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Mathematical Analysis I

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Functions

Functions crop up regularly in everyday life (for instance: each student of the Polytechnic of Turin has a unique identification number), in physics (to each point of a region in space occupied by a fluid we may associate the velocity of the particle passing through that point at a given moment), in economy (each working day at Milan's stock exchange is tagged with the Mibtel index), and so on.

The mathematical notion of a function subsumes all these situations.

2.1 Definitions and first examples

Let X and Y be two sets. A **function f defined on X with values in Y** is a correspondence associating to each element $x \in X$ *at most* one element $y \in Y$. This is often shortened to 'a function from X to Y'. A synonym for function is **map**. The set of $x \in X$ to which f associates an element in Y is the **domain** of f; the domain is a subset of X, indicated by $\operatorname{dom} f$. One writes

$$\boxed{f : \operatorname{dom} f \subseteq X \to Y.}$$

If $\operatorname{dom} f = X$, one says that f is defined **on** X and writes simply $f : X \to Y$.

The element $y \in Y$ associated to an element $x \in \operatorname{dom} f$ is called the **image of** x **by** or **under** f and denoted $y = f(x)$. Sometimes one writes

$$f : x \mapsto f(x).$$

The set of images $y = f(x)$ of all points in the domain constitutes the **range of** f, a subset of Y indicated by $\operatorname{im} f$.

The **graph** of f is the subset $\Gamma(f)$ of the Cartesian product $X \times Y$ made of pairs $(x, f(x))$ when x varies in the domain of f, i.e.,

$$\boxed{\Gamma(f) = \{(x, f(x)) \in X \times Y \ : \ x \in \operatorname{dom} f\}.} \qquad (2.1)$$

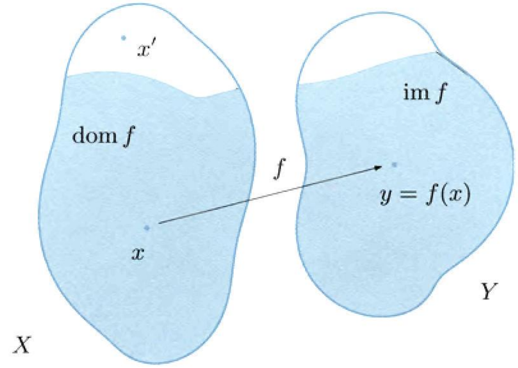

Figure 2.1. Naive representation of a function using Venn diagrams

In the sequel we shall consider maps between sets of numbers most of the time. If $Y = \mathbb{R}$, the function f is said **real** or **real-valued**. If $X = \mathbb{R}$, the function is of **one real variable**. Therefore the graph of a real function is a subset of the Cartesian plane $\mathbb{R}^2$.

A remarkable special case of map arises when $X = \mathbb{N}$ and the domain contains a set of the type $\{n \in \mathbb{N} \ : \ n \geq n_0\}$ for a certain natural number $n_0 \geq 0$. Such a function is called **sequence**. Usually, indicating by a the sequence, it is preferable to denote the image of the natural number n by the symbol a_n rather than $a(n)$; thus we shall write $a : n \mapsto a_n$. A common way to denote sequences is $\{a_n\}_{n \geq n_0}$ (ignoring possible terms with $n < n_0$) or even $\{a_n\}$.

Examples 2.1

Let us consider examples of real functions of real variable.

i) $f : \mathbb{R} \to \mathbb{R}, \ f(x) = ax + b$ (a, b real coefficients), whose graph is a straight line (Fig. 2.2, top left).

ii) $f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2$, whose graph is a parabola (Fig. 2.2, top right).

iii) $f : \mathbb{R}\backslash\{0\} \subset \mathbb{R} \to \mathbb{R}, \ f(x) = \frac{1}{x}$, has a rectangular hyperbola in the coordinate system of its asymptotes as graph (Fig. 2.2, bottom left).

iv) A real function of a real variable can be defined by multiple expressions on different intervals, in which case is it called a **piecewise function**. An example is given by $f : [0, 3] \to \mathbb{R}$

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1, \\ 4 - x & \text{if } 1 < x \leq 2, \\ x - 1 & \text{if } 2 < x \leq 3, \end{cases} \tag{2.2}$$

drawn in Fig. 2.2, bottom right.

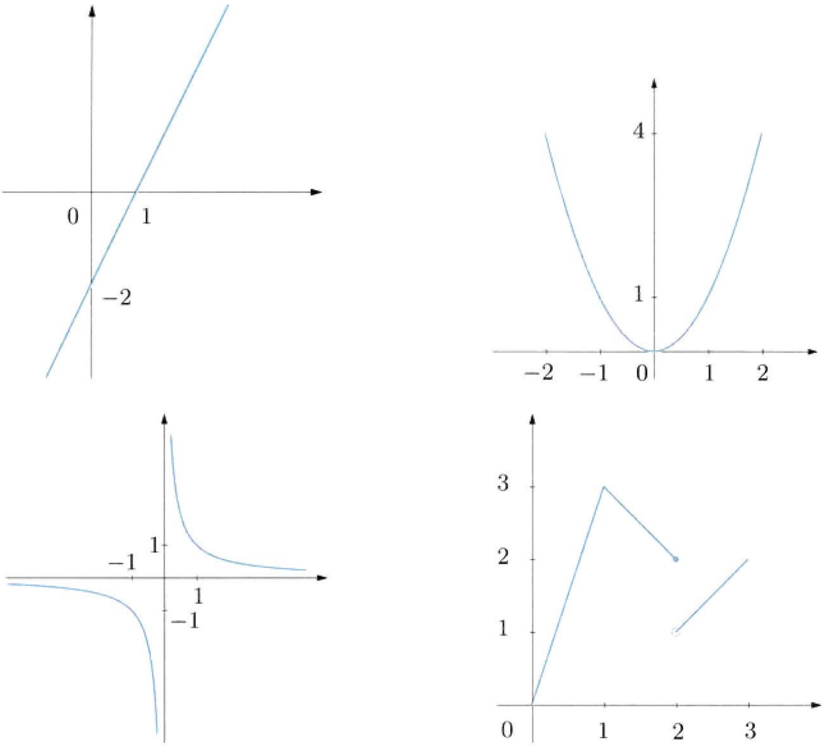

Figure 2.2. Graphs of the maps $f(x) = 2x-2$ (top left), $f(x) = x^2$ (top right), $f(x) = \dfrac{1}{x}$ (bottom left) and of the piecewise function (2.2) (bottom right)

Among piecewise functions, the following are particularly important:

v) the **absolute value** (Fig. 2.3, top left)

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0; \end{cases}$$

vi) the **sign** (Fig. 2.3, top right)

$$f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \operatorname{sign}(x) = \begin{cases} +1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0; \end{cases}$$

vii) the **integer part** (Fig. 2.3, bottom left), also known as **floor function**,

$$f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = [x] = \text{ the greatest integer } \leq x$$

(for example, $[4] = 4$, $[\sqrt{2}] = 1$, $[-1] = -1$, $[-\frac{3}{2}] = -2$); notice that

$$[x] \leq x < [x] + 1\,, \qquad \forall x \in \mathbb{R}\,;$$

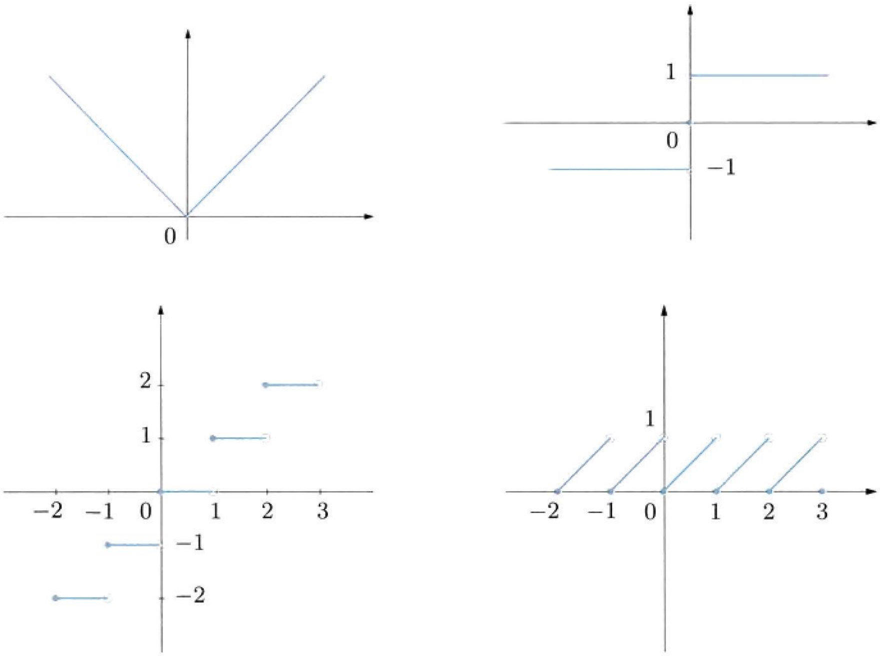

Figure 2.3. Clockwise from top left: graphs of the functions: absolute value, sign, mantissa and integer part

viii) the **mantissa** (Fig. 2.3, bottom right)

$$\boxed{f : \mathbb{R} \to \mathbb{R}, \quad f(x) = M(x) = x - [x]}$$

(the property of the floor function implies $0 \le M(x) < 1$).

Let us give some examples of sequences now.

ix) The sequence

$$a_n = \frac{n}{n+1} \tag{2.3}$$

is defined for all $n \ge 0$. The first few terms read

$$a_0 = 0, \quad a_1 = \frac{1}{2} = 0.5, \quad a_2 = \frac{2}{3} = 0.\overline{6}, \quad a_3 = \frac{3}{4} = 0.75.$$

Its graph is shown in Fig. 2.4 (top left).

x) The sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n \tag{2.4}$$

is defined for $n \ge 1$. The first terms are

$$a_1 = 2, \quad a_2 = \frac{9}{4} = 2.25, \quad a_3 = \frac{64}{27} = 2.37\overline{037}, \quad a_4 = \frac{625}{256} = 2.44140625.$$

Fig. 2.4 (top right) shows the graph of such sequence.

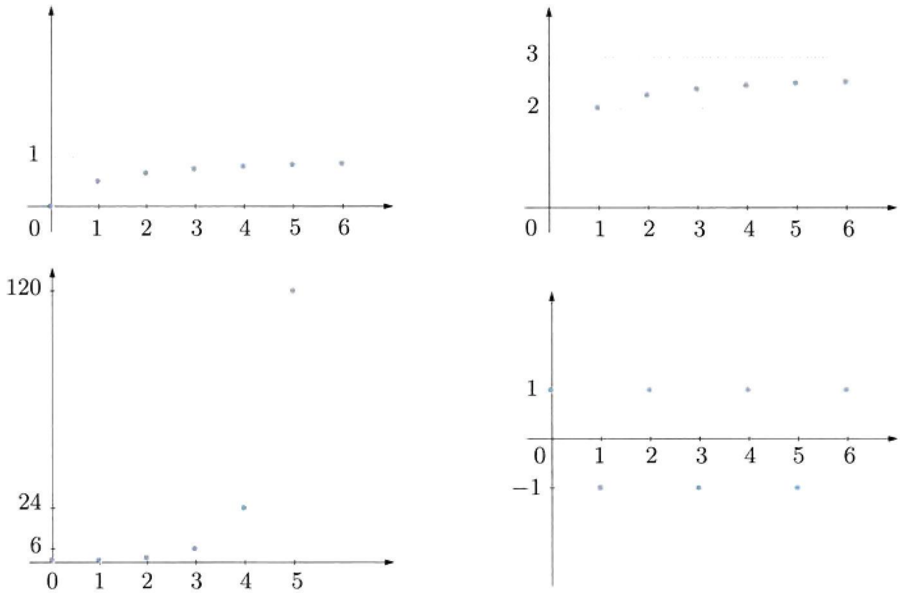

Figure 2.4. Clockwise: graphs of the sequences (2.3), (2.4), (2.6), (2.5)

xi) The sequence

$$a_n = n! \tag{2.5}$$

associates to each natural number its factorial, defined in (1.9). The graph of this sequence is shown in Fig. 2.4 (bottom left); as the values of the sequence grow rapidly as n increases, we used different scalings on the coordinate axes.

xii) The sequence

$$a_n = (-1)^n = \begin{cases} +1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd,} \end{cases} \qquad (n \geq 0) \tag{2.6}$$

has alternating values $+1$ and -1, according to the parity of n. The graph of the sequence is shown in Fig. 2.4 (bottom right).

At last, here are two maps defined on $\mathbb{R}^2$ (functions of *two real variables*).

xiii) The function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \sqrt{x^2 + y^2}$$

maps a generic point P of the plane with coordinates (x, y) to its distance from the origin.

xiv) The map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (y, x)$$

associates to a point P the point P' symmetric to P with respect to the bisectrix of the first and third quadrants. □

Consider a map from X to Y. One should take care in noting that the symbol for an element of X (to which one refers as the *independent variable*) and the symbol for an element in Y (*dependent variable*), are completely arbitary. What really determines the function is the way of associating each element of the domain to its corresponding image. For example, if x, y, z, t are symbols for real numbers, the expressions $y = f(x) = 3x$, $x = f(y) = 3y$, or $z = f(t) = 3t$ denote the *same* function, namely the one mapping each real number to its triple.

2.2 Range and pre-image

Let A be a subset of X. The **image of A under f** is the set

$$f(A) = \{f(x) \ : \ x \in A\} \subseteq \operatorname{im} f$$

of all the images of elements of A. Notice that $f(A)$ is empty if and only if A contains no elements of the domain of f. The image $f(X)$ of the whole set X is the range of f, already denoted by $\operatorname{im} f$.

Let y be any element of Y; the **pre-image of y by f** is the set

$$f^{-1}(y) = \{x \in \operatorname{dom} f \ : \ f(x) = y\}$$

of elements in X whose image is y. This set is empty precisely when y does not belong to the range of f. If B is a subset of Y, the **pre-image of B under f** is defined as the set

$$f^{-1}(B) = \{x \in \operatorname{dom} f \ : \ f(x) \in B\},$$

union of all pre-images of elements of B.

It is easy to check that $A \subseteq f^{-1}(f(A))$ for any subset A of $\operatorname{dom} f$, and $f(f^{-1}(B)) = B \cap \operatorname{im} f \subseteq B$ for any subset B of Y.

Example 2.2

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. The image under f of the interval $A = [1, 2]$ is the interval $B = [1, 4]$. Yet the pre-image of B under f is the union of the intervals $[-2, -1]$ and $[1, 2]$, namely, the set

$$f^{-1}(B) = \{x \in \mathbb{R} \ : \ 1 \le |x| \le 2\}$$

(see Fig. 2.5). □

The notions of infimum, supremum, maximum and minimum, introduced in Sect. 1.3.1, specialise in the case of images of functions.

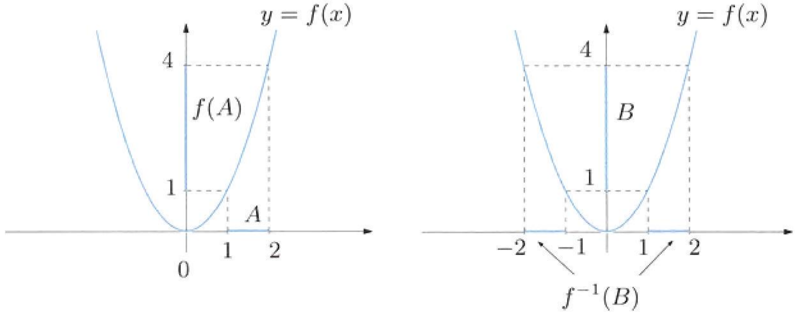

Figure 2.5. Image (left) and pre-image (right) of an interval relative to the function $f(x) = x^2$

Definition 2.3 *Let f be a real map and A a subset of* dom f. *One calls* **supremum of f on A** *(or* **in** *A) the supremum of the image of A under f*

$$\sup_{x \in A} f(x) = \sup f(A) = \sup\{f(x) \mid x \in A\}.$$

Then f is **bounded from above on** *A if the set $f(A)$ is bounded from above, or equivalently, if* $\sup_{x \in A} f(x) < +\infty$.
If $\sup_{x \in A} f(x)$ *is finite and belongs to $f(A)$, then it is the maximum of this set. This number is the* **maximum value** *(or simply, the* **maximum***)* **of f on** *A and is denoted by* $\max_{x \in A} f(x)$.
The concepts of **infimum** *and of* **minimum** *of f on A are defined similarly. Eventually, f is said* **bounded on** *A if the set $f(A)$ is bounded.*

At times, the shorthand notations $\sup_A f$, $\max_A f$, et c. are used.
The maximum value $M = \max_A f$ of f on the set A is characterised by the conditions:

i) M is a value assumed by the function on A, i.e.,

$$\text{there exists } x_M \in A \text{ such that } f(x_M) = M;$$

ii) M is greater or equal than any other value of the map on A, so

$$\text{for any } x \in A, \ f(x) \leq M.$$

Example 2.4

Consider the function $f(x)$ defined in (2.2). One verifies easily

$$\max_{x \in [0,2]} f(x) = 3, \qquad \min_{x \in [0,2]} f(x) = 0, \qquad \max_{x \in [1,3]} f(x) = 3, \qquad \inf_{x \in [1,3]} f(x) = 1.$$

The map does not assume the value 1 anywhere in the interval $[1, 3]$, so there is no minimum on that set. □

2.3 Surjective and injective functions; inverse function

A map with values in Y is called **onto** if $\operatorname{im} f = Y$. This means that each $y \in Y$ is the image of one element $x \in X$ at least. The term **surjective** (**on** Y) has the same meaning. For instance, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ with $a \neq 0$ is surjective on $\mathbb{R}$, or onto: the real number y is the image of $x = \frac{y-b}{a}$. On the contrary, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not onto, because its range coincides with the interval $[0, +\infty)$.

A function f is called **one-to-one** (or **1-1**) if every $y \in \operatorname{im} f$ is the image of a unique element $x \in \operatorname{dom} f$. Otherwise put, if $y = f(x_1) = f(x_2)$ for some elements x_1, x_2 in the domain of f, then necessarily $x_1 = x_2$. This, in turn, is equivalent to

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$$

for all $x_1, x_2 \in \operatorname{dom} f$ (see Fig. 2.6). Again, the term **injective** may be used. If a map f is one-to-one, we can associate to each element y in the range the unique x in the domain with $f(x) = y$. Such correspondence determines a function defined on Y and with values in X, called **inverse function** of f and denoted by the symbol f^{-1}. Thus

$$x = f^{-1}(y) \quad \Longleftrightarrow \quad y = f(x)$$

(the notation mixes up deliberately the pre-image of y under f with the unique element this set contains). The inverse function f^{-1} has the image of f as its domain, and the domain of f as range:

$$\operatorname{dom} f^{-1} = \operatorname{im} f, \qquad \operatorname{im} f^{-1} = \operatorname{dom} f.$$

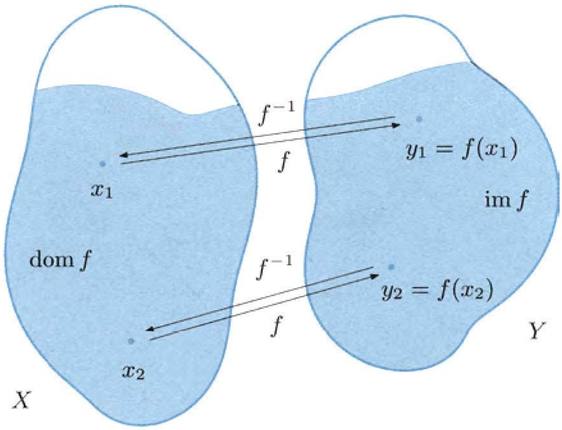

Figure 2.6. Representation of a one-to-one function and its inverse

A one-to-one map is therefore **invertible**; the two notions (injectivity and invertibility) coincide.

What is the link between the graphs of f, defined in (2.1), and of the inverse function f^{-1}? One has

$$\begin{aligned}\Gamma(f^{-1}) &= \{(y, f^{-1}(y)) \in Y \times X \ : \ y \in \mathrm{dom}\, f^{-1}\} \\ &= \{(f(x), x) \in Y \times X \ : \ x \in \mathrm{dom}\, f\}.\end{aligned}$$

Therefore, the graph of the inverse map may be obtained from the graph of f by *swapping* the components in each pair. For real functions of one real variable, this corresponds to a reflection in the Cartesian plane with respect to the bisectrix $y = x$ (see Fig. 2.7: a) is reflected into b)). On the other hand, finding the explicit expression $x = f^{-1}(y)$ of the inverse function could be hard, if possible at all.

Provided that the inverse map in the form $x = f^{-1}(y)$ can be determined, often one prefers to denote the independent variable (of f^{-1}) by x, and the dependent variable by y, thus obtaining the expression $y = f^{-1}(x)$. This is merely a change of notation (see the remark at the end of Sect. 2.1). The procedure allows to draw the graph of the inverse function in the same frame system of f (see Fig. 2.7, from b) to c)).

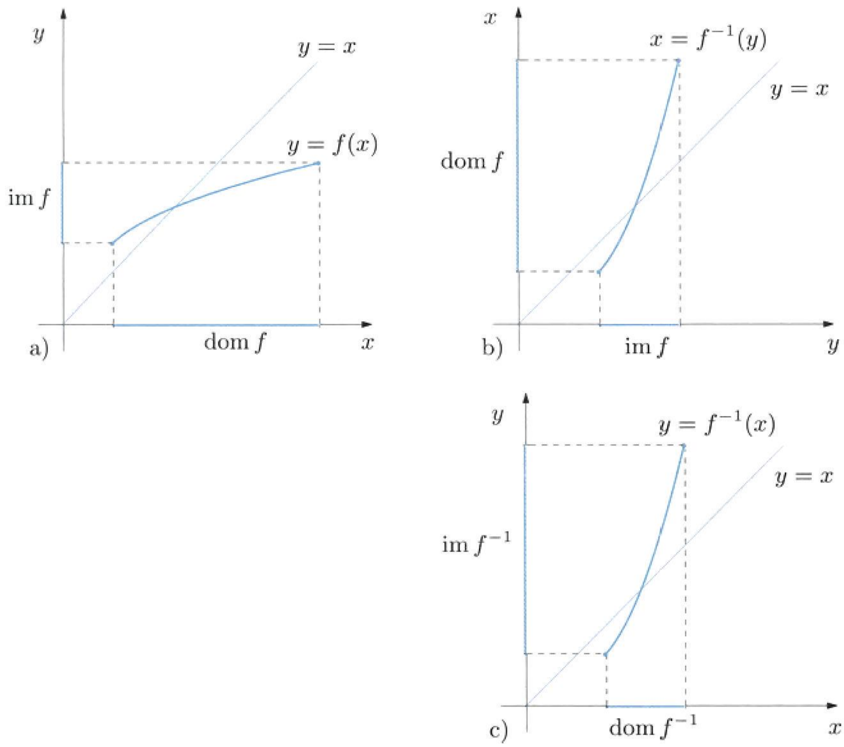

Figure 2.7. From the graph of a function to the graph of its inverse

Examples 2.5

i) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ is one-to-one for all $a \neq 0$ (in fact, $f(x_1) = f(x_2) \Rightarrow ax_1 = ax_2 \Rightarrow x_1 = x_2$). Its inverse is $x = f^{-1}(y) = \frac{y-b}{a}$, or $y = f^{-1}(x) = \frac{x-b}{a}$.

ii) The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not one-to-one because $f(x) = f(-x)$ for any real x. Yet if we consider only values ≥ 0 for the independent variable, i.e., if we **restrict** f to the interval $[0, +\infty)$, then the function becomes 1-1 (in fact, $f(x_1) = f(x_2) \Rightarrow x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = 0 \Rightarrow x_1 = x_2$). The inverse function $x = f^{-1}(y) = \sqrt{y}$ is also defined on $[0, +\infty)$. Conventionally one says that the 'squaring' map $y = x^2$ has the function 'square root' $y = \sqrt{x}$ for inverse (on $[0, +\infty)$). Notice that the restriction of f to the interval $(-\infty, 0]$ is 1-1, too; the inverse in this case is $y = -\sqrt{x}$.

iii) The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is one-to-one. In fact $f(x_1) = f(x_2) \Rightarrow x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \Rightarrow x_1 = x_2$ since $x_1^2 + x_1x_2 + x_2^2 = \frac{1}{2}[x_1^2 + x_2^2 + (x_1 + x_2)^2] > 0$ for any $x_1 \neq x_2$. The inverse function is the 'cubic root' $y = \sqrt[3]{x}$, defined on all $\mathbb{R}$. □

As in Example ii) above, if a function f is not injective over the whole domain, it might be so on a subset $A \subseteq \operatorname{dom} f$. The **restriction of f to A** is the function

$$f_{|A} : A \to Y \qquad \text{such that} \qquad f_{|A}(x) = f(x)\,, \quad \forall x \in A\,,$$

and is therefore invertible.

Let f be defined on X with values Y. If f is one-to-one and onto, it is called a **bijection** (or **bijective function**) from X to Y. If so, the inverse map f^{-1} is defined on Y, and is one-to-one and onto (on X); thus, f^{-1} is a bijection from Y to X.

For example, the functions $f(x) = ax + b$ ($a \neq 0$) and $f(x) = x^3$ are bijections from $\mathbb{R}$ to itself. The function $f(x) = x^2$ is a bijection on $[0, +\infty)$ (i.e., from $[0, +\infty)$ to $[0, +\infty)$).

If f is a bijection between X and Y, the sets X and Y are in **bijective correspondence** through f: each element of X is assigned to one and only one element of Y, and vice versa. The reader should notice that two *finite* sets (i.e., containing a finite number of elements) are in bijective correspondence if and only if they have the same number of elements. On the contrary, an infinite set can correspond bijectively to a proper subset; the function (sequence) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$, for example, establishes a bijection between $\mathbb{N}$ and the subset of even numbers.

To conclude the section, we would like to mention a significant interpretation of the notions of 1-1, onto, and bijective maps just introduced. Both in pure Mathematics and in applications one is frequently interested in solving a problem, or an equation, of the form

$$f(x) = y\,,$$