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# Mathematical Analysis II

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UNITEXT

*where $d$, $\alpha_i$, $p_j$, $q_j$ are real and $r_i$, $s_j$ integers such that*

$$r_1 + \cdots + r_h + 2s_1 + \cdots + 2s_k = m\,.$$

*The $\alpha_i$, all distinct, are the real roots of $Q$ counted with multiplicity $r_i$. The factors $x^2+2p_jx+q_j$ are pairwise distinct and irreducible over $\mathbb{R}$, i.e., $p_j^2-q_j < 0$, and have two complex(-conjugate) roots $\beta_{j,\pm}$ of multiplicity $s_j$.*

Using the factorisation (9.11) of $Q(x)$ we can now write $g(x)$ as sum of partial fractions

$$\frac{R(x)}{Q(x)} = \frac{1}{d}\left[F_1(x) + \cdots + F_h(x) + \bar{F}_1(x) + \cdots + \bar{F}_k(x)\right], \qquad (9.12)$$

where each $F_i(x)$ takes the form

$$F_i(x) = \frac{A_{i1}}{x-\alpha_i} + \frac{A_{i2}}{(x-\alpha_i)^2} + \cdots + \frac{A_{ir_i}}{(x-\alpha_i)^{r_i}},$$

while $\bar{F}_j(x)$ are like

$$\bar{F}_j(x) = \frac{B_{j1}x + C_{j1}}{x^2+2p_jx+q_j} + \frac{B_{j2}x + C_{j2}}{(x^2+2p_jx+q_j)^2} + \cdots + \frac{B_{j\bar{r}_j}x + C_{j\bar{r}_j}}{(x^2+2p_jx+q_j)^{s_j}},$$

for suitable constants $A_{i\ell}, B_{j\mu}, C_{j\mu}$. Note the total number of constants is $r_1 + \cdots r_h + 2s_1 + \cdots + 2s_k = m$.

To recover the undetermined coefficients we can transform the right-hand side of (9.12) into one fraction, whose denominator is clearly $Q(x)$. The numerator $\mathcal{R}(x)$ is a polynomial of degree $\leq m-1$ that must coincide with $R(x)$, and its coefficients are linear combinations of the unknown constants we are after. To find these numbers, the following principle on identity of polynomials is at our disposal.

**Theorem 9.16** *Two polynomials of degree $m-1$ coincide if and only if either of the next conditions holds*

*a) the coefficients of corresponding monomials coincide;*
*b) the polynomials assume the same values at $m$ distinct points.*

The first equivalence is easily derived from Proposition 7.5.

Going back to the $m$ unknowns $A_{i\ell}, B_{j\mu}, C_{j\mu}$, we could impose that the coefficients of each monomial in $\mathcal{R}(x)$ and $R(x)$ be the same, or else choose $m$ values of $x$ where the polynomials must agree. In the latter case the best choice falls on the real zeroes of $Q(x)$; should these be less than $m$ in number, we could also take $x = 0$.

Once these coefficients have been determined, we can start integrating the right-hand side of (9.12) and rely on the fundamental cases i)–v) above.

As usual, the technique is best illustrated with a few examples.

## Examples 9.17

i) Let us integrate

$$f(x) = \frac{2x^3 + x^2 - 4x + 7}{x^2 + x - 2}.$$

The numerator has greater degree than the denominator, so we divide the polynomials

$$f(x) = 2x - 1 + \frac{x+5}{x^2 + x - 2}.$$

The denominator factorises as $Q(x) = (x-1)(x+2)$. Therefore the coefficients to be found, $A_1 = A_{11}$ and $A_2 = A_{21}$, should satisfy

$$\frac{x+5}{x^2+x-2} = \frac{A_1}{x-1} + \frac{A_2}{x+2},$$

that is to say

$$x + 5 = A_1(x+2) + A_2(x-1), \tag{9.13}$$

hence

$$x + 5 = (A_1 + A_2)x + (2A_1 - A_2).$$

Comparing coefficients yields the linear system

$$\begin{cases} A_1 + A_2 = 1, \\ 2A_1 - A_2 = 5, \end{cases}$$

solved by $A_1 = 2$, $A_2 = -1$. Another possibility is to compute (9.13) at the zeroes $x = 1$, $x = -2$ of $Q(x)$, obtaining $6 = 3A_1$ and $3 = -3A_2$, whence again $A_1 = 2, A_2 = -1$. Therefore,

$$\begin{aligned} \int f(x)\,dx &= \int (2x-1)\,dx + 2\int \frac{1}{x-1}\,dx - \int \frac{1}{x+2}\,dx \\ &= x^2 - x + 2\log|x-1| - \log|x+2| + c. \end{aligned}$$

ii) Determine a primitive of the function

$$f(x) = \frac{x^2 - 3x + 3}{x^3 - 2x^2 + x}.$$

The denominator splits as $Q(x) = x(x-1)^2$, so we must search for $A_1 = A_{11}$, $A_{21}$ and $A_{22}$ such that

$$\frac{x^2 - 3x + 3}{x^3 - 2x^2 + x} = \frac{A_1}{x} + \frac{A_{21}}{x-1} + \frac{A_{22}}{(x-1)^2},$$

or

$$x^2 - 3x + 3 = A_1(x-1)^2 + A_{21}x(x-1) + A_{22}x.$$

Putting $x = 0$ yields $A_1 = 3$, with $x = 1$ we find $A_{22} = 1$. The remaining $A_{21}$ is determined by picking a third value $x \neq 0, 1$. For instance $x = -1$ gives $7 = 12 + 2A_{21} - 1$, so $A_{21} = -2$.

In conclusion,

$$\int f(x)\,\mathrm{d}x = 3\int \frac{1}{x}\,\mathrm{d}x - 2\int \frac{1}{x-1}\,\mathrm{d}x + \int \frac{1}{(x-1)^2}\,\mathrm{d}x$$
$$= 3\log|x| - 2\log|x-1| - \frac{1}{x-1} + c.$$

iii) Integrate

$$f(x) = \frac{3x^2 + x - 4}{x^3 + 5x^2 + 9x + 5}.$$

The point $x = -1$ annihilates the denominator (the sum of the odd-degree coefficients equals those of even degree), so the denominator splits $Q(x) = (x+1)(x^2+4x+5)$ by Ruffini's rule. The unknown coefficients are $A = A_{11}$, $B = B_{11}$, $C = C_{11}$ so that

$$\frac{3x^2 + x - 4}{x^3 + 5x^2 + 9x + 5} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 4x + 5},$$

hence

$$3x^2 + x - 4 = A(x^2 + 4x + 5) + (Bx + C)(x+1).$$

Choosing $x = -1$, and then $x = 0$, produces $A = -1$ and $C = 1$. The last coefficient $B = 4$ is found by taking $x = -1$. Thus

$$\int f(x)\,\mathrm{d}x = -\int \frac{1}{x+1}\,\mathrm{d}x + \int \frac{4x+1}{x^2+4x+5}\,\mathrm{d}x$$
$$= -\int \frac{1}{x+1}\,\mathrm{d}x + 2\int \frac{2x+4}{x^2+4x+5}\,\mathrm{d}x - 7\int \frac{1}{1+(x+2)^2}\,\mathrm{d}x$$
$$= -\log|x+1| + 2\log(x^2+4x+5) - 7\arctan(x+2) + c.$$ □

Note that many functions $f(x)$ that are not rational in the variable $x$ can be transformed – by an appropriate change $t = \varphi(x)$ – into a rational map in the new variable $t$. Special cases thereof include:

i) $f$ is a rational function of $\sqrt[p]{x-a}$ for some integer $p$ and $a$ real. Then one lets

$$t = \sqrt[p]{x-a}, \qquad \text{whence } x = a + t^p \text{ and } \mathrm{d}x = pt^{p-1}\mathrm{d}t.$$

ii) $f$ is rational in $\mathrm{e}^{ax}$ for some real $a \neq 0$. The substitution

$$t = \mathrm{e}^{ax} \qquad \text{gives } x = \frac{1}{a}\log t \text{ and } \mathrm{d}x = \frac{1}{at}\,\mathrm{d}t.$$

iii) $f$ is rational in $\sin x$ and/or $\cos x$. In this case

$$t = \tan\frac{x}{2},$$

together with the identities

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \tag{9.14}$$