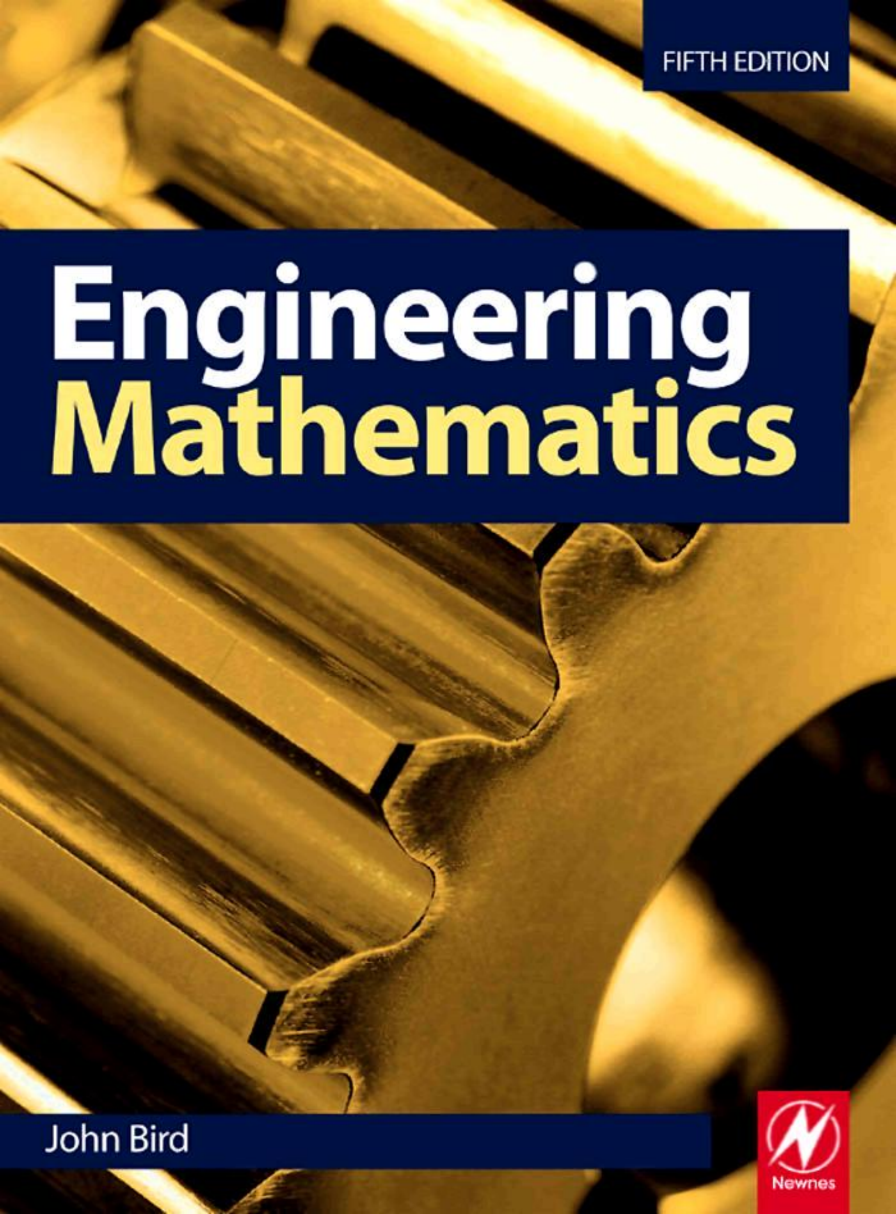
FIFTH EDITION

Engineering Mathematics



John Bird

Newnes

Chapter 53

Integration by parts

53.1 Introduction

From the product rule of differentiation:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

where u and v are both functions of x.

Rearranging gives: $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

Integrating both sides with respect to x gives:

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$

ie $$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

or $$\int u\,dv = uv - \int v\,du$$

This is known as the **integration by parts formula** and provides a method of integrating such products of simple functions as $\int xe^x dx$, $\int t\sin t\,dt$, $\int e^\theta \cos\theta\,d\theta$ and $\int x\ln x\,dx$.

Given a product of two terms to integrate the initial choice is: 'which part to make equal to u' and 'which part to make equal to dv'. The choice must be such that the 'u part' becomes a constant after successive differentiation and the 'dv part' can be integrated from standard integrals. Invariable, the following rule holds: 'If a product to be integrated contains an algebraic term (such as x, t^2 or 3θ) then this term is chosen as the u part. The one exception to this rule is when a '$\ln x$' term is involved; in this case $\ln x$ is chosen as the 'u part'.

53.2 Worked problems on integration by parts

Problem 1. Determine: $\int x\cos x\,dx$

From the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

Let $u = x$, from which $\frac{du}{dx} = 1$, i.e. $du = dx$ and let $dv = \cos x\,dx$, from which $v = \int \cos x\,dx = \sin x$.

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\int \boxed{u}\ \boxed{dv} = \boxed{u}\ \boxed{v} - \int \boxed{v}\ \boxed{du}$$
$$\int \boxed{x}\ \boxed{\cos x\,dx} = \boxed{(x)}\ \boxed{(\sin x)} - \int \boxed{(\sin x)}\ \boxed{(dx)}$$

i.e. $$\int x\cos x\,dx = x\sin x - (-\cos x) + c$$

$$= \mathbf{x\sin x + \cos x + c}$$

[This result may be checked by differentiating the right hand side,

i.e. $\frac{d}{dx}(x\sin x + \cos x + c)$

$= [(x)(\cos x) + (\sin x)(1)] - \sin x + 0$

using the product rule

$= x\cos x$, which is the function being integrated

Problem 2. Find: $\int 3te^{2t}\,dt$

Let $u=3t$, from which, $\frac{du}{dt}=3$, i.e. $du=3\,dt$ and let $dv=e^{2t}\,dt$, from which, $v=\int e^{2t}dt=\frac{1}{2}e^{2t}$

Substituting into $\int u\,dv=u\,v-\int v\,du$ gives:

$$\int 3te^{2t}\,dt=(3t)\left(\frac{1}{2}e^{2t}\right)-\int\left(\frac{1}{2}e^{2t}\right)(3\,dt)$$

$$=\frac{3}{2}te^{2t}-\frac{3}{2}\int e^{2t}\,dt$$

$$=\frac{3}{2}te^{2t}-\frac{3}{2}\left(\frac{e^{2t}}{2}\right)+c$$

Hence $\int \mathbf{3te^{2t}\,dt=\frac{3}{2}e^{2t}\left(t-\frac{1}{2}\right)+c}$,

which may be checked by differentiating.

Problem 3. Evaluate $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$

Let $u=2\theta$, from which, $\frac{du}{d\theta}=2$, i.e. $du=2\,d\theta$ and let $dv=\sin\theta\,d\theta$, from which,

$$v=\int\sin\theta\,d\theta=-\cos\theta$$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int 2\theta\sin\theta\,d\theta=(2\theta)(-\cos\theta)-\int(-\cos\theta)(2\,d\theta)$$

$$=-2\theta\cos\theta+2\int\cos\theta\,d\theta$$

$$=-2\theta\cos\theta+2\sin\theta+c$$

Hence $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$

$$=\Big[2\theta\cos\theta+2\sin\theta\Big]_0^{\frac{\pi}{2}}$$

$$=\left[-2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2}+2\sin\frac{\pi}{2}\right]-[0+2\sin 0]$$

$$=(-0+2)-(0+0)=\mathbf{2}$$

since $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$

Problem 4. Evaluate: $\int_0^1 5xe^{4x}dx$, correct to 3 significant figures

Let $u=5x$, from which $\frac{du}{dx}=5$, i.e. $du=5\,dx$ and let $dv=e^{4x}\,dx$, from which, $v=\int e^{4x}dx=\frac{1}{4}e^{4x}$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int 5xe^{4x}\,dx=(5x)\left(\frac{e^{4x}}{4}\right)-\int\left(\frac{e^{4x}}{4}\right)(5\,dx)$$

$$=\frac{5}{4}xe^{4x}-\frac{5}{4}\int e^{4x}\,dx$$

$$=\frac{5}{4}xe^{4x}-\frac{5}{4}\left(\frac{e^{4x}}{4}\right)+c$$

$$=\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)+c$$

Hence $\int_0^1 5xe^{4x}\,dx$

$$=\left[\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)\right]_0^1$$

$$=\left[\frac{5}{4}e^4\left(1-\frac{1}{4}\right)\right]-\left[\frac{5}{4}e^0\left(0-\frac{1}{4}\right)\right]$$

$$=\left(\frac{15}{16}e^4\right)-\left(-\frac{5}{16}\right)$$

$$=51.186+0.313=51.499=\mathbf{51.5},$$

correct to 3 significant figures.

Problem 5. Determine: $\int x^2\sin x\,dx$

Let $u=x^2$, from which, $\frac{du}{dx}=2x$, i.e. $du=2x\,dx$, and let $dv=\sin x\,dx$, from which, $v=\int\sin x\,dx=-\cos x$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int x^2\sin x\,dx=(x^2)(-\cos x)-\int(-\cos x)(2x\,dx)$$

$$=-x^2\cos x+2\left[\int x\cos x\,dx\right]$$

The integral, $\int x\cos x\,dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.

From Problem 1, $\int x\cos x\,dx = x\sin x + \cos x$

Hence $\int x^2 \sin x\,dx$

$$= -x^2\cos x + 2\{x\sin x + \cos x\} + c$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

$$= \mathbf{(2 - x^2)\cos x + 2x\sin x + c}$$

In general, if the algebraic term of a product is of power n, then the integration by parts formula is applied n times.

Now try the following exercise

Exercise 186 Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int xe^{2x}dx$ $\quad\left[\frac{e^{2x}}{2}\left(x-\frac{1}{2}\right)+c\right]$
2. $\int \frac{4x}{e^{3x}}dx$ $\quad\left[-\frac{4}{3}e^{-3x}\left(x+\frac{1}{3}\right)+c\right]$
3. $\int x\sin x\,dx$ $\quad[-x\cos x + \sin x + c]$
4. $\int 5\theta\cos 2\theta\,d\theta$ $\quad\left[\frac{5}{2}\left(\theta\sin 2\theta+\frac{1}{2}\cos 2\theta\right)+c\right]$
5. $\int 3t^2e^{2t}\,dt$ $\quad\left[\frac{3}{2}e^{2t}\left(t^2-t+\frac{1}{2}\right)+c\right]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_0^2 2xe^x\,dx$ $\quad[16.78]$
7. $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$ $\quad[0.2500]$
8. $\int_0^{\frac{\pi}{2}} t^2\cos t\,dt$ $\quad[0.4674]$
9. $\int_1^2 3x^2e^{\frac{x}{2}}\,dx$ $\quad[15.78]$

53.3 Further worked problems on integration by parts

Problem 6. Find: $\int x\ln x\,dx$

The logarithmic function is chosen as the 'u part' Thus when $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$

Letting $dv = x\,dx$ gives $v = \int x\,dx = \frac{x^2}{2}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x\ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\frac{dx}{x}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

Hence $\displaystyle\int x\ln x\,dx = \mathbf{\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c}$

or $\mathbf{\frac{x^2}{4}(2\ln x - 1) + c}$

Problem 7. Determine: $\int \ln x\,dx$

$\int \ln x\,dx$ is the same as $\int (1)\ln x\,dx$

Let $u = \ln x$, from which, $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$ and let $dv = 1\,dx$, from which, $v = \int 1\,dx = x$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int \ln x\,dx = (\ln x)(x) - \int x\frac{dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + c$$

Hence $\displaystyle\int \mathbf{\ln x\,dx = x(\ln x - 1) + c}$

Problem 8. Evaluate: $\int_1^9 \sqrt{x}\ln x\,dx$, correct to 3 significant figures

Let $u = \ln x$, from which $du = \dfrac{dx}{x}$
and let $dv = \sqrt{x}\,dx = x^{\frac{1}{2}}\,dx$, from which,

$$v = \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int \sqrt{x}\ln x\,dx = (\ln x)\left(\frac{2}{3}x^{\frac{3}{2}}\right) - \int\left(\frac{2}{3}x^{\frac{3}{2}}\right)\left(\frac{dx}{x}\right)$$
$$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$$
$$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + c$$
$$= \frac{2}{3}\sqrt{x^3}\left[\ln x - \frac{2}{3}\right] + c$$

Hence $\displaystyle\int_1^9 \sqrt{x}\ln x\,dx = \left[\frac{2}{3}\sqrt{x^3}\left(\ln x - \frac{2}{3}\right)\right]_1^9$

$$= \left[\frac{2}{3}\sqrt{9^3}\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\sqrt{1^3}\left(\ln 1 - \frac{2}{3}\right)\right]$$
$$= \left[18\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\left(0 - \frac{2}{3}\right)\right]$$
$$= 27.550 + 0.444 = 27.994 = \mathbf{28.0},$$

correct to 3 significant figures.

Problem 9. Find: $\int e^{ax}\cos bx\,dx$

When integrating a product of an exponential and a sine or cosine function it is immaterial which part is made equal to 'u'.

Let $u = e^{ax}$, from which $\dfrac{du}{dx} = ae^{ax}$, i.e. $du = ae^{ax}\,dx$
and let $dv = \cos bx\,dx$, from which,

$$v = \int \cos bx\,dx = \frac{1}{b}\sin bx$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int e^{ax}\cos bx\,dx$$
$$= (e^{ax})\left(\frac{1}{b}\sin bx\right) - \int\left(\frac{1}{b}\sin bx\right)(ae^{ax}dx)$$
$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[\int e^{ax}\sin bx\,dx\right] \qquad (1)$$

$\int e^{ax}\sin bx\,dx$ is now determined separately using integration by parts again:

Let $u = e^{ax}$ then $du = ae^{ax}\,dx$, and let $dv = \sin bx\,dx$, from which

$$v = \int \sin bx\,dx = -\frac{1}{b}\cos bx$$

Substituting into the integration by parts formula gives:

$$\int e^{ax}\sin bx\,dx = (e^{ax})\left(-\frac{1}{b}\cos bx\right) - \int\left(-\frac{1}{b}\cos bx\right)(ae^{ax}\,dx)$$
$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

Substituting this result into equation (1) gives:

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right]$$
$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

The integral on the far right of this equation is the same as the integral on the left hand side and thus they may be combined.

$$\int e^{ax}\cos bx\,dx + \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

i.e. $$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

i.e. $$\left(\frac{b^2 + a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{b^2}(b\sin bx + a\cos bx)$$