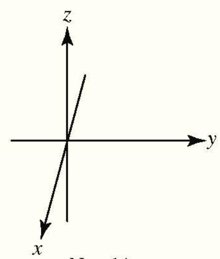
Drawing Lesson

How to Draw Three-Dimensional Objects to Look Three-Dimensional

1. Make the angle between the positive x-axis and the positive y-axis large enough.

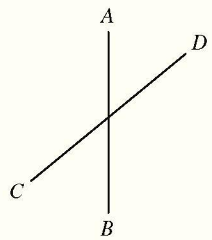


This

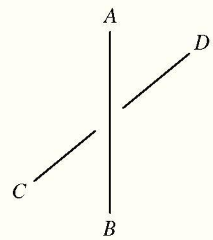


Not this

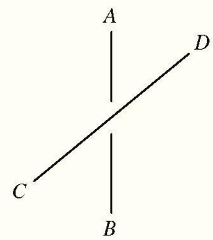
2. Break lines. When one line passes behind another, break it to show that it doesn't touch and that part of it is hidden.



Intersecting

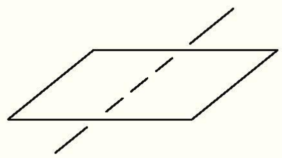


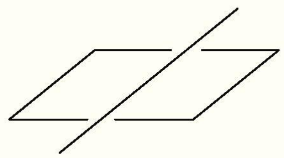
CD behind *AB*

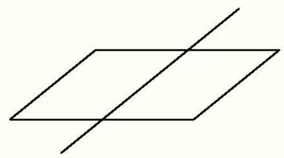


AB behind *CD*

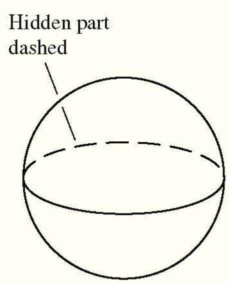
3. Dash or omit hidden portions of lines. Don't let the line touch the boundary of the parallelogram that represents the plane, unless the line lies in the plane.


Line below plane

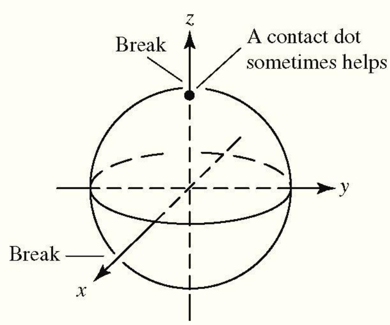

Line above plane


Line *in* plane

4. Spheres: Draw the sphere first (outline and equator); draw axes, if any, later. Use line breaks and dashed lines.



Sphere first



Axes later

EXAMPLE 3 Finding the Standard Equation of a Sphere

The standard equation of the sphere with center $(2, 0, -3)$ and radius 7 is

$$(x - 2)^2 + y^2 + (z + 3)^2 = 49.$$

Now try Exercise 13.

Planes and Other Surfaces

In Section P.4, we learned that every line in the Cartesian plane can be written as a first-degree (linear) equation in two variables; that is, every line can be written as

$$Ax + By + C = 0,$$

where A and B are not both zero. Conversely, every first-degree equation in two variables represents a line in the Cartesian plane.

In an analogous way, every **plane** in Cartesian space can be written as a **first-degree equation in three variables**:

Equation for a Plane in Cartesian Space

Every plane can be written as

$$Ax + By + Cz + D = 0,$$

where A, B, and C are not all zero. Conversely, every first-degree equation in three variables represents a plane in Cartesian space.

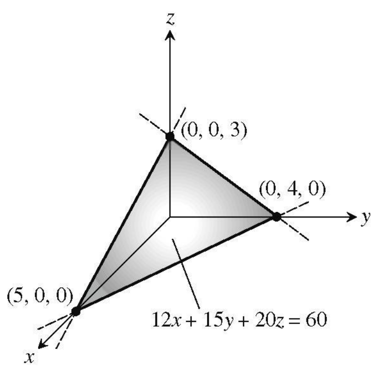


FIGURE 8.49 The intercepts $(5, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$ determine the plane $12x + 15y + 20z = 60$. (Example 4)

EXAMPLE 4 Sketching a Plane in Space

Sketch the graph of $12x + 15y + 20z = 60$.

SOLUTION Because this is a first-degree equation, its graph is a plane. Three points determine a plane. To find three points, we first divide both sides of $12x + 15y + 20z = 60$ by 60:

$$\frac{x}{5} + \frac{y}{4} + \frac{z}{3} = 1.$$

In this form, it is easy to see that the points $(5, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$ satisfy the equation. These are the points where the graph crosses the coordinate axes. Figure 8.49 shows the completed sketch.

Now try Exercise 17.

Equations in the three variables x, y, and z generally graph as surfaces in three-dimensional space. Just as in the plane, second-degree equations are of particular interest. Recall that second-degree equations in two variables yield conic sections in the Cartesian plane. In space, second-degree equations in *three* variables yield **quadric surfaces**: The paraboloids, ellipsoids, and hyperboloids of revolution that have special reflective properties are all quadric surfaces, as are such exotic-sounding surfaces as hyperbolic paraboloids and elliptic hyperboloids.

Other surfaces of interest include graphs of **functions of two variables**, whose equations have the form $z = f(x, y)$. Some examples are $z = x \ln y$, $z = \sin(xy)$, and $z = \sqrt{1 - x^2 - y^2}$. The last equation graphs as a hemisphere (see Exercise 63). Equations of the form $z = f(x, y)$ can be graphed using some graphing calculators and most computer algebra software. Quadric surfaces and functions of two variables are studied in most university-level calculus course sequences.

Vectors in Space

In space, just as in the plane, the sets of equivalent directed line segments (or arrows) are *vectors*. They are used to represent forces, displacements, and velocities in three dimensions. In space, we use ordered triples to denote vectors:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

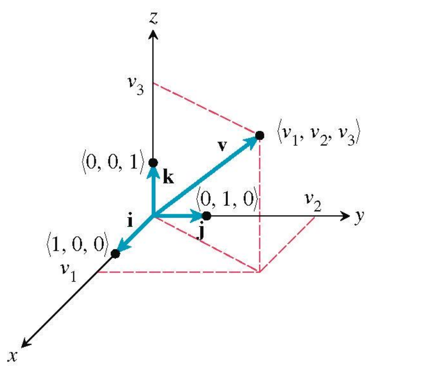


FIGURE 8.50 The vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

The **zero vector** is $\mathbf{0} = \langle 0, 0, 0 \rangle$, and the **standard unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. As shown in Figure 8.50, the vector **v** can be expressed in terms of these standard unit vectors:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

The vector **v** that is represented by the arrow from $P(a, b, c)$ to $Q(x, y, z)$ is

$$\mathbf{v} = \overrightarrow{PQ} = \langle x - a, y - b, z - c \rangle = (x - a)\mathbf{i} + (y - b)\mathbf{j} + (z - c)\mathbf{k}.$$

A vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be multiplied by a scalar (real number) c as follows:

$$c\mathbf{v} = c\langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle.$$

Many other properties of vectors generalize in a natural way when we move from two to three dimensions:

Vector Relationships in Space

For vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$,

- **Equality:** $\mathbf{v} = \mathbf{w}$ if and only if $v_1 = w_1$, $v_2 = w_2$, and $v_3 = w_3$
- **Addition:** $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
- **Subtraction:** $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$
- **Magnitude:** $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- **Dot product:** $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$
- **Unit vector:** $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$, $\mathbf{v} \neq \mathbf{0}$, is the unit vector in the direction of **v**.