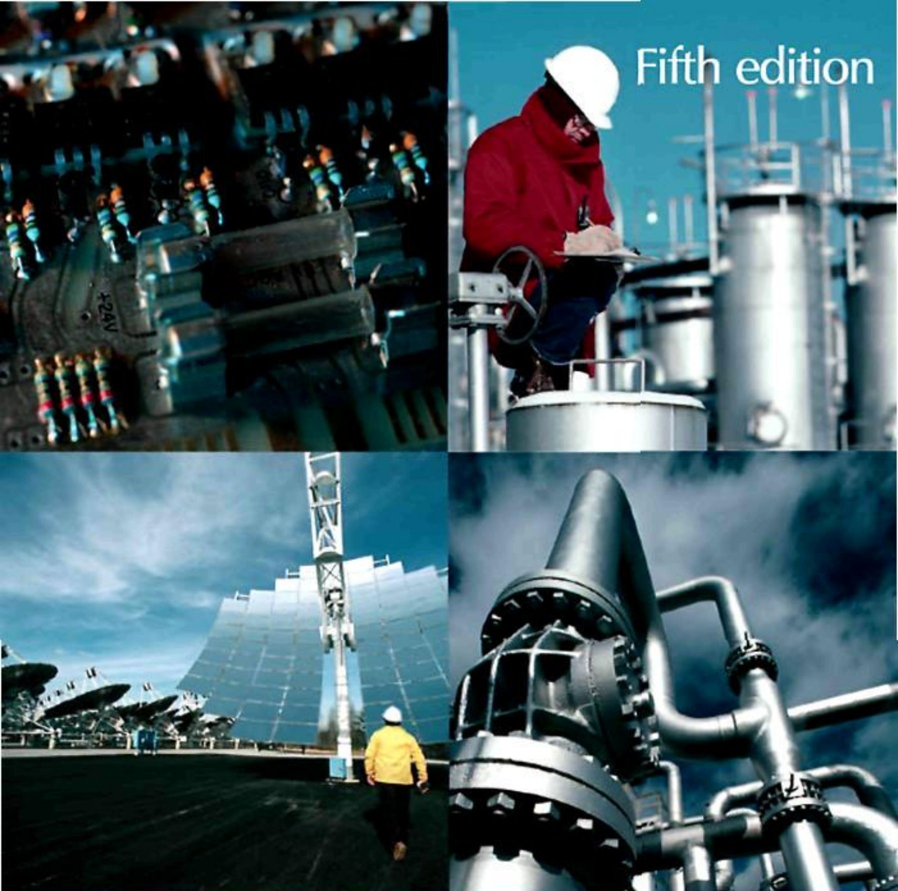

Fifth edition

HIGHER ENGINEERING MATHEMATICS

John Bird

Newnes

40

Integration using trigonometric and hyperbolic substitutions

40.1 Introduction

Table 40.1 gives a summary of the integrals that require the use of **trigonometric and hyperbolic substitutions** and their application is demonstrated in Problems 1 to 27.

40.2 Worked problems on integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

Problem 1. Evaluate $\int_0^{\frac{\pi}{4}} 2\cos^2 4t\,dt$.

Since $\cos 2t = 2\cos^2 t - 1$ (from Chapter 18),

then $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ and

$$\cos^2 4t = \frac{1}{2}(1 + \cos 8t)$$

Hence $\int_0^{\frac{\pi}{4}} 2\cos^2 4t\,dt$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 8t)\,dt$$

$$= \left[t + \frac{\sin 8t}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} + \frac{\sin 8\left(\frac{\pi}{4}\right)}{8}\right] - \left[0 + \frac{\sin 0}{8}\right]$$

$$= \boldsymbol{\frac{\pi}{4}} \text{ or } \mathbf{0.7854}$$

Problem 2. Determine $\int \sin^2 3x\,dx$.

Since $\cos 2x = 1 - 2\sin^2 x$ (from Chapter 18),

then $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and

$$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$$

Hence $\int \sin^2 3x\,dx = \int \frac{1}{2}(1 - \cos 6x)\,dx$

$$= \boldsymbol{\frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c}$$

Problem 3. Find $3\int \tan^2 4x\,dx$.

Since $1 + \tan^2 x = \sec^2 x$, then $\tan^2 x = \sec^2 x - 1$ and $\tan^2 4x = \sec^2 4x - 1$.

Hence $3\int \tan^2 4x\,dx = 3\int (\sec^2 4x - 1)\,dx$

$$= \boldsymbol{3\left(\frac{\tan 4x}{4} - x\right) + c}$$

Problem 4. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta\,d\theta$.

Since $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$, then $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and $\cot^2 2\theta = \operatorname{cosec}^2 2\theta - 1$.

Hence $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta\,d\theta$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\operatorname{cosec}^2 2\theta - 1)\,d\theta = \frac{1}{2}\left[\frac{-\cot 2\theta}{2} - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{-\cot 2\left(\frac{\pi}{3}\right)}{2} - \frac{\pi}{3}\right) - \left(\frac{-\cot 2\left(\frac{\pi}{6}\right)}{2} - \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}[(0.2887 - 1.0472) - (-0.2887 - 0.5236)]$$

$$= \mathbf{0.0269}$$

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Table 40.1 Integrals using trigonometric and hyperbolic substitutions

$f(x)$	$\int f(x)\mathrm{d}x$	Method	See problem
1. $\cos^2 x$	$\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$	Use $\cos 2x = 2\cos^2 x - 1$	1
2. $\sin^2 x$	$\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$	Use $\cos 2x = 1 - 2\sin^2 x$	2
3. $\tan^2 x$	$\tan x - x + c$	Use $1 + \tan^2 x = \sec^2 x$	3
4. $\cot^2 x$	$-\cot x - x + c$	Use $\cot^2 x + 1 = \mathrm{cosec}^2 x$	4
5. $\cos^m x \sin^n x$	(a) If either m or n is odd (but not both), use $\cos^2 x + \sin^2 x = 1$ (b) If both m and n are even, use either $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$		5, 6 7, 8
6. $\sin A \cos B$		Use $\frac{1}{2}[\sin(A+B) + \sin(A-B)]$	9
7. $\cos A \sin B$		Use $\frac{1}{2}[\sin(A+B) - \sin(A-B)]$	10
8. $\cos A \cos B$		Use $\frac{1}{2}[\cos(A+B) + \cos(A-B)]$	11
9. $\sin A \sin B$		Use $-\frac{1}{2}[\cos(A+B) - \cos(A-B)]$	12
10. $\frac{1}{\sqrt{(a^2 - x^2)}}$	$\sin^{-1}\frac{x}{a} + c$	Use $x = a\sin\theta$ substitution	13, 14
11. $\sqrt{(a^2 - x^2)}$	$\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{(a^2 - x^2)} + c$	Use $x = a\sin\theta$ substitution	15, 16
12. $\frac{1}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$	Use $x = a\tan\theta$ substitution	17–19
13. $\frac{1}{\sqrt{(x^2 + a^2)}}$	$\sinh^{-1}\frac{x}{a} + c$ or $\ln\left\{\frac{x + \sqrt{(x^2 + a^2)}}{a}\right\} + c$	Use $x = a\sinh\theta$ substitution	20–22
14. $\sqrt{(x^2 + a^2)}$	$\frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{(x^2 + a^2)} + c$	Use $x = a\sinh\theta$ substitution	23
15. $\frac{1}{\sqrt{(x^2 - a^2)}}$	$\cosh^{-1}\frac{x}{a} + c$ or $\ln\left\{\frac{x + \sqrt{(x^2 - a^2)}}{a}\right\} + c$	Use $x = a\cosh\theta$ substitution	24, 25
16. $\sqrt{(x^2 - a^2)}$	$\frac{x}{2}\sqrt{(x^2 - a^2)} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c$	Use $x = a\cosh\theta$ substitution	26, 27

Now try the following exercise.

Exercise 156 Further problems on integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

In Problems 1 to 4, integrate with respect to the variable.

1. $\sin^2 2x$ $\quad \left[\frac{1}{2}\left(x - \frac{\sin 4x}{4}\right) + c\right]$
2. $3\cos^2 t$ $\quad \left[\frac{3}{2}\left(t + \frac{\sin 2t}{2}\right) + c\right]$
3. $5\tan^2 3\theta$ $\quad \left[5\left(\frac{1}{3}\tan 3\theta - \theta\right) + c\right]$
4. $2\cot^2 2t$ $\quad [-(\cot 2t + 2t) + c]$

In Problems 5 to 8, evaluate the definite integrals, correct to 4 significant figures.

5. $\int_0^{\frac{\pi}{3}} 3\sin^2 3x\, dx$ $\quad \left[\frac{\pi}{2} \text{ or } 1.571\right]$
6. $\int_0^{\frac{\pi}{4}} \cos^2 4x\, dx$ $\quad \left[\frac{\pi}{8} \text{ or } 0.3927\right]$
7. $\int_0^{1} 2\tan^2 2t\, dt$ $\quad [-4.185]$
8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^2 \theta\, d\theta$ $\quad [0.6311]$

40.3 Worked problems on powers of sines and cosines

Problem 5. Determine $\int \sin^5 \theta\, d\theta$.

Since $\cos^2\theta + \sin^2\theta = 1$ then $\sin^2\theta = (1 - \cos^2\theta)$.

Hence $\int \sin^5\theta\, d\theta$

$$= \int \sin\theta(\sin^2\theta)^2\, d\theta = \int \sin\theta(1 - \cos^2\theta)^2\, d\theta$$

$$= \int \sin\theta(1 - 2\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta)\, d\theta$$

$$= \mathbf{-\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + c}$$

[Whenever a power of a cosine is multiplied by a sine of power 1, or vice-versa, the integral may be determined by inspection as shown.

In general, $\int \cos^n\theta \sin\theta\, d\theta = \frac{-\cos^{n+1}\theta}{(n+1)} + c$

and $\int \sin^n\theta\cos\theta\, d\theta = \frac{\sin^{n+1}\theta}{(n+1)} + c$

Problem 6. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\, dx$.

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)(1 - \sin^2 x)(\cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x\cos x - \sin^4 x\cos x)\, dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{\left(\sin\frac{\pi}{2}\right)^5}{5}\right] - [0 - 0]$$

$$= \frac{1}{3} - \frac{1}{5} = \mathbf{\frac{2}{15}} \text{ or } \mathbf{0.1333}$$

Problem 7. Evaluate $\int_0^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta$, correct to 4 significant figures.

$$\int_0^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta = 4\int_0^{\frac{\pi}{4}} (\cos^2\theta)^2\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1 + \cos 2\theta)\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

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$$= \left[\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\left(\frac{\pi}{4}\right) + \sin\frac{2\pi}{4} + \frac{\sin 4(\pi/4)}{8}\right] - [0]$$

$$= \frac{3\pi}{8} + 1 = \mathbf{2.178},$$

correct to 4 significant figures

Problem 8. Find $\int \sin^2 t \cos^4 t \, dt$.

$$\int \sin^2 t \cos^4 t \, dt = \int \sin^2 t (\cos^2 t)^2 \, dt$$

$$= \int \left(\frac{1 - \cos 2t}{2}\right)\left(\frac{1 + \cos 2t}{2}\right)^2 dt$$

$$= \frac{1}{8}\int (1 - \cos 2t)(1 + 2\cos 2t + \cos^2 2t)\, dt$$

$$= \frac{1}{8}\int (1 + 2\cos 2t + \cos^2 2t - \cos 2t - 2\cos^2 2t - \cos^3 2t)\, dt$$

$$= \frac{1}{8}\int (1 + \cos 2t - \cos^2 2t - \cos^3 2t)\, dt$$

$$= \frac{1}{8}\int \left[1 + \cos 2t - \left(\frac{1 + \cos 4t}{2}\right) - \cos 2t(1 - \sin^2 2t)\right] dt$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \frac{\cos 4t}{2} + \cos 2t \sin^2 2t\right) dt$$

$$= \mathbf{\frac{1}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8} + \frac{\sin^3 2t}{6}\right) + c}$$

Now try the following exercise.

Exercise 157 Further problems on integration of powers of sines and cosines

In Problems 1 to 6, integrate with respect to the variable.

1. $\sin^3 \theta$ $\quad \left[(a) -\cos\theta + \frac{\cos^3\theta}{3} + c\right]$

2. $2\cos^3 2x$ $\quad \left[\sin 2x - \frac{\sin^3 2x}{3} + c\right]$

3. $2\sin^3 t \cos^2 t$ $\quad \left[\frac{-2}{3}\cos^3 t + \frac{2}{5}\cos^5 t + c\right]$

4. $\sin^3 x \cos^4 x$ $\quad \left[\frac{-\cos^5 x}{5} + \frac{\cos^7 x}{7} + c\right]$

5. $2\sin^4 2\theta$ $\quad \left[\frac{3\theta}{4} - \frac{1}{4}\sin 4\theta + \frac{1}{32}\sin 8\theta + c\right]$

6. $\sin^2 t \cos^2 t$ $\quad \left[\frac{t}{8} - \frac{1}{32}\sin 4t + c\right]$

40.4 Worked problems on integration of products of sines and cosines

Problem 9. Determine $\int \sin 3t \cos 2t \, dt$.

$$\int \sin 3t \cos 2t \, dt$$

$$= \int \frac{1}{2}[\sin(3t + 2t) + \sin(3t - 2t)]\, dt,$$

from 6 of Table 40.1, which follows from Section 18.4, page 183,

$$= \frac{1}{2}\int (\sin 5t + \sin t)\, dt$$

$$= \mathbf{\frac{1}{2}\left(\frac{-\cos 5t}{5} - \cos t\right) + c}$$

Problem 10. Find $\int \frac{1}{3}\cos 5x \sin 2x \, dx$.

$$\int \frac{1}{3}\cos 5x \sin 2x \, dx$$

$$= \frac{1}{3}\int \frac{1}{2}[\sin(5x + 2x) - \sin(5x - 2x)]\, dx,$$

from 7 of Table 40.1

$$= \frac{1}{6}\int (\sin 7x - \sin 3x)\, dx$$

$$= \mathbf{\frac{1}{6}\left(\frac{-\cos 7x}{7} + \frac{\cos 3x}{3}\right) + c}$$

Problem 11. Evaluate $\int_0^1 2\cos 6\theta \cos\theta \, d\theta$, correct to 4 decimal places.

$$\int_0^1 2\cos 6\theta \cos\theta \, d\theta$$

$$= 2\int_0^1 \frac{1}{2}[\cos(6\theta+\theta) + \cos(6\theta-\theta)]\, d\theta,$$

from 8 of Table 40.1

$$= \int_0^1 (\cos 7\theta + \cos 5\theta)\, d\theta = \left[\frac{\sin 7\theta}{7} + \frac{\sin 5\theta}{5}\right]_0^1$$

$$= \left(\frac{\sin 7}{7} + \frac{\sin 5}{5}\right) - \left(\frac{\sin 0}{7} + \frac{\sin 0}{5}\right)$$

'sin 7' means 'the sine of 7 radians' ($\equiv 401°4'$) and $\sin 5 \equiv 286°29'$.

Hence $\int_0^1 2\cos 6\theta \cos\theta \, d\theta$

$$= (0.09386 + (-0.19178)) - (0)$$

$$= \mathbf{-0.0979}, \text{ correct to 4 decimal places}$$

Problem 12. Find $3\int \sin 5x \sin 3x \, dx$.

$$3\int \sin 5x \sin 3x \, dx$$

$$= 3\int -\frac{1}{2}[\cos(5x+3x) - \cos(5x-3x)]\, dx,$$

from 9 of Table 40.1

$$= -\frac{3}{2}\int (\cos 8x - \cos 2x)\, dx$$

$$= \mathbf{-\frac{3}{2}\left(\frac{\sin 8}{8} - \frac{\sin 2x}{2}\right) + c} \quad \text{or}$$

$$\mathbf{\frac{3}{16}(4\sin 2x - \sin 8x) + c}$$

Now try the following exercise.

Exercise 158 Further problems on integration of products of sines and cosines

In Problems 1 to 4, integrate with respect to the variable.

1. $\sin 5t \cos 2t$ $\quad \left[-\frac{1}{2}\left(\frac{\cos 7t}{7} + \frac{\cos 3t}{3}\right) + c\right]$

2. $2\sin 3x \sin x$ $\quad \left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4} + c\right]$

3. $3\cos 6x \cos x$ $\quad \left[\frac{3}{2}\left(\frac{\sin 7x}{7} + \frac{\sin 5x}{5}\right) + c\right]$

4. $\frac{1}{2}\cos 4\theta \sin 2\theta$ $\quad \left[\frac{1}{4}\left(\frac{\cos 2\theta}{2} - \frac{\cos 6\theta}{6}\right) + c\right]$

In Problems 5 to 8, evaluate the definite integrals.

5. $\int_0^{\frac{\pi}{2}} \cos 4x \cos 3x \, dx$ $\quad \left[\text{(a) } \frac{3}{7} \text{ or } 0.4286\right]$

6. $\int_0^1 2\sin 7t \cos 3t \, dt$ $\quad [0.5973]$

7. $-4\int_0^{\frac{\pi}{3}} \sin 5\theta \sin 2\theta \, d\theta$ $\quad [0.2474]$

8. $\int_1^2 3\cos 8t \sin 3t \, dt$ $\quad [-0.1999]$

40.5 Worked problems on integration using the sin θ substitution

Problem 13. Determine $\int \frac{1}{\sqrt{(a^2-x^2)}}\, dx$.

Let $x = a\sin\theta$, then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\, d\theta$.

Hence $\int \frac{1}{\sqrt{(a^2-x^2)}}\, dx$

$$= \int \frac{1}{\sqrt{(a^2 - a^2\sin^2\theta)}}\, a\cos\theta\, d\theta$$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{[a^2(1-\sin^2\theta)]}}$$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{(a^2\cos^2\theta)}}, \quad \text{since } \sin^2\theta + \cos^2\theta = 1$$

$$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta + c$$

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Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$.

Hence $\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \boldsymbol{\sin^{-1}\frac{x}{a} + c}$

Problem 14. Evaluate $\int_0^3 \frac{1}{\sqrt{(9 - x^2)}}\,dx$.

From Problem 13, $\int_0^3 \frac{1}{\sqrt{(9 - x^2)}}\,dx$

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3, \quad \text{since } a = 3$$

$$= (\sin^{-1} 1 - \sin^{-1} 0) = \boldsymbol{\frac{\pi}{2}} \text{ or } \mathbf{1.5708}$$

Problem 15. Find $\int \sqrt{(a^2 - x^2)}\,dx$.

Let $x = a\sin\theta$ then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$.

Hence $\int \sqrt{(a^2 - x^2)}\,dx$

$$= \int \sqrt{(a^2 - a^2\sin^2\theta)}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{[a^2(1 - \sin^2\theta)]}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{(a^2\cos^2\theta)}\,(a\cos\theta\,d\theta)$$

$$= \int (a\cos\theta)(a\cos\theta\,d\theta)$$

$$= a^2\int \cos^2\theta\,d\theta = a^2\int \left(\frac{1 + \cos 2\theta}{2}\right)d\theta$$

(since $\cos 2\theta = 2\cos^2\theta - 1$)

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + c$$

since from Chapter 18, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Also, $\cos^2\theta + \sin^2\theta = 1$, from which,

$$\cos\theta = \sqrt{(1 - \sin^2\theta)} = \sqrt{\left[1 - \left(\frac{x}{a}\right)^2\right]}$$

$$= \sqrt{\left(\frac{a^2 - x^2}{a^2}\right)} = \frac{\sqrt{(a^2 - x^2)}}{a}$$

Thus $\int \sqrt{(a^2 - x^2)}\,dx = \frac{a^2}{2}[\theta + \sin\theta\cos\theta]$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \left(\frac{x}{a}\right)\frac{\sqrt{(a^2 - x^2)}}{a}\right] + c$$

$$= \boldsymbol{\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{(a^2 - x^2)} + c}$$

Problem 16. Evaluate $\int_0^4 \sqrt{(16 - x^2)}\,dx$.

From Problem 15, $\int_0^4 \sqrt{(16 - x^2)}\,dx$

$$= \left[\frac{16}{2}\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{(16 - x^2)}\right]_0^4$$

$$= \left[8\sin^{-1}1 + 2\sqrt{(0)}\right] - [8\sin^{-1}0 + 0]$$

$$= 8\sin^{-1}1 = 8\left(\frac{\pi}{2}\right) = \boldsymbol{4\pi} \text{ or } \mathbf{12.57}$$

Now try the following exercise.

Exercise 159 Further problems on integration using the sine θ substitution

1. Determine $\int \frac{5}{\sqrt{(4 - t^2)}}\,dt$
$\left[5\sin^{-1}\frac{x}{2} + c\right]$

2. Determine $\int \frac{3}{\sqrt{(9 - x^2)}}\,dx$
$\left[3\sin^{-1}\frac{x}{3} + c\right]$

3. Determine $\int \sqrt{(4 - x^2)}\,dx$
$\left[2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{(4 - x^2)} + c\right]$

4. Determine $\int \sqrt{(16-9t^2)}\,dt$

$$\left[\frac{8}{3}\sin^{-1}\frac{3t}{4}+\frac{t}{2}\sqrt{(16-9t^2)}+c\right]$$

5. Evaluate $\int_0^4 \frac{1}{\sqrt{(16-x^2)}}\,dx$

$$\left[\frac{\pi}{2} \text{ or } 1.571\right]$$

6. Evaluate $\int_0^1 \sqrt{(9-4x^2)}\,dx$ [2.760]

40.6 Worked problems on integration using tan θ substitution

Problem 17. Determine $\int \frac{1}{(a^2+x^2)}\,dx$.

Let $x = a\tan\theta$ then $\frac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta\,d\theta$.

Hence $\int \frac{1}{(a^2+x^2)}\,dx$

$$= \int \frac{1}{(a^2+a^2\tan^2\theta)}(a\sec^2\theta\,d\theta)$$

$$= \int \frac{a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}, \quad \text{since } 1+\tan^2\theta = \sec^2\theta$$

$$= \int \frac{1}{a}\,d\theta = \frac{1}{a}(\theta)+c$$

Since $x = a\tan\theta$, $\theta = \tan^{-1}\frac{x}{a}$

Hence $\int \frac{1}{(a^2+x^2)}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$.

Problem 18. Evaluate $\int_0^2 \frac{1}{(4+x^2)}\,dx$.

From Problem 17, $\int_0^2 \frac{1}{(4+x^2)}\,dx$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 \quad \text{since } a = 2$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4}-0\right)$$

$$= \frac{\pi}{8} \text{ or } \mathbf{0.3927}$$

Problem 19. Evaluate $\int_0^1 \frac{5}{(3+2x^2)}\,dx$, correct to 4 decimal places.

$$\int_0^1 \frac{5}{(3+2x^2)}\,dx = \int_0^1 \frac{5}{2[(3/2)+x^2]}\,dx$$

$$= \frac{5}{2}\int_0^1 \frac{1}{[\sqrt{(3/2)}]^2+x^2}\,dx$$

$$= \frac{5}{2}\left[\frac{1}{\sqrt{(3/2)}}\tan^{-1}\frac{x}{\sqrt{(3/2)}}\right]_0^1$$

$$= \frac{5}{2}\sqrt{\left(\frac{2}{3}\right)}\left[\tan^{-1}\sqrt{\left(\frac{2}{3}\right)} - \tan^{-1}0\right]$$

$$= (2.0412)[0.6847-0]$$

$= \mathbf{1.3976}$, correct to 4 decimal places

Now try the following exercise.

Exercise 160 Further problems on integration using the tan θ substitution

1. Determine $\int \frac{3}{4+t^2}\,dt$ $\left[\frac{3}{2}\tan^{-1}\frac{t}{2}+c\right]$

2. Determine $\int \frac{5}{16+9\theta^2}\,d\theta$

$$\left[\frac{5}{12}\tan^{-1}\frac{3\theta}{4}+c\right]$$

3. Evaluate $\int_0^1 \frac{3}{1+t^2}\,dt$ [2.356]

4. Evaluate $\int_0^3 \frac{5}{4+x^2}\,dx$ [2.457]

40.7 Worked problems on integration using the sinh θ substitution

Problem 20. Determine $\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx$.

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