Trigonometrik funksiyalarni integrallash

- Reja:

- 1. trigonometrik funksiyalarni integrallashda foydalaniladigan formulalar.
- 2. Trigonometrik funksiyalarni integrallash
- 3.Misollar yechish.

Trigonometrik funksiyalarni integrallash.

(1) ko'rinishdagi integrallarni qaraymiz. Bu integral almashtirish yordami bilan hamma vaqt ratsional funksiyaning integraliga keltirilishi mumkin ekanini ko'rsatamiz. *Sinx* va *cosx* funksiyalarni (2) bilan, ya'ni t bilan ifoda etamiz:

$$sinx = \frac{2sin\frac{x}{2}cos\frac{x}{2}}{1} = \frac{2sin\frac{x}{2}cos\frac{x}{2}}{sin^2\frac{x}{2}cos^2\frac{x}{2}} = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$cosx = \frac{cos^2\frac{x}{2} - sin^2\frac{x}{2}}{1} = \frac{cos^2\frac{x}{2} - sin^2\frac{x}{2}}{cos^2\frac{x}{2} + sin^2\frac{x}{2}} = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$= \frac{1-t^2}{1+t^2}$$

- Endi (2) tenglikdan:
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- $x = 2\, arc\, tg\, t, \qquad dx = \frac{2dt}{1+t^2}$
- Shunday qilib, *sinx, cosx* va *dx* lar *t* bilan ratsional ifodalandi, ammo ratsional funksiyalarning raysional funksiyasi o'z navbatida yana ratsiona funksiya bo'lgani uchun hosil qilingan ifodalarni berilgan (1) integralga qo'yib ratsional funksiyaning integralini hosil qilamiz:
- $\int R(sinx, cosx)dx = \int R\left[\frac{2t}{1+t^2},\ \frac{1-t^2}{1+t^2}\right]\frac{2dt}{1+t^2}$

- Yuqoridagi almashtirish har qanday trigonometric funksiyani integrallash imkonini beradi. Shuning uchun uni ba’zan «universal trigonometric almashtiris» deb ataladi. Lekin amalda bu almashtirish ko’pincha ancha murakkab ratsional funksiyaga olib keladi. Shunung uchun «universal» almashtirish bilan bir qatorda ba’zi hollar uchun maqsadga tez olib keladigan boshqa almashtirishlar ham qo’llaniladi.
- Agar integral ko’rinishida bo’lsa, u holda sinx = t, cosxdx = dt almashtirish bu integralni ko’rinishiga olib keladi.
- Agar integral ko’rinishida bo’lsa, u holda cosx = t, sinxdx = -dt almashtirish yordamida bu integral ratsional funksiyaning integraliga keltiriladi.

- Agar integral ostidagi funksiya faqat tgx ga bog'liq bo'lsa, u holda
- *tgx = t, x = arctgt*, dx = almashtirish yordamida bu integral ratsional funksiyaning integraliga keltiriladi.
- Agar integral ostidagi funksiya *R(sinx, cosx)* ko'rinishida bo'lsa, ammo bunda sinx va cosx larning faqat juft darajalari kirsa, u holda
- *tgx = t* (3) almashtirish tatbiq etiladi. Sin^2x va cos^2x lar tgx bilan ratsional ifoda etiladi.

- *Ba’zi trigonometric funksiyalarni tgx orqali ifodalanishi.*
- $cos^2x = \frac{1}{1+tg^2x} = \frac{1}{1+t^2}$
- $sin^2x = \frac{tg^2x}{1+tg^2} = \frac{t^2}{1+t^2}$
- $dx = \frac{dt}{1+t^2}$
- 2 misol.
- $\int \frac{sin^3x}{2+cosx} dx$
- Integral hisoblansin.
- Yechish. Bu integralni
- ko’rinishiga keltiriladi.

- $\int \frac{sin^3 x}{2+cosx} dx = \int \frac{sin^2 x \, sinx \, dx}{2+cos \, x} = \int \frac{1-cos^2 x}{2+cosx} \sin x \, dx$
- Cosx =z almashtirishni bajaramiz. Bu holda *sinxdx = -dz*
- Demak,
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- $\int \frac{sin^3 x}{2+cosx} dx = \int \frac{1-z^2}{2+z}(-dz) = \int \frac{z^2-1}{z+2} dz = \int \left(z - 2 + \frac{3}{z+2}\right) = \frac{z^2}{2} - 2z + 3\ln(z+2) + C = \frac{cos^2 x}{2} - 2cosx + 3\ln(cosx + 2) + C$

- 5) ko‘rinishdagi integrallarda uchta holni ko‘ramiz.
- a) integralda m va n larning kamida bittasi toq bo’lsin. Aniqliq uchun n toq son deb faraz qilamiz. $n = 2p + 1$ deb olib, integralni o’zgartiramiz.
- o’zgaruvchini almashtiramiz: $\sin x = t$, $\cos x dx = dt$
- yangi o’zgaruvchini berilgan integralga qo’yamiz

- , bu esa *t* ning ratsional funksiyasining integralidir.
- b) integralda m va n manfiy bo’lmagan juft son. $m = 2p$, $n = 2q$ deb qaraymiz. Trigonometriyada ma’lum bo’lgan formulalarni yozamiz: (4)
- bularni berilgan integralga qo’yamiz:

- Darajaga ko’tarib hamda qavslarni ochib, *cos2x* ning juft ba toq darajalarini o’z ichiga olgan hadlarni hosil qilamiz.
- Toq darajali hadlar a) holda ko’rsatilgandek, integrallanadi.
- Darajaning juft ko’rsatkichlarini (4) formulalariga ko’ra yana pasaytiramiz. Daraja ko’rsatkichlarini pasaytiriwni oson integrallanadigan ko’rinishdagi hadlar hosil bo’lgunicha shunday davom ettiramiz.

3-misol. $\int \sin 2x \cos 7x dx$ integralni hisoblang.

yechish. Yuqoridagi formulalarning birinchisidan

$$\sin 2x \cos 7x = \frac{1}{2}\left[\sin(2x + 7x) + \sin(2x - 7x)\right] = \frac{1}{2}(\sin 9x \sin 5x),$$

$$\int \sin 2x \cos 7x dx = \frac{1}{2}\int (\sin 9x - \sin 5x) dx = \frac{1}{2}\int \sin 9x dx - \frac{1}{2}\int \sin 5x dx =$$

$$= \frac{1}{2}\cdot\frac{1}{9}(-\cos 9x) - \frac{1}{2}\cdot\frac{1}{5}(-\cos 5x) + C = \frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x + C.$$

natijaga ega bo‘lamiz.

- 4-misol.   integralni hisoblang.
- yechish. Bu integralni izoḱlarsiz ḱisoblaymiz:
- .
- 5-misol.   integralni hisoblang.
- yechish. Trigonometrik funksiyalarning darajalarini pasaytirish formulalaridan foydalanib, quyidagi natijaga kelamiz:

$$\int \sin^2 x\cos^4 x dx = \int \frac{1-\cos 2x}{2}\cdot\left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)^2 dx =$$

$$= \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)dx = \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int (\sin^2 2x\cos 2x dx) =$$

$$= \frac{1}{16}\int (1-\cos 4x)dx + \frac{1}{16}\int \sin^2 2x d\sin 2x = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{16}\frac{\sin^3 2x}{3} + C =$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$$

6-misol. $\int \sin^3 x \cos^4 x dx$ integralni hisoblang.

yechish. $\sin x dx = -d(\cos x)$ va $\sin^2 x = 1 - \cos^2 x$ ekanligini hamda $\cos x = z$ almashtirish kiritib, quyidagini hosil qilamiz:

$$\int \sin^3 x \cos^4 x dx = \int \sin^2 x \cos^4 x \sin x dx = \int (1 - \cos^2 x) \cos^4 x (-d \cos x) =$$

$$= -\int (1 - z^2) z^4 dz = -\int (z^4 - z^6) dz = -\frac{z^5}{5} + \frac{z^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.$$

Har xil argumentli sinus va kosinuslar ko‘paytmalari shaklidagi funksiyalarni integrallash.

$$\int \sin mx \cos nxdx, \quad \int \sin mx \sin nxdx, \quad \int \cos mx \cos nxdx \qquad (1)$$

ko‘rinishdagi integrallarni karaymiz. Trigonometrik funksiyalarning ko‘paytmadan yig‘indiga keltirish formulalaridan foydalanamiz.

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha+\beta)\right], \quad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right],$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

formulalardan foydalanib, (1) ko‘rinishdagi integrallarni

$$\int \sin axdx, \quad \int \cos bxdx$$

integrallardan biriga keltirib itegrallanadi.

8-misol. $\int \sin 2x \cos 7x dx$ integralni hisoblang.

yechish. Yuqoridagi formulalarning birinchisidan foydalanamiz

$$\sin 2x \cos 7x = \frac{1}{2}\left[\sin(2x+7x)+\sin(2x-7x)\right] = \frac{1}{2}(\sin 9x \sin 5x),$$

$$\int \sin 2x \cos 7x dx = \frac{1}{2}\int(\sin 9x - \sin 5x)dx = \frac{1}{2}\int \sin 9x dx - \frac{1}{2}\int \sin 5x dx =$$

$$= \frac{1}{2}\cdot\frac{1}{9}(-\cos 9x) - \frac{1}{2}\cdot\frac{1}{5}(-\cos 5x) + C = \frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x + C.$$

natijaga ega bo‘lamiz.